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ON SOME PROPERTIES OF AGGREGATION-BASED EXTENSIONS OF FUZZY MEASURES

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ABSTRACT. In this paper, we analyse properties of aggregation-based extensions of fuzzy measures depending on properties of aggregation functions which they are based on. We mainly focus on properties possessed by the well-known Lovász and Owen extensions. Moreover, we characterize aggregation functions suitable for extension of particular subclasses of fuzzy measures.

1. Introduction

The key problem of multicriteria decision making theory is how to assign a single value to each score vector achieved in set $N = \{1, \dots, n\}$ of criteria. In case of Boolean vectors, a function $m\colon \{0,1\}^n \to [0,1]$ is needed. Considering one-to-one correspondence between the sets $\{0,1\}^n$ and 2^N and adding requirements of $m(\mathbf{0}) = 0$, $m(\mathbf{1}) = 1$ and nondecreasingness of m, one can regard the required utility function m as a fuzzy measure defined on 2^N. Switching from $\{0,1\}$ to $[0,1]$-scale vectors, a nondecreasing utility function $F\colon [0,1]^n \to [0,1]$ fulfilling $F(\mathbf{0}) = 0$, $F(\mathbf{1}) = 1$ can be regarded as an aggregation function which is an extension of the fuzzy measure m corresponding to the restriction of F to $\{0,1\}^n$, i.e., $m = F \upharpoonright \{0,1\}^n$.

There are two well-known extensions of a fuzzy measure defined on $\{0,1\}^n$ yielding an aggregation function defined on $[0,1]^n$—the L o v á s z extension [5] and the O w e n extension [7]. Both can be expressed by means of Möbius transform and some auxiliary aggregation function. Generalization of this approach was recently used by K o l e s á r o v á et al. [4], where the construction method of extension of a fuzzy measure based on an aggregation function and Möbius transform was proposed and the characterization of all aggregation functions suitable

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for this construction was given. Spizzichino has shown that taking a copula in rôle of basic aggregation function such an extension has a simple probabilistic meaning (see [8]).

The aim of this paper is to study the link between properties of an underlying aggregation function and properties of the extension of fuzzy measure based on this aggregation function. We start with investigation of properties possessed by extensions for each fuzzy measure and then we focus on properties of extensions for some subclasses of fuzzy measures. Next, we find a characterization of aggregation functions suitable for aggregation-based extensions of particular subclasses of fuzzy measures.

The paper is organized as follows. In Section 2, we recall some basic notions and definitions needed throughout the paper. In Sections 3 and 4, our main results are confined. Finally, some concluding remarks are provided.

2. Preliminaries

In this section we recall some definitions and results which will be used in the sequel. We also fix the notation, mostly according to [2], [9], wherein more information concerning the theory of aggregation functions and the fuzzy measure theory can be found.

Let $n \in \mathbb{N}$ and $N = \{1, \ldots, n\}$.

Definition 2.1. A set function $m\colon 2^N \to [0,1]$ is said to be a *fuzzy measure* iff $m(\emptyset) = 0$, $m(N) = 1$ and it is nondecreasing, i.e., $m(C) \leq m(D)$ whenever $C \subseteq D$. The class of all fuzzy measures on 2^N will be denoted by $\mathcal{M}_{(n)}$.

Definition 2.2. Let $E, F \subseteq N$. A fuzzy measure m is:

- *additive*, if $m(E \cup F) = m(E) + m(F)$ whenever $E \cap F = \emptyset$;
- *submodular*, if $m(E \cup F) + m(E \cap F) \leq m(E) + m(F)$;
- *supermodular*, if $m(E \cup F) + m(E \cap F) \geq m(E) + m(F)$;
- *symmetric*, if $m(E) = m(F)$ whenever $|E| = |F|$;
- *k-additive*, if it holds

$$\sum_{i=1}^{k+1} (-1)^{k+1-i} \left(\sum_{\substack{I \subseteq \{1,\ldots,k+1\} \\ |I| = i}} m\left(\bigcup_{j \in I} A_j \right) \right) = 0$$

for any system $(A_i)_{i=1}^{k+1}$ of pairwise disjoint subsets of N.

