ON THE DISCREPANCY OF RANDOM WALKS ON THE CIRCLE

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ABSTRACT. Let $X_1, X_2, \dots$ be i.i.d. absolutely continuous random variables, let $S_k = \sum_{j=1}^k X_j \pmod{1}$ and let D_N^* denote the star discrepancy of the sequence $(S_k)_{1 \leq k \leq N}$. We determine the limit distribution of $\sqrt{N}D_N^*$ and the weak limit of the sequence $\sqrt{N}(F_N(t) - t)$ in the Skorohod space $D[0, 1]$, where $F_N(t)$ denotes the empirical distribution function of the sequence $(S_k)_{1 \leq k \leq N}$.

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1. Introduction

Let $X_1, X_2, \dots$ be i.i.d. absolutely continuous random variables and let $S_k = \sum_{j=1}^k X_j \pmod{1}$. Then $S_k, k = 0, 1, \dots$ is a random walk on the circle and by a classical result of Lévy [6], the distribution of S_k converges weakly to the uniform distribution on $(0, 1)$. Schatte [8] proved that the speed of convergence

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is exponential, and letting

$$D_N^* := \sup_{0 \leq a < 1} \left| \frac{1}{N} \sum_{k=1}^N I_{(0,a)}(S_k) - a \right| \quad (N = 1, 2, \dots)$$

denote the star discrepancy of the sequence $(S_k)_{1 \leq k \leq N}$, he also proved in [9] the law of the iterated logarithm

$$\limsup_{N \rightarrow \infty} \sqrt{\frac{N}{\log \log N}} D_N^* = \gamma \quad \text{a.s.}, \quad (1)$$

where

$$\gamma = \sup_{x \in [0,1]} \sigma^2(x)$$

with

$$\sigma^2(x) = x - x^2 + 2 \sum_{j=1}^{\infty} (\mathbb{E} I_{(0,x)}(U) I_{(0,x)}(U + X_j) - x^2). \quad (2)$$

Here U is a uniform $(0, 1)$ random variable independent of the sequence $(X_n)_{n \geq 1}$ and for $0 \leq a < b \leq 1$, $I_{(a,b)}$ denotes the indicator function of (a, b) , extended with period 1. Letting

$$F_N(t) = F_N(t, \omega) = \frac{1}{N} \sum_{k=1}^N I_{(0,t)}(S_k) \quad (0 \leq t \leq 1) \quad (3)$$

denote the empirical distribution function of the first N terms of the sequence $(S_n)_{n \geq 1}$, Berkes and Raseto [2] proved a Strassen type functional LIL for $F_N(t)$, yielding precise asymptotics for several functionals of the empirical process. The purpose of the present paper is to prove the following result, determining the limit distribution of $\sqrt{N} D_N^*$.

THEOREM 1. *Let $X_1, X_2, \dots$ be i.i.d. random variables and assume X_1 is bounded with bounded density. Let*

$$\Gamma(s, t) = s(1 - t) + \sum_{k=1}^{\infty} \mathbb{E} f_s(U) f_t(U + S_k) + \sum_{k=1}^{\infty} \mathbb{E} f_t(U) f_s(U + S_k), \quad (4)$$

where U is a $U(0, 1)$ variable independent of $(X_n)_{n \in \mathbb{N}}$ and $f_s = I_{(0,s)} - s$. Then the series in (4) are absolutely convergent and

$$\sqrt{N} D_N^* \xrightarrow{d} \sup_{0 \leq t \leq 1} |K(t)|, \quad (5)$$

where $K(s)$ a mean zero Gaussian process with covariance function $\Gamma(s, t)$.

Actually, Theorem 1 will be deduced from a more general functional result describing the weak limit behaviour of the empirical distribution function $F_N(t)$.

THEOREM 2. *Under the conditions of Theorem 1 we have*

$$\sqrt{N}(F_N(t) - t) \xrightarrow{D[0,1]} K(t) \quad \text{as } N \rightarrow \infty. \quad (6)$$

Relation (6) expresses weak convergence in the Skorohod space $D[0, 1]$, see Billingsley [4] for basic definitions and facts for weak convergence of probability measures on metric spaces.

