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# OSCILLATION TESTS FOR FRACTIONAL DIFFERENCE EQUATIONS

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ABSTRACT. In this paper, we study the oscillatory behavior of solutions of the fractional difference equation of the form

$$\Delta\Big(r(t)g\big(\Delta^\alpha x(t)\big)\Big) + p(t)f\bigg(\sum_{s=t_0}^{t-1+\alpha}(t-s-1)^{(-\alpha)}x(s)\bigg) = 0, \qquad t \in \mathbb{N}_{t_0+1-\alpha},$$

where $\Delta^\alpha$ denotes the Riemann-Liouville fractional difference operator of order $\alpha$, $0 < \alpha \le 1$, $\mathbb{N}_{t_0+1-\alpha} = \{t_0+1-\alpha, t_0+2-\alpha, \dots\}$, $t_0 > 0$ and $\gamma > 0$ is a quotient of odd positive integers. We establish some oscillatory criteria for the above equation, using the Riccati transformation and Hardy type inequalities. Examples are provided to illustrate the theoretical results.

## 1. Introduction

Fractional difference equations are generalizations of classical difference equations of integer order and can find their applications in many fields of science and machinery. In the last few decades, extensive research has been made on various aspects of fractional difference equations. The existence, uniqueness, stability, oscillation, numerical studies, and various scientific models are some examples. Fractional calculus finds significant application in the fields of capacitor theory, electrical circuits, viscoelasticity, electro-analytical chemistry, tumor growth models, neurology, control theory and statistics. Significant progress has been made in the study of fractional differential equations. On the contrary, very little development has been made in the theory of fractional difference equations, see [5]–[8], [10]. In particular, we notice that the oscillatory results for fractional difference equations has been studied by several authors [1], [3], [4], [9], [11]–[14].


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T. Abdeljawad [2] extensively studied the properties and relations between Riemann-Liouville and Caputo fractional differences.

W. N. Li [11] obtained interesting results about T. Kalaimani's oscillatory behavior of solutions of the fractional difference equation of the form

$$\big(1+p(t)\big)\Delta\big(\Delta^{\alpha}x(t)\big)+p(t)\Delta^{\alpha}x(t)+f\big(t,x(t)\big)=g(t), \qquad t\in\mathbb{N}_0,$$

where $\Delta^{\alpha}$ is the Riemann–Liouville fractional difference operator of order $\alpha$ with $0<\alpha\le 1$.

A. Secer and H. Adiguzel [14] considered the oscillation of the following fractional difference equation

$$\Delta\left(a(t)\left[\Delta\left(r(t)\big(\Delta^{\alpha}x(t)\big)^{\gamma_1}\right)\right]^{\gamma_2}\right)+q(t)f\left(\sum_{s=t_0}^{t-1+\alpha}(t-s-1)^{(-\alpha)}x(s)\right)=0,$$

where $t\in\mathbb{N}_{t_0+1-\alpha}$, $\gamma_1$, $\gamma_2$ are the quotients of two odd positive numbers and $\Delta^{\alpha}$ denotes the Riemann–Liouville fractional difference operator of order $\alpha$, $0<\alpha\le 1$ with the condition $\sum_{s=t_0}^{\infty}\frac{1}{a^{\frac{1}{\gamma_2}}(s)}=\infty$.

G. E. Chatzarakis et al. [4] studied the oscillatory behavior of the following fractional difference equation

$$\Delta\big(\Delta^{\alpha}x(t)\big)^{\gamma}+q(t)f\big(x(t)\big)=0, \qquad t\in\mathbb{N}_{t_0+1-\alpha},$$

where $\Delta^{\alpha}$ denotes the Riemann–Liouville fractional difference operator of order $\alpha$, $0<\alpha\le 1$, $\gamma>0$ is a quotient of odd positive integers with the condition $\sum_{s=t_0}^{\infty}q^{\frac{1}{\gamma}}(s)=\infty$.