Definition 2.3. Let $m \in \mathcal{M}_{(n)}$. The set function $M_m \colon 2^N \to \mathbb{R}$ defined by

$$M_m(I) = \sum_{K \subseteq I} (-1)^{|I \setminus K|} m(K)$$

for $I \subseteq N$ is said to be *Möbius transform* corresponding to a fuzzy measure m.

Definition 2.4. A function $A \colon [0,1]^n \to [0,1]$ is said to be an *aggregation function* if and only if it is nondecreasing in each variable and $A(0,\dots,0) = 0$, $A(1,\dots,1) = 1$. The class of all n-ary aggregation functions will be denoted by $\mathcal{A}_{(n)}$.

Definition 2.5. An aggregation function $A : [0,1]^n \to [0,1]$ is:

- *internal*, if for each $\mathbf{x} \in [0,1]^n$ it holds $\mathrm{Min}(\mathbf{x}) \leq A(\mathbf{x}) \leq \mathrm{Max}(\mathbf{x})$;
- *translation invariant*, if $A(x_1 + c, \dots, x_n + c) = c + A(x_1, \dots, x_n)$ for all $c \in]0,1]$ and $(x_1, \dots, x_n) \in [0,1]^n$ such that $(x_1 + c, \dots, x_n + c) \in [0,1]^n$;
- *idempotent*, if $A(x, \dots, x) = x$ for each $x \in [0,1]$;
- *positively homogeneous*, if $A(c\mathbf{x}) = cA(\mathbf{x})$ for each $\mathbf{x} \in [0,1]^n$ and $c > 0$ such that $c\mathbf{x} \in [0,1]^n$;
- *comonotone additive*, if $A(\mathbf{x}+\overline{\mathbf{x}}) = A(\mathbf{x})+A(\overline{\mathbf{x}})$ for all comonotone vectors $\mathbf{x}, \overline{\mathbf{x}} \in [0,1]^n$ such that $\mathbf{x} + \overline{\mathbf{x}} \in [0,1]^n$ (vectors $\mathbf{x}, \overline{\mathbf{x}}$ are comonotone, if and only if $(x_i - x_j)(\overline{x}_i - \overline{x}_j) \geq 0$ for all $i, j \in \{1, \dots, n\}$);
- *conjunctive*, if for each $\mathbf{x} \in [0,1]^n$ it holds $\mathrm{Min}(\mathbf{x}) \geq A(\mathbf{x})$;
- *k-additive*, if

$$\sum_{i=1}^{k+1} (-1)^{k+1-i} \left(\sum_{\substack{I \subseteq \{1,\dots,k+1\} \\ |I| = i}} A\left(\sum_{j \in I} \mathbf{x}_j \right) \right) = 0$$

for all $(k+1)$-tuples of vectors $\mathbf{x}_1, \dots, \mathbf{x}_{k+1} \in [0,1]^n$ with $\sum_{i=1}^{k+1} \mathbf{x}_i \in [0,1]^n$.

For $K \subseteq N$, define the vector $\mathbf{1}_K$ by $(\mathbf{1}_K)_i = 1$ if $i \in K$ and $(\mathbf{1}_K)_i = 0$ otherwise, which establishes a one-to-one correspondence between the sets 2^N and $\{0,1\}^n$.

Definition 2.6. A function $F \colon [0,1]^n \to [0,1]$ is said to be an *extension* of a fuzzy measure $m \colon 2^N \to [0,1]$ if and only if

$$F(\mathbf{1}_K) = m(K) \qquad \text{for all} \quad K \subseteq N.$$

The following construction method for extension of the given fuzzy measure based on aggregation function was proposed in [4].