By a classical result of Donsker [5], if $X_1, X_2, \dots$ are i.i.d. random variables with distribution function F and F_N denotes the empirical distribution function of the sample $(X_1, \dots, X_N)$, then

$$\sqrt{N}(F_N(t) - F(t)) \xrightarrow{D[0,1]} B(F(t))$$

where B is Brownian bridge. Note the substantial difference caused by considering mod 1 sums in the present case.

If X_1 has a lattice distribution, the situation changes essentially. For example, in [1] it is shown that if α is irrational and X_1 takes the values α and 2α with probability $1/2 - 1/2$, then up to logarithmic factors, the order of magnitude of D_N^* is

$$O(N^{-1/2}) \quad \text{or} \quad O(N^{-1/\gamma})$$

according as $\gamma < 2$ or $\gamma > 2$, where γ is the Diophantine rank of α , i.e., the supremum of numbers c such that $|\alpha - p/q| < q^{-c-1}$ holds for infinitely many fractions p/q . The asymptotic distribution of D_N^* in this case remains open.

2. Proofs

The proof of our theorems uses, similarly to that of the functional LIL in [2], a traditional blocking argument combined with a coupling lemma of Schatte, see Lemma 1 below. The substantial new difficulty is to prove the tightness of the sequence $\sqrt{N}(F_N(t) - t)$, since the standard maximal inequalities (e.g., Billingsley's inequalities in [4], Section 2.12) are not applicable here. We circumvent this difficulty by proving a Chernoff type exponential bound (Lemma 6) for the considered partial sums which, combined with the chaining method of Philipp [7], yields the desired fluctuation inequality (Lemma 7).

LEMMA 1. *Let*

$$\ell \geq 1 \quad \text{and} \quad I_1, I_2, \dots$$

be disjoint blocks of integers with $\geq \ell$ integers between consecutive blocks. Then there exists a sequence $\delta_1, \delta_2, \dots$ of random variables such that

$$|\delta_n| \leq Ce^{-\lambda \ell}$$

with some positive constants C, λ and the random vectors

$$\{S_i, i \in I_1\}, \quad \{S_i - \delta_1, i \in I_2\}, \dots, \{S_i - \delta_{n-1}, i \in I_n\}, \dots$$

are independent and have, except for the first one, uniformly distributed components.

Proof. For the proof, see [2]. The uniformity statement is implicit in the proof; see also Lemma 4.3 of [3]. $\square$

In what follows, $C, \lambda, \gamma, \gamma' \dots$ will denote positive constants, possibly different at different places, depending (at most) on the distribution of X_1 . The relation $\ll$ will mean the same as the big O notation, with a constant depending on the distribution of X_1 .

Let $\mathcal{F}$ denote the class of functions f of the form

$$f = I_{(a,b)} - (b - a) \quad (0 \leq a < b \leq 1),$$

extended with period 1. For $f \in \mathcal{F}$ we put

$$A^{(f)} := \|f\|^2 + 2 \sum_{k=1}^{\infty} \mathbb{E} f(U) f(U + S_k), \quad (7)$$

where U is a uniform $(0, 1)$ random variable, independent of $(X_j)_{j \geq 1}$ and $\|f\|$ denotes the $L^2(0, 1)$ norm of f . Put further

$$\tilde{m}_k = \sum_{j=1}^k \left\lfloor j^{1/2} \right\rfloor, \quad \hat{m}_k = \sum_{j=1}^k \left\lfloor j^{1/4} \right\rfloor$$

and let $m_k = \tilde{m}_k + \hat{m}_k$. Using Lemma 1 we can construct sequences $(\Delta_k)_{k \in \mathbb{N}}$, $(\Pi_k)_{k \in \mathbb{N}}$ of random variables such that $\Delta_0 = 0$, $\Pi_0 = 0$,

$$|\Delta_k| \leq Ce^{-\lambda k^{1/4}}, \quad |\Pi_k| \leq Ce^{-\lambda \sqrt{k}} \quad (8)$$

and

$$\begin{aligned} T_k^{(f)} &:= \sum_{j=m_{k-1}+1}^{m_k-1+\lfloor \sqrt{k} \rfloor} f(S_j - \Delta_{k-1}), \quad k = 1, 2, \dots, \\ T_k^{*(f)} &= \sum_{j=m_{k-1}+\lfloor \sqrt{k} \rfloor+1}^{m_k} f(S_j - \Pi_{k-1}), \quad k = 1, 2, \dots \end{aligned}$$

are sequences of independent random variables.