The objective of this paper is to study the oscillatory behavior of the solutions of fractional difference equations of the form

$$\Delta\Big(r(t)g\big(\Delta^{\alpha}x(t)\big)\Big)+p(t)f\left(\sum_{s=t_0}^{t-1+\alpha}(t-s-1)^{(-\alpha)}x(s)\right)=0, \quad t\in\mathbb{N}_{t_0+1-\alpha}. \tag{1}$$

Here $\Delta^{\alpha}$ denotes the Riemann–Liouville fractional difference operator defined in [11]. In the paper, we assume the following conditions:

(H1) $r(t)$ and $p(t)$ are positive sequences and $f,g\colon\mathbb{R}\longrightarrow\mathbb{R}$ are continuous functions with $\frac{f(x)}{x}\ge k_1$ and $\frac{x}{g(x)}\ge k_2$ for some constants $k_1,k_2$ and for all $x\ne 0$.

(H2) $g\in C(\mathbb{R},\mathbb{R})$ is a continuous function with $vg(v)>0$ for $v\ne 0$ and there exists a positive constant $\mu$ such that $g(vw)\le \mu vg(w)$ for $vw\ne 0$.

A solution $x(t)$ of (1) is said to be *oscillatory* if it has no last zero, i.e., if $x(t_1)=0$, then there exists a $t_2>t_1$ such that $x(t_2)=0$. Otherwise, the solution is said to be *nonoscillatory*. An equation is *oscillatory* if all its solutions oscillate.

## 2. Preliminaries

In this section, we present some preliminary results from discrete fractional calculus. We will make use of these results, throughout the paper.

**Definition 2.1** (See [14]). Let $\nu > 0$. The $\nu$th fractional sum $f$ is defined by

$$\Delta^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{(\nu-1)} f(s),$$

where $f$ is defined for $s \equiv a \bmod (1)$, $\Delta^{-\nu} f(t)$ is defined for $t \equiv (a+\nu) \bmod (1)$ and $t^{(\nu)} = \frac{\Gamma(t+1)}{\Gamma(t-\nu+1)}$. The fractional sum $\Delta^{-\nu} f$ maps functions defined in $\mathbb{N}_a$ to functions defined in $\mathbb{N}_{a+\nu}$.

**Definition 2.2.** (See [11]). Let $\mu > 0$ and $m-1 < \mu < m$, where $m$ denotes a positive integer, $m = \lceil \mu \rceil$. Set $\nu = m - \mu$. The $\mu$th order fractional difference is defined as

$$\Delta^{\mu} f(t) = \Delta^{m-\nu} f(t) = \Delta^{m} \Delta^{-\nu} f(t).$$

**Lemma 2.1** (See [14]). *Let $x(t)$ be a solution of* (1) *and*

$$G(t) = \sum_{s=t_0}^{t-1+\alpha} (t-s-1)^{(-\alpha)} x(s).$$

*Then*

$$\Delta\big(G(t)\big) = \Gamma(1-\alpha)\Delta^{\alpha}\big(x(t)\big). \tag{2}$$

**Lemma 2.2** (See [10]). *If $X$ and $Y$ are nonnegative, then*

$$mXY^{m-1} - X^m \le (m-1)Y^m \qquad \text{for} \quad m > 1. \tag{3}$$

## 3. Main results

**Theorem 3.1.** *Suppose that* (H1) *and* (H2) *hold and*

$$\sum_{s=t_1}^{\infty} g\big(1/r(s)\big) = \infty. \tag{4}$$

*Furthermore, assume that there exists a positive sequence $b(t)$ such that*

$$\limsup_{t\to\infty} \sum_{s=t_1}^{t-1} \left[ k_1 b(s) p(s) - \frac{1}{k_2} R(s) \right] = \infty, \tag{5}$$

*where*

$$R(s) = \frac{(\Delta b_+(s))^2 r(s+1)}{4b(s+1)\Gamma(1-\alpha)}, \quad \Delta b_+(s) = \max\big\{\Delta b(s), 0\big\}.$$