Let $m \in \mathcal{M}_{(n)}$ be a fuzzy measure, $A \in \mathcal{A}_{(n)}$ be an aggregation function. Define $F_{m,A}\colon [0,1]^n \to \mathbb{R}$ by

$$F_{m,A}(x_1,\dots,x_n) = \sum_{I \subseteq N} M_m(I) A(\mathbf{x}_I), \tag{1}$$

where $(\mathbf{x}_I)_i = x_i$ whenever $i \in I$ and $(\mathbf{x}_I)_i = 1$ otherwise.

In particular, taking the aggregation function Min defined by

$$\mathrm{Min}(x_1,\dots,x_n) = \min\{x_1,\dots,x_n\}$$

in rôle of A, we obtain the *Lovász extension of* m:

$$C_m(x_1,\dots,x_n) = \sum_{I \subseteq N} M_m(I) \min_{i \in I} x_i.$$

Note that the Lovász extension is nothing but the discrete Choquet integral.

Similarly, taking the product aggregation function $\Pi(x_1,\dots,x_n) = \prod_{i \in I} x_i$ in rôle of A, we obtain the *Owen extension of* m:

$$O_m(x_1,\dots,x_n) = \sum_{I \subseteq N} M_m(I) \prod_{i \in I} x_i.$$

Both well-known extensions of fuzzy measure m are aggregation functions. However, it is not true in general, i.e., for an aggregation function $A \in \mathcal{A}_{(n)}$ and a fuzzy measure $m \in \mathcal{M}_{(n)}$, the function $F_{m,A}$ defined by (1) need not to be nor an aggregation function neither an extension of m. An aggregation function A will be called *suitable* for aggregation-based extension for a fuzzy measure $m \in \mathcal{M}_{(n)}$ if and only if $F_{m,A}$ defined by (1) is an aggregation function extending m. The following characterization of all aggregation functions $A \in \mathcal{A}_{(n)}$ suitable for all $m \in \mathcal{M}_{(n)}$ was given in [4].

Theorem 2.1. *Let $A \in \mathcal{A}_{(n)}$. $F_{m,A}$ is an aggregation function extending m for every $m \in \mathcal{M}_{(n)}$ if and only if A is an aggregation function with zero annihilator and for each $[\mathbf{a},\mathbf{b}] \subseteq [0,1]^n$ such that $\{0,1\} \cap \{a_1,\dots,a_n,b_1,\dots,b_n\} \neq \emptyset$ the A-volume $V_A([\mathbf{a},\mathbf{b}])$ is non-negative.*

Recall that the A-volume of the n-box $[\mathbf{a},\mathbf{b}]$ is defined by

$$V_A([\mathbf{a},\mathbf{b}]) = \sum (-1)^{\alpha(\mathbf{c})} A(\mathbf{c}),$$

where the sum is taken over all vertices $\mathbf{c}$ of $[\mathbf{a},\mathbf{b}]$ and $\alpha(\mathbf{c})$ is the number of indices k such that $c_k = a_k$.

Clearly, due to Theorem 2.1, each copula C is a suitable aggregation function for every $m \in \mathcal{M}_{(n)}$. Moreover, it was shown in [4] that for $n = 2$, A is a suitable aggregation function for every $m \in \mathcal{M}_{(2)}$ if and only if $A(x,y) = Q\big(f(x), g(x)\big)$ for some 2-quasi-copula Q and f, g nondecreasing endomorphisms of $[0,1]$ satisfying $f(0) = g(0) = 0$ and $f(1) = g(1) = 1$.