Since $\int_0^1 f(x) dx = 0$ for $f \in \mathcal{F}$, the uniformity statement in Lemma 1 implies that

$$\mathbb{E}T_k^{(f)} = \mathbb{E}T_k^{*(f)} = 0 \quad \text{for } k \geq 2.$$

The following asymptotic estimates for the variances of $T_k^{(f)}$ and $T_k^{*(f)}$ are from [2].

LEMMA 2. *For $f \in \mathcal{F}$ we have*

$$\sum_{k=1}^n \mathbb{V}\text{ar} \left(T_k^{(f)} \right) \sim A^{(f)} \tilde{m}_n, \quad \sum_{k=1}^n \mathbb{V}\text{ar} \left(T_k^{*(f)} \right) \sim A^{(f)} \hat{m}_n,$$

where $A^{(f)}$ is defined by (7).

Since

$$\text{Cov} \left(T_k^{(f)}, T_k^{(g)} \right) = \frac{1}{4} \left(\mathbb{V}\text{ar} \left(T_k^{(f+g)} \right) - \mathbb{V}\text{ar} \left(T_k^{(f-g)} \right) \right),$$

Lemma 2 implies

$$\sum_{k=1}^n \text{Cov} \left(T_k^{(f)}, T_k^{(g)} \right) \sim \frac{1}{4} \left(A^{(f+g)} - A^{(f-g)} \right) \tilde{m}_n \quad (9)$$

and

$$\sum_{k=1}^n \text{Cov} \left(T_k^{*(f)}, T_k^{*(g)} \right) \sim \frac{1}{4} \left(A^{(f+g)} - A^{(f-g)} \right) \hat{m}_n. \quad (10)$$

From (7) it follows that

$$A^{(f+g)} - A^{(f-g)} = 4\langle f, g \rangle + 4 \sum_{k=1}^{\infty} \mathbb{E}f(U)g(U+S_k) + 4 \sum_{k=1}^{\infty} \mathbb{E}g(U)f(U+S_k). \quad (11)$$

LEMMA 3. *Let $f \in \mathcal{F}$, $h > 0$ and let ξ be a random variable with $|\xi| < h$. Then for any $n \geq 1$ we have*

$$\mathbb{E}|f(S_n + \xi) - f(S_n)|^2 \leq Ch.$$

Proof. Since X_1 is bounded with bounded density, Theorem 1 of [8] implies that the sums $S_n = \sum_{k=1}^n X_k \pmod{1}$ have a uniformly bounded density and thus

$$\mathbb{P}(S_n \in J) \leq C|J| \quad \text{for any interval } J. \quad (12)$$

Now if $f = I_{(a,b)} - (b-a)$, then $|f(S_n + \xi) - f(S_n)| = |I_{(a,b)}(S_n + \xi) - I_{(a,b)}(S_n)|$ is different from 0 only if one of $S_n + \xi$ and S_n lies in (a, b) and the other does not, which, in view of $|\xi| < h$, implies that S_n lies closer to the boundary of (a, b) than h , i.e., $S_n \in (a, a+h)$ or $S_n \in (b-h, b)$. Since $|f(S_n + \xi) - f(S_n)| \leq 2$, Lemma 3 follows from (12). $\square$

LEMMA 4. *For $f \in \mathcal{F}$ and any $M \geq 0$, $N \geq 1$ we have*

$$\mathbb{E} \left(\sum_{k=M+1}^{M+N} f(S_k) \right)^2 \leq C \|f\| N. \quad (13)$$

Proof. We first show

$$|\mathbb{E}f(S_k)f(S_\ell)| \leq Ce^{-\lambda(\ell-k)} \|f\| \quad (k < \ell). \quad (14)$$

Indeed, by the proof of Lemma 1 in [2], there exists a r.v. Δ with $|\Delta| \leq Ce^{-\lambda(\ell-k)}$ such that $S_\ell - \Delta$ is a uniform r.v. independent of S_k . Hence

$$\mathbb{E}f(S_\ell - \Delta) = \int_0^1 f(t) dt = 0$$

and thus

$$\mathbb{E}f(S_k)f(S_\ell - \Delta) = \mathbb{E}f(S_k)\mathbb{E}f(S_\ell - \Delta) = 0. \quad (15)$$