3. Properties of aggregation-based extensions

In this section we study properties of extensions $F_{m,A}$ defined by (1) depending on properties of A. We look for aggregation functions A yielding $F_{m,A}$ with particular properties possessed by the Lovász and Owen extensions. The Lovász extension is an aggregation function which is comonotone additive, idempotent, translation invariant, positively homogeneous, internal, linear w.r.t. the fuzzy measure. Moreover, if m is symmetric, then so is C_m. Finally, m is submodular if and only if C_m is convex. Next, we know that the Owen extension is an aggregation function which is n-additive and linear w.r.t. the fuzzy measure. For $k \leq n$ the Owen extension is k-additive if and only if m is k-additive (see [3]).

In the following proposition we claim that the property of being linear with respect to the fuzzy measure is fulfilled by $F_{m,A}$ based on an arbitrary aggregation function A.

Proposition 3.1. *Let $A \in \mathcal{A}_{(n)}$. $F_{m,A}$ is linear w.r.t. the fuzzy measure, i.e., for every $m_1, m_2 \in \mathcal{M}_{(n)}$ and $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $\lambda_1 m_1 + \lambda_2 m_2 \in \mathcal{M}_{(n)}$ it holds*

$$F_{\lambda_1 m_1 + \lambda_2 m_2, A} = \lambda_1 F_{m_1,A} + \lambda_2 F_{m_1,A}.$$

P r o o f. Let $I \subseteq N$. For the Möbius transform of the set I we get

$$\begin{aligned} M_{\lambda_1 m_1 + \lambda_2 m_2}(I) &= \sum_{K \subseteq I} (-1)^{|I \setminus K|} (\lambda_1 m_1 + \lambda_2 m_2)(K) \\ &= \lambda_1 \sum_{K \subseteq I} (-1)^{|I \setminus K|} m_1(K) + \lambda_2 \sum_{K \subseteq I} (-1)^{|I \setminus K|} m_2(K) \\ &= \lambda_1 M_{m_1}(I) + \lambda_2 M_{m_2}(I). \end{aligned}$$

Therefore,

$$\begin{aligned} F_{\lambda_1 m_1 + \lambda_2 m_2, A}(\mathbf{x}) &= \sum_{I \subseteq N} M_{\lambda_1 m_1 + \lambda_2 m_2}(I) A(\mathbf{x}_I) \\ &= \lambda_1 \sum_{I \subseteq N} M_{m_1}(I) A(\mathbf{x}_I) + \lambda_2 \sum_{I \subseteq N} M_{m_2}(I) A(\mathbf{x}_I) \\ &= \lambda_1 F_{m_1,A}(\mathbf{x}) + \lambda_2 F_{m_2,A}(\mathbf{x}). \end{aligned}$$

□

Proposition 3.2. *Let $A \in \mathcal{A}_{(n)}$. If $F_{m,A}$ for all $m \in \mathcal{M}_{(n)}$ has one of the following properties: idempotency, positive homogeneity, k-additivity, translation invariance, conjunctivity, internality; then so has A.*

P r o o f.

$$\begin{aligned} F_{m,A}(x_1,\dots,x_n) &= \sum_{I\subseteq N} M_m(I)A(\mathbf{x}_I) \\ &= \sum_{E\subseteq N} m(E) \sum_{I\supseteq E} A(\mathbf{x}_I)(-1)^{|I\setminus E|} \\ &= \sum_{E\subsetneq N} m(E) \sum_{I\supseteq E} A(\mathbf{x}_I)(-1)^{|I\setminus E|} + A(x_1,\dots,x_n). \end{aligned}$$

For $m = m_*$ defined by $m_*(E) = 1$ if $E = N$ and $m_*(E) = 0$ otherwise, we have $F_{m_*,A} = A$ and the claim follows. □

Note that by the same reasoning $F_{m,A}$ cannot have any property for all $m \in \mathcal{M}_{(n)}$ not possessed by A.