On the other hand,

$$\begin{aligned} & |\mathbb{E}f(S_k)f(S_\ell) - \mathbb{E}f(S_k)f(S_\ell - \Delta)| \\ & \leq \mathbb{E}(|f(S_k)| |f(S_\ell) - f(S_\ell - \Delta)|) \\ & \leq (\mathbb{E}f^2(S_k))^{1/2} (\mathbb{E}|f(S_\ell) - f(S_\ell - \Delta)|^2)^{1/2}. \end{aligned} \quad (16)$$

Using (12) we get

$$\mathbb{E}f^2(S_k) \leq C \int_0^1 f^2(t) dt = C \|f\|^2. \quad (17)$$

Also, $|\Delta| \leq Ce^{-\lambda(\ell-k)}$ and Lemma 3 imply

$$\mathbb{E}|f(S_\ell) - f(S_\ell - \Delta)|^2 \leq Ce^{-\lambda(\ell-k)} \quad (18)$$

which, together with (16)–(18), gives

$$|\mathbb{E}f(S_k)f(S_\ell) - \mathbb{E}f(S_k)f(S_\ell - \Delta)| \leq Ce^{-\lambda(\ell-k)}.$$

Thus using (15) we get (14). Now by (14)

$$\left| \sum_{M+1 \leq k < \ell \leq M+N} \mathbb{E}f(S_k)f(S_\ell) \right| \leq CN \|f\| \sum_{\ell \geq 1} e^{-\lambda\ell} \leq CN \|f\|$$

which, together with (17), completes the proof of Lemma 4. $\square$

Let $0 < t_1 < \dots < t_r \leq 1$ and put

$$\mathbf{Y}_k = (f_{(0,t_1)}(S_k), f_{(0,t_2)}(S_k), \dots, f_{(0,t_r)}(S_k)),$$

where

$$f_{(a,b)} = I_{(a,b)} - (b - a),$$

with the indicator $I_{(a,b)}$ extended with period 1, as before.

LEMMA 5. *We have*

$$N^{-1/2} \sum_{k=1}^N \mathbf{Y}_k \xrightarrow{d} N(\mathbf{0}, \Sigma), \quad (19)$$

where

$$\Sigma = (\Gamma(t_i, t_j))_{1 \leq i, j \leq r}.$$

Proof. Let

$$\begin{aligned} \mathbf{T}_k &= (T_k^{(f_{(0,t_1)})}, \dots, T_k^{(f_{(0,t_r)})}), \\ \mathbf{T}_k^* &= (T_k^{*(f_{(0,t_1)})}, \dots, T_k^{*(f_{(0,t_r)})}), \end{aligned}$$

and let $\Sigma_{\mathbf{k}}$ denote the covariance matrix of the vector $\mathbf{T}_{\mathbf{k}}$. From (9), (10) and (11) it follows that

$$m_n^{-1} (\Sigma_1 + \dots + \Sigma_{\mathbf{n}}) \longrightarrow \Sigma.$$

Clearly,

$$|\mathbf{T}_k| \leq C_r k^{1/2} = o(m_k^{1/2}),$$

where C_r is a constant depending on r , showing that the sequence $(\mathbf{T}_k)_{k \geq 1}$ of independent, mean $\mathbf{0}$ random vectors satisfies the Lindeberg condition and thus

$$m_n^{-1/2} \sum_{k=1}^n \mathbf{T}_k \xrightarrow{d} N(\mathbf{0}, \Sigma). \quad (20)$$

A similar statement holds for the sequence $(\mathbf{T}_k^*)_{k \geq 1}$, implying that

$$\left| \sum_{k=1}^n \mathbf{T}_k^* \right| = O_P(\widehat{m}_n) = o_P(m_n^{1/2}), \quad (21)$$

and consequently,

$$m_n^{-1/2} \sum_{k=1}^n (\mathbf{T}_k + \mathbf{T}_k^*) \xrightarrow{d} N(\mathbf{0}, \Sigma). \quad (22)$$

Now, using (8) and Lemma 3 we get

$$\left\| T_k^{(f)} - \sum_{j=m_{k-1}+1}^{m_{k-1}+\lfloor \sqrt{k} \rfloor} f(S_j) \right\| \ll \sqrt{k} e^{-\lambda k^{1/4}}$$

and

$$\left\| T_k^{*(f)} - \sum_{j=m_{k-1}+\lfloor \sqrt{k} \rfloor+1}^{m_k} f(S_j) \right\| \ll k^{1/4} e^{-\lambda k^{1/2}},$$

and thus

$$\left\| \sum_{k=1}^{m_n} \mathbf{Y}_k - \sum_{k=1}^n (\mathbf{T}_k + \mathbf{T}_k^*) \right\| = O(1).$$