Proposition 3.3. *Let $A \in \mathcal{A}_{(n)}$ be an aggregation function suitable for aggregation-based extension for all $m \in \mathcal{M}_{(n)}$. Then the following statements are equivalent:*

(i) *$F_{m,A}$ is idempotent for all $m \in \mathcal{M}_{(n)}$.*
(ii) *$A = \mathrm{Min}$.*

P r o o f. Let $\mathbf{x} = (x,\dots,x)$.
(i) ⇒ (ii). According to the Proposition 3.2, A is idempotent. We get

$$\begin{aligned} F_{m,A}(\mathbf{x}) &= \sum_{E\subsetneq N} m(E) \sum_{I\supseteq E} A(\mathbf{x}_I)(-1)^{|I\setminus E|} + A(\mathbf{x}) \\ &= \sum_{E\subsetneq N} m(E) \sum_{I\supseteq E} A(\mathbf{x}_I)(-1)^{|I\setminus E|} + x. \end{aligned}$$

Next,

$$\sum_{I\supseteq E} A(\mathbf{x}_I)(-1)^{|I\setminus E|} = V_A[\widetilde{\mathbf{x}}_E, \mathbf{x}_E],$$

where

$$(\widetilde{\mathbf{x}}_E)_i = \begin{cases} 0, & \text{if } i \in E; \\ x, & \text{otherwise.} \end{cases}$$

Due to [4, Thm. 4], $V_A[\widetilde{\mathbf{x}}_E, \mathbf{x}_E] \ge 0$. Hence, $F_{m,A}(\mathbf{x}) = x$ for all $x \in [0,1]$ if and only if

$$\sum_{I\supseteq E} A(\mathbf{x}_I)(-1)^{|I\setminus E|} = 0 \qquad \text{for all} \quad E \subseteq N.$$

Let $E \subseteq N$, $|N \setminus E| = 1$. Then

$$A(\mathbf{x}_E) - A(\mathbf{x}) = 0,$$

$$A(\mathbf{x}_E) = A(\mathbf{x}) = x.$$

Let $E \subseteq N$, $|N \setminus E| = 2$, $N \setminus E = \{a, b\}$. Then

$$A(\mathbf{x}_E) - A(\mathbf{x}_{E\cup\{a\}}) - A(\mathbf{x}_{E\cup\{b\}}) + A(\mathbf{x}) = 0,$$
$$A(\mathbf{x}_E) - x - x + x = 0,$$
$$A(\mathbf{x}_E) = x.$$

By induction, $A(\mathbf{x}_E) = A(\mathbf{x}) = x$ for all $E \subseteq N$. Using the [2, Prop. 2.51] we get that A is a conjuctive aggregation function. Since it is also idempotent, it follows that $A = \mathrm{Min}$ [2, Prop. 3.1].
(ii) $\Rightarrow$ (i). It is trivial, therefore proof is omitted. □

Corollary 3.1. *Let $A \in \mathcal{A}_{(n)}$ be an aggregation function suitable for aggregation-based extension for all $m \in \mathcal{M}_{(n)}$. Then the following statements are equivalent:*

(i) *$F_{m,A}$ is internal (translation invariant, comonotone additive, positively homogeneous) for all $m \in \mathcal{M}_{(n)}$.*
(ii) *$A = \mathrm{Min}$.*

P r o o f. (i) $\Rightarrow$ (ii). Due to [2], for nondecreasing function $F\colon [0,1]^n \to \mathbb{R}$ internality is equivalent to idempotency. Similarly, translation invariance, comonotone additivity and positive homogeneity imply idempotency. Hence the claim follows from Proposition 3.3.
(ii) $\Rightarrow$ (i). It is trivial, therefore proof is omitted. □

We have shown that apart from the Lovász extension no aggregation-based extension is internal for all $m \in \mathcal{M}_{(n)}$. Now, we ask whether there is any aggregation-based extension which is conjunctive for all $m \in \mathcal{M}_{(n)}$. For a nontrivial fuzzy measure m there exists a subset $K \subsetneq N$ such that $m(K) \neq 0$ and therefore for any extension F of m it should hold $F(\mathbf{1}_K) \neq 0 = \mathrm{Min}(\mathbf{1}_K)$. We obtain the following observation.