Together with (22) this shows that (19) holds for the indices $N = m_n$. To get (19) for all N , observe that $m_k \sim ck^{3/2}$ and thus for $m_k \leq N < m_{k+1}$ we have

$$\left| \sum_{j=1}^N \mathbf{Y}_j - \sum_{j=1}^{m_k} \mathbf{Y}_j \right| = O(m_{k+1} - m_k) = O(k^{1/2}) = O(m_k^{1/3}) = O(N^{1/3}).$$

This completes the proof of Lemma 5. $\square$

LEMMA 6. For $f \in \mathcal{F}$, any $N \geq 1$, $t \geq 1$ and $\|f\| \geq \frac{1}{5}N^{-5/18}$ we have

$$\mathbb{P} \left\{ \left| \sum_{k=1}^N f(S_k) \right| \geq t \|f\|^{1/4} \sqrt{N} \right\} \ll \exp \left(-Ct \|f\|^{-7/20} \right) + t^{-2} \exp \left(-CN^{1/3} \right). \quad (23)$$

REMARK. The constants $1/5, 5/18, 1/4, 7/20, 1/3$ in (23) are not sharp and the inequality could be easily improved. However, the present form of Lemma 6 will suffice for the chaining argument in Lemma 7.

Proof. Put

$$\psi(n) = \sup_{0 \leq x \leq 1} |\mathbb{P}(S_n \leq x) - x|.$$

By Theorem 1 of [8] we have

$$\psi(n) \leq Ce^{-\gamma n} \quad (n \geq 1)$$

for some constant $\gamma > 0$. Divide the interval $[1, N]$ into subintervals $I_1, \dots, I_L$, with $L \sim N^{2/3}$, where each interval I_ν contains $\sim N^{1/3}$ terms. We set

$$\sum_{k=1}^N f(S_k) = \eta_1 + \dots + \eta_L, \quad \text{where} \quad \eta_\nu = \sum_{k \in I_\nu} f(S_k).$$

We deal with the sums $\sum \eta_{2j}$ and $\sum \eta_{2j+1}$, separately. Since there is a separation $\sim N^{1/3}$ between the even block sums η_{2j} , we can apply Lemma 1 to get

$$\eta_{2j} = \eta_{2j}^* + \eta_{2j}^{**},$$

where

$$\begin{aligned} \eta_{2j}^* &= \sum_{k \in I_{2j}} f(S_k - \Delta_j), \\ \eta_{2j}^{**} &= \sum_{k \in I_{2j}} (f(S_k) - f(S_k - \Delta_j)). \end{aligned}$$

Here the Δ_j are r.v.'s with $|\Delta_j| \leq \psi(N^{1/3}) \leq C \exp(-\gamma N^{1/3})$ and the r.v.'s η_{2j}^* $j = 1, 2, \dots$ are independent. Conditionally on Δ_j , the distribution of S_k in a term of η_{2j}^{**} is the same as the (unconditional) distribution of an $S_{k_1} + c$ with $k_1 < k$ and a constant c and thus by Lemma 3, the L_2 norm of each summand in η_{2j}^{**} is $\leq C\psi^{1/2}(N^{1/3}) \leq C \exp(-\gamma N^{1/3})$ and thus for $\|f\| \geq N^{-1}$ we have

$$\begin{aligned} \|\eta_{2j}^{**}\| &\leq CN \exp(-\gamma N^{1/3}) \leq C\|f\|N^2 \exp(-\gamma N^{1/3}) \\ &\leq C\|f\| \exp(-\gamma' N^{1/3}). \end{aligned} \tag{24}$$

Thus

$$\left\| \sum \eta_{2j}^{**} \right\| \leq C\|f\| \exp(-\gamma'' N^{1/3})$$

and therefore by the Markov inequality

$$\begin{aligned} \mathbb{P} \left(\left| \sum \eta_{2j}^{**} \right| \geq t\|f\|^{1/4} \sqrt{N} \right) &\leq Ct^{-2} \|f\|^{-1/2} N^{-1} \|f\|^2 \exp(-2\gamma'' N^{1/3}) \\ &\leq Ct^{-2} \exp(-2\gamma'' N^{1/3}). \end{aligned}$$