Proposition 3.4. *There is no aggregation function A such that $F_{m,A}$ is conjunctive function extending fuzzy measure m for all $m \in \mathcal{M}_{(n)}$.*

The only case when $F_{m,A}$ is conjunctive is if $m = m_*$ and A is conjunctive ($F_{m_*,A} = A$).

It was shown in [3] that an n-ary aggregation function A is k-additive if and only if A is a polynomial with degree not exceeding k with coefficients fulfilling certain conditions imposed by nondecreasingness and boundary conditions of A. It means that for $k < n$ there is no k-additive n-ary aggregation function with zero annihilator. The only n-additive n-ary aggregation function with zero annihilator is $A(\mathbf{x}) = \prod_{i=1}^{n} x_i$. Taking into account Proposition 3.2 we get the following claim.

Proposition 3.5. *Let $k < n$. Then $F_{m,A}$ is not k-additive for all $m \in \mathcal{M}_{(n)}$ for any aggregation function $A \in \mathcal{A}_{(n)}$ suitable for aggregation-based extension. Moreover, $F_{m,A}$ is n-additive for all $m \in \mathcal{M}_{(n)}$ if and only if $A(\mathbf{x}) = \prod_{i=1}^n x_i$.*

Note that for the subclass of all k-additive measures $\mathcal{M}^k_{(n)}$ and $A(\mathbf{x}) = \prod_{i=1}^n x_i$ we have

$$F_{m,A}(\mathbf{x}) = \sum_{I \subseteq N} M_m(I) \prod_{i \in I} x_i = \sum_{\substack{I \subseteq N \\ |I| \leq k}} M_m(I) \prod_{i \in I} x_i,$$

for $m \in \mathcal{M}^k_{(n)}$, since $M_m(E) = 0$ whenever $|E| > k$ (see, e.g., [1]). Therefore, $F_{m,A}$ is a sum of polynomials with degree not exceeding k, so it is a k-additive aggregation function.

For the subclass of all symmetric fuzzy measures the property of being symmetric inherits from A to $F_{m,A}$.

Proposition 3.6. *Let $m \in \mathcal{M}_{(n)}$ be a symmetric fuzzy measure. Then $F_{m,A}$ is symmetric if and only if A is symmetric.*

Proof. Denote $m_k = m(E)$ for some E such that $|E| = k$. Then

$$\begin{aligned} F_{m,A}(\mathbf{x}) &= \sum_{E \subseteq N} m(E) \sum_{I \supseteq E} A(\mathbf{x}_I)(-1)^{|I \setminus E|} \\ &= \sum_{k=1}^{n} m_k \sum_{\substack{E \subseteq N \\ |E| = k}} \sum_{I \supseteq E} A(\mathbf{x}_I)(-1)^{|I \setminus E|}. \end{aligned}$$

Since the symmetry of $\sum_{\substack{E \subseteq N \\ |E|=k}} \sum_{I \supseteq E} A(\mathbf{x}_I)(-1)^{|I \setminus E|}$ for every $k \in N$ is equivalent to the symmetry of A, the claim follows. □

Proposition 3.7. *Let $m \in \mathcal{M}_{(2)}$ be a submodular fuzzy measure and $A \in \mathcal{A}_{(2)}$ be such that $A(x, 1) = x$ and $A(1, y) = y$. Then the following holds:*

(i) *$F_{m,A}$ is convex if and only if A is concave.*

(ii) *$F_{m,A}$ is concave if and only if A is convex.*

Proof. We prove just the claim (i), the proof of (ii) is similar.

Let $m(\{1\}) = a, m(\{2\}) = b$. Then

$$F_{m,A}(x, y) = aA(x, 1) + bA(1, y) + (1 - a - b)A(x, y).$$

For $\lambda \in [0,1]$, $(x_1,y_1),(x_2,y_2) \in [0,1]^2$ we have

$$\begin{aligned}
&F_{m,A}\big(\lambda(x_1,y_1)+(1-\lambda)(x_2,y_2)\big)\\
&= F_{m,A}\big(\lambda x_1+(1-\lambda)x_2, \lambda y_1+(1-\lambda)y_2\big)\\
&= aA(\lambda x_1+(1-\lambda)x_2, 1)+bA\big(1, \lambda y_1+(1-\lambda)y_2\big)\\
&\quad +(1-a-b)A\big(\lambda x_1+(1-\lambda)x_2, \lambda y_1+(1-\lambda)y_2\big)\\
&= a\big(\lambda x_1+(1-\lambda)x_2\big)+b\big(\lambda y_1+(1-\lambda)y_2\big)\\
&\quad +(1-a-b)A\big(\lambda x_1+(1-\lambda)x_2, \lambda y_1+(1-\lambda)y_2\big).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\lambda F_{m,A}(x_1,y_1)+(1-\lambda)F_{m,A}(x_2,y_2)\\
&= a\big(\lambda x_1+(1-\lambda)x_2\big)+b\big(\lambda y_1+(1-\lambda)y_2\big)\\
&\quad +(1-a-b)\big(\lambda A(x_1,y_1)+(1-\lambda)A(x_2,y_2)\big).
\end{aligned}$$

Summarizing and taking into account submodularity of m we obtain that convexity of $F_{m,A}$ is equivalent to concavity of A. □

Note, that a similar consideration can be done for a supermodular fuzzy measure $m \in \mathcal{M}_{(2)}$ resulting in equivalence of convexity (concavity) of $F_{m,A}$ and A.

As a consequence of the previous proposition we can conclude that whence the only copula which is concave is the Frechét-Hoeffding upper bound Min (see [6]), there is no copula A apart from Min yielding a convex aggregation function $F_{m,A}$ for a submodular measure m.

4. Suitable aggregation functions for some subclasses of $\mathcal{M}_{(n)}$

Recall that an aggregation function A is called suitable for a fuzzy measure m, if the function $F_{m,A}$ defined by (1) is an aggregation function extending m. Theorem 2.1 gives a complete characterization of aggregation functions suitable for all $m \in \mathcal{M}_{(n)}$. According to [4, Thm. 1], $F_{m,A}$ is an extension of m for each fuzzy measure m if and only if an aggregation function A has zero annihilator. But having zero annihilator need not be a necessary condition for $F_{m,A}$ being an extension of m, if we consider a fuzzy measure m just from some proper subclass of $\mathcal{M}_{(n)}$.

Example 4.1. Let $N=\{1,2\}$. Define the fuzzy measure m by $m(\emptyset)=m(\{1\})=0$, $m(\{2\})=m(N)=1$. Then $F_{m,A}(x,y)=A(1,y)$ and thus it is an extension of m whenever it holds $A(x,0)=0$ for all $x \in [0,1]$. Moreover, $F_{m,A}$ is an aggregation function for any aggregation function A.

Let $\delta_{\{i\}}$ be the Dirac measure, i.e., $\delta_{\{i\}}(E) = 1$ whenever $i \in E$ and it vanishes elsewhere. The following claim follows directly from the proof of [4, Thm. 1].

Proposition 4.1. *Let $M \subset \mathcal{M}_{(n)}$ such that $\delta_{\{i\}} \in M$ for every $i \in N$. If an aggregation function A is suitable for all $m \in M$ then A has zero annihilator.*

Proposition 4.2. *Let $M \subset \mathcal{M}_{(n)}$ be the subclass of all additive measures. Then an aggregation function A is suitable for all $m \in M$ if and only if A has zero annihilator.*

Proof. Let A has zero annihilator. Denote $m_i = m(\{i\})$. Since for additive measure m its Möbius transform vanishes whenever $|E| > 1$, we have $F_{m,A}(\mathbf{x}) = \sum_{i \in N} m_i.A(\mathbf{x}_{\{i\}})$. Therefore, $F_{m,A}$ is nondecreasing. It is also extension of m due to [4, Proof of Thm.1].