Let now $|\lambda| \leq dN^{-1/3}$ with a sufficiently small constant $d > 0$. Then $|\lambda \eta_{2j}^*| \leq 1/2$ for all N and thus using $e^x \leq 1 + x + x^2$ for $|x| \leq 1/2$ we get, using $E\eta_{2j}^* = 0$ for $j \geq 2$,

$$\begin{aligned} \mathbb{E} \left(\exp \lambda \left(\sum_j \eta_{2j}^* \right) \right) &= \prod_j \mathbb{E}(e^{\lambda \eta_{2j}^*}) \leq \prod_j \mathbb{E}(1 + \lambda \eta_{2j}^* + \lambda^2 \eta_{2j}^{*2}) \\ &= \prod_j (1 + \lambda^2 \mathbb{E} \eta_{2j}^{*2}) \leq \exp \left(\lambda^2 \sum_j \mathbb{E} \eta_{2j}^{*2} \right). \end{aligned} \tag{25}$$

Here, and in the rest of the proof of the lemma, the sums and products are extended for $j \geq 2$. By Lemma 4

$$\|\eta_{2j}\| \leq C\|f\|^{1/2} N^{1/6},$$

which, together with (24) and the Minkowski inequality, implies

$$\|\eta_{2j}^*\| \leq C\|f\|^{1/2} N^{1/6}.$$

Thus the last expression in (25) cannot exceed

$$\exp\left(\lambda^2 C \|f\| \sum_j N^{1/3}\right) \leq \exp(\lambda^2 C \|f\| N).$$

We choose now

$$\lambda = \frac{d}{2} N^{-1/2} \|f\|^{-3/5}$$

with the number d introduced before and note that by $\|f\| \geq \frac{1}{5} N^{-5/18}$ we have

$$|\lambda| \leq d N^{-1/3}.$$

Thus using the Markov inequality, we get

$$\begin{aligned} \mathbb{P}\left\{\left|\sum_j \eta_{2j}^*\right| \geq t \|f\|^{1/4} \sqrt{N}\right\} &\leq 2 \exp\left\{-\lambda t \|f\|^{1/4} \sqrt{N} + \lambda^2 C \|f\| N\right\} \\ &= 2 \exp\left(-\|f\|^{-7/20} t + C \|f\|^{-1/5}\right) \\ &\leq 2 \exp\left(-C \|f\|^{-7/20} t\right). \end{aligned}$$

Recall that the sum here is extended for $j \geq 2$. However, the term corresponding to $j = 1$ is $O(N^{1/3})$ and since $\|f\|^{1/4} \sqrt{N} \geq N^{0.4}$ for $N \geq N_0$ by the assumptions of the lemma, the last chain of estimates remains valid by including the term $j = 1$ in the sum in the first probability and changing t to $2t$. A similar argument applies for the odd blocks η_{2j+1}^* (note that $\mathbb{E}\eta_1^*$ can be different from 0, but this causes no problem), completing the proof of Lemma 6. $\square$

LEMMA 7. *For any $N \geq 1$, $0 < \delta < 1$ we have*

$$\mathbb{P}\left(\sup_{0 \leq a \leq \delta} \left|\sum_{k=1}^N (I_{(0,a)}(S_k) - a)\right| \gg \delta^{1/8} \sqrt{N} + N^{4/9}\right) \ll \delta^4 + N^{-2}.$$

Proof. For any $h \geq 1$, $1 \leq j \leq 2^h$ let $\varphi_h^{(j)}$ denote the indicator function of the interval $[(j-1)2^{-h}, j2^{-h})$ and put

$$F(N, j, h) = \left|\sum_{k=1}^N (\varphi_h^{(j)}(S_k) - 2^{-h})\right|.$$

Clearly, $\|\varphi_h^{(j)}\| = 2^{-h/2}$.