Opposite direction is a consequence of the previous proposition, since every Dirac measure $\delta_{\{i\}}$ is additive. □

Proposition 4.3. *Let $M \subset \mathcal{M}_{(n)}$ be the subclass of all symmetric measures. If an aggregation function A is suitable for all $m \in M$ then A has zero annihilator.*

Proof. For $E \subseteq N$ with $|E| = k$ denote $m(E) = m_k$. Then

$$\begin{aligned} F_{m,A}(\mathbf{0}) &= \sum_{E \subseteq N} m(E) \sum_{I \supseteq E} A(\mathbf{0}_I)(-1)^{|I \setminus E|} \\ &= \sum_{k=1}^{n} m_k \sum_{\substack{E \subseteq N \\ |E|=k}} \sum_{I \supseteq E} A(\mathbf{0}_I)(-1)^{|I \setminus E|}. \end{aligned}$$

Similarly, as in the proof of Proposition 3.3, it can be shown by induction w.r.t. to the cardinality $|E|$ that $A(\mathbf{0}_E) = 0$ for all $E \subseteq N$ for ensuring that $F_{m,A}$ is an extension of m and the claim follows. □

Proposition 4.4. *Let $n = 2$, $M \subset \mathcal{M}_{(2)}$ be the subclass of all symmetric supermodular measures. Then an aggregation function A is suitable for all $m \in M$ if and only if A has zero annihilator.*

Proof. Since m is symmetric, we have

$$F_{m,A}(x, y) = a\big(A(x, 1) + A(1, y) - 2A(x, y)\big) + A(x, y),$$

where $m(\{1\}) = m(\{2\}) = a$. Clearly, having zero annihilator is necessary and sufficient condition for $F_{m,A}$ being an extension of m.

To show nondecreasingness of $F_{m,A}$ in the first coordinate let $x_1 \geq x_2$. Then

$$\begin{aligned}
&F_{m,A}(x_1, y) - F_{m,A}(x_2, y)\\
&= a\big(A(x_1, 1) + A(1, y) - 2A(x_1, y)\big) + A(x_1, y)\\
&\quad - a\big(A(x_2, 1) + A(1, y) - 2A(x_2, y)\big) - A(x_2, y)\\
&= a\big(A(x_1, 1) - A(x_2, 1) - 2A(x_1, y) + 2A(x_2, y)\big) + A(x_1, y) - A(x_2, y)\\
&= a\big(A(x_1, 1) - A(x_2, 1)\big) + (1 - 2a)\big(A(x_1, y) - A(x_2, y)\big).
\end{aligned}$$

Thus, $F_{m,A}(x_1, y) - F_{m,A}(x_2, y) \geq 0$ due to supermodularity of m and nondecreasingness of A. Similarly, nondecreasingness in second coordinate can be shown. $\square$

5. Concluding remarks

We have shown that among all aggregation-based extensions only the extension based on Min, i.e., the Lovász extension, has the property of being idempotent (internal, comonotone additive, positive homogeneous, translation invariant) for all fuzzy measures. Similarly, only the Owen extension has the property of being n-additive for all fuzzy measures. On the other hand, for $n = 2$ we have shown that for the subclass of all submodular measures there is a larger class of aggregation functions (class of all concave aggregation functions satisfying certain boundary conditions) yielding convex aggregation-based extensions. By our conjecture, it is also the case for $n > 2$ and in the further work we plan to look at this problem. We have characterized all aggregation functions suitable for aggregation-based extension for the subclass of all additive fuzzy measures and, for n=2, for the subclass of all symmetric supermodular measures.

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