We observe that if $0 \leq a \leq 1$ has the dyadic expansion

$$a = \sum_{j=1}^{\infty} \varepsilon_j 2^{-j}, \quad \varepsilon_j = 0, 1$$

and $H \geq 1$ is an arbitrary integer, then $g_a = I_{(0,a)}$ satisfies

$$\sum_{h=1}^H \varrho_h(x) \leq g_a(x) \leq \sum_{h=1}^H \varrho_h(x) + \sigma_H(x), \quad (26)$$

where ϱ_h is the indicator function of

$$\left[\sum_{j=1}^{h-1} \varepsilon_j 2^{-j}, \sum_{j=1}^h \varepsilon_j 2^{-j} \right)$$

and σ_H is the indicator function of

$$\left[\sum_{j=1}^H \varepsilon_j 2^{-j}, \sum_{j=1}^H \varepsilon_j 2^{-j} + 2^{-H} \right).$$

For $\varepsilon_h = 0$, clearly, $\varrho_h \equiv 0$ and thus (26) remains valid if in the sums we keep only those terms where $\varepsilon_h = 1$. Also, for $\varepsilon_h = 1$, ϱ_h coincides with one of the $\varphi_h^{(j)}$ and σ_H also coincides with some of the $\varphi_H^{(j)}$. It follows that

$$\begin{aligned} g_a(x) - a &\leq \sum_{1 \leq h \leq H}^* (\varrho_h(x) - \varepsilon_h 2^{-h}) \\ &\quad + (\sigma_H(x) - 2^{-H}) + 2^{-H} \end{aligned}$$

and

$$g_a(x) - a \geq \sum_{1 \leq h \leq H}^* (\varrho_h(x) - \varepsilon_h 2^{-h}) - 2^{-H},$$

where $\sum^*$ means that the summation is extended only for those h such that $\varepsilon_h = 1$. Setting $x = S_k$ and summing for $1 \leq k \leq N$ we get

$$\begin{aligned} \sum_{k \leq N} (g_a(S_k) - a) &\leq \sum_{k \leq N} \sum_{1 \leq h \leq H}^* (\varrho_h(S_k) - \varepsilon_h 2^{-h}) \\ &\quad + \sum_{k \leq N} (\sigma_H(S_k) - 2^{-H}) + N 2^{-H} \end{aligned}$$

and

$$\sum_{k \leq N} (g_a(S_k) - a) \geq \sum_{k \leq N} \sum_{1 \leq h \leq H}^* (\varrho_h(S_k) - \varepsilon_h 2^{-h}) - N 2^{-H}.$$

Hence it follows that for any $N \geq 1$, $H \geq 1$ there exist suitable integers $1 \leq j_h \leq 2^h$, $1 \leq h \leq H$ such that

$$\begin{aligned} \left| \sum_{k \leq N} (g_a(S_k) - a) \right| &\leq 2 \sum_{h \leq H} \left| \sum_{k \leq N} \varphi_h^{(j_h)}(S_k) - 2^{-h} \right| + N2^{-H} \\ &= 2 \sum_{h \leq H} F(N, j_h, h) + N2^{-H}. \end{aligned} \quad (27)$$

Introduce the events

$$\begin{aligned} G(N, j, h) &= \left\{ F(N, j, h) \geq 2^{-h/8} \sqrt{N} \right\}, \\ G_N &= \bigcup_{A \leq h \leq B \log_2 N} \bigcup_{j \leq 2^h} G(N, j, h) \quad \text{with} \quad A = \log_2 \frac{1}{a}, \quad B = \frac{5}{9}. \end{aligned}$$

For $h \leq B \log_2 N$ we have

$$\|\varphi_j^{(h)} - 2^{-h}\| \geq 2^{-h/2} - 2^{-h} \geq 2^{-h/2} (1 - 1/\sqrt{2}) \geq \frac{1}{5} N^{-5/18}$$

and thus applying (23) with $t = 1$, we get

$$\mathbb{P}(G(N, h, j)) \ll \exp(-C2^{7h/40}) + N^{-3},$$

and consequently,

$$\mathbb{P}(G_N) \ll \sum_{h=A}^{\infty} 2^h \exp(-C2^{7h/40}) + N^{-3} \sum_{h \leq B \log_2 N} 2^h. \quad (28)$$

Clearly, the second term on the right-hand side of (28) is $\ll N^{-2}$. On the other hand, the terms of the first sum in (28) decrease superexponentially and thus the sum can be bounded by a constant times its first term, i.e., the sum is

$$\ll 2^A \exp(-C2^{7A/40}) \ll 2^{-4A} \ll a^4.$$

Hence

$$\mathbb{P}(G_N) \ll a^4 + N^{-2}.$$

Note that when breaking the interval $(0, a)$ into dyadic intervals of length 2^{-h} we automatically have $h \geq \log \frac{1}{a} = A$ and thus choosing

$$H = [B \log_2 N],$$

it follows that with the exception of a set with probability $\ll a^4 + N^{-2}$, for any $0 < a \leq \delta$ the expression in the second line of (27) is

$$\begin{aligned} \ll \sum_{A \leq h \leq H} 2^{-h/8} \sqrt{N} + N2^{-H} &\ll 2^{-A/8} \sqrt{N} + N^{4/9} \ll \\ &a^{1/8} \sqrt{N} + N^{4/9} \ll \delta^{1/8} \sqrt{N} + N^{4/9}. \end{aligned}$$

This proves Lemma 7. $\square$

Lemma 5 implies the convergence of the finite dimensional distributions of the sequence $\sqrt{N}(F_N(t) - t)$ in (5) to those of K and to prove Theorem 2 it remains to prove the tightness of the sequence in $D[0, 1]$. To this end, fix $\varepsilon > 0$ and choose h so that $2^{-h} \leq \varepsilon < 2^{-(h-1)}$. Note that for $j = 0, 1, \dots, 2^h - 1$ we have

$$\mathbb{P} \left(\sup_{0 \leq a \leq 2^{-h}} \left| \sum_{k=1}^N (f_{j2^{-h}+a}(S_k) - f_{j2^{-h}}(S_k)) \right| \gg 2^{-h/8} \sqrt{N} + N^{4/9} \right) \ll \frac{1}{2^{-4h} + N^{-2}}. \quad (29)$$

For $j = 0$ relation (29) is identical with Lemma 7 and for $j = 1, 2, \dots$ the proof is the same. It follows that

$$\mathbb{P} \left(\max_{0 \leq j \leq 2^h - 1} \sup_{0 \leq a \leq 2^{-h}} \left| \sum_{k=1}^N (f_{j2^{-h}+a}(S_k) - f_{j2^{-h}}(S_k)) \right| \gg 2^{-h/8} \sqrt{N} + N^{4/9} \right) \ll \frac{1}{2^{-3h} + 2^h N^{-2}}. \quad (30)$$

Then (30) implies that with the exception of a set with probability

$$\ll 2^{-3h} + 2^h N^{-2} \ll \varepsilon^3 + N^{-2} \varepsilon^{-1}$$

the fluctuation of the process $\sqrt{N}(F_N(t) - t)$ over any subinterval of $(0, 1)$ with length $\leq \varepsilon$ is

$$\ll \varepsilon^{1/8} + N^{-1/18}.$$

By Theorem 15.5 of Billingsley [4, p. 127], the sequence $\sqrt{N}(F_N(t) - t)$ is tight in $D[0, 1]$. This completes the proof of Theorem 2; Theorem 1 follows immediately from Theorem 2.

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REFERENCES

- [1] BERKES, I.—BORDA, B.: *On the discrepancy of random subsequences of $\{n\alpha\}$* , Acta Arithmetica (to appear).
- [2] BERKES, I.—RASETA, M.: *On the discrepancy and empirical distribution function of $\{n_k\alpha\}$* , Unif. Distr. Theory **10** (2015), no. 1, 1–17.
- [3] BERKES, I.—WEBER, M.: *On the convergence of $\sum c_k f(n_k x)$* , Mem. Amer. Math. Soc. **201** (2009), no. 943, viii+72 pp.
- [4] BILLINGSLEY, P.: *Convergence of probability measures*. Wiley, 1968.
- [5] DONSKER, M.: *Justification and extension of Doob's heuristic approach to the Kolmogorov-Smirnov theorems*, Ann. Math. Statist. **23** (1952), 277–281.

- [6] LÉVY, P.: *L'addition des variables aléatoires définies sur une circonference*, Bull. Soc. Math. France **67** (1939), 1–40.
- [7] PHILIPP, W.: *Limit theorems for lacunary series and uniform distribution mod 1*, Acta Arith. **26** (1975), 241–251.
- [8] SCHATTE, P.: *On the asymptotic uniform distribution of sums reduced mod 1*, Math. Nachr. **115** (1984), 275–281.
- [9] SCHATTE, P.: *On a uniform law of the iterated logarithm for sums mod 1 and Benford's law*, Lithuanian Math. J. **31** (1991), 133–142.

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