*Then all solutions of* (1) *are oscillatory.*

P r o o f. Assume, for the sake of contradiction, that $x(t)$ is a nonoscillatory solution of (1). Without loss of generality, we can assume that $x(t)$ is an eventually positive sequence of (1). Then there exists $t_1 > t_0$ such that

$$x(t) > 0, \quad G(t) > 0, \quad f\big(G(t)\big) > 0 \qquad \text{for} \quad t \geq t_1.$$

From (1) we have

$$\Delta\Big(r(t)g\big(\Delta^\alpha x(t)\big)\Big) = -p(t)f\big(G(t)\big) < 0 \qquad \text{for} \quad t \geq t_1.$$

Thus, $r(t)g\big(\Delta^\alpha x(t)\big)$ is an eventually nonincreasing sequence. Next we show that $r(t)g\big(\Delta^\alpha x(t)\big)$ is eventually positive sequence. Suppose there exists an integer $t_1 > t_0$ such that

$$r(t_1)g\big(\Delta^\alpha x(t_1)\big) = c < 0 \qquad \text{for} \quad t_1 > t_0,$$

so that

$$r(t)g\big(\Delta^\alpha x(t)\big) \leq c \qquad \text{for} \quad t \geq t_1,$$

or

$$\Delta^\alpha x(t) \leq g\big(c/r(t)\big) \leq \mu c g\big(1/r(t)\big).$$

Thus

$$\Delta(G(t)) \leq \Gamma(1-\alpha)\mu c g\big(1/r(t)\big).$$

Summing both sides of the last inequality from $t_1$ to $t-1$, we get

$$\sum_{s=t_1}^{t-1} \Delta\big(G(s)\big) \leq \sum_{s=t_1}^{t-1} \Gamma(1-\alpha)\mu c g\big(1/r(s)\big),$$

or

$$G(t) \leq G(t_1) + \sum_{s=t_1}^{t-1} \Gamma(1-\alpha)\mu c g\big(1/r(s)\big) \to -\infty \qquad \text{as} \quad t \to \infty,$$

which contradicts the fact that $G(t) > 0$. Hence, $r(t)g\big(\Delta^\alpha x(t)\big)$ is eventually positive.

Define the function $w(t)$ by the Riccati substitution

$$w(t) = \frac{b(t)r(t)g(\Delta^\alpha x(t))}{G(t)}.$$

Since $b(t) > 0$, $x(t) > 0$ and $r(t)g\big(\Delta^\alpha x(t)\big) > 0$, we have $w(t) > 0$.

Now,

$$\begin{aligned}
\Delta w(t) &= \Delta\left(\frac{b(t)r(t)g(\Delta^\alpha x(t))}{G(t)}\right)\\
&= b(t)\Delta\left(\frac{r(t)g(\Delta^\alpha x(t))}{G(t)}\right) + \frac{r(t+1)g(\Delta^\alpha x(t+1))}{G(t+1)}\Delta b(t)\\
&\le b(t)\left(\frac{G(t)(-p(t)f(G(t))) - r(t)g(\Delta^\alpha x(t))\Delta G(t)}{G(t)G(t+1)}\right) + \frac{w(t+1)}{b(t+1)}\Delta b(t)\\
&\le -\frac{b(t)p(t)G(t)f(G(t))}{G(t)G(t+1)}\\
&\quad - \frac{b(t)r(t)g(\Delta^\alpha x(t))\Delta G(t)w(t)w(t+1)}{b(t)b(t+1)r(t)r(t+1)g(\Delta^\alpha x(t))g(\Delta^\alpha x(t+1))} + \frac{w(t+1)}{b(t+1)}\Delta b(t)\\
&\le -b(t)p(t)k_1 - \frac{\Delta G(t)w(t)w(t+1)}{b(t+1)r(t+1)g(\Delta^\alpha x(t+1))} + \Delta b_+(t)\frac{w(t+1)}{b(t+1)}\\
&= -b(t)p(t)k_1 - \frac{\Gamma(1-\alpha)\Delta^\alpha x(t)w(t)w(t+1)}{b(t+1)r(t+1)g(\Delta^\alpha x(t+1))} + \Delta b_+(t)\frac{w(t+1)}{b(t+1)},
\end{aligned}$$

or

$$\Delta w(t) \le \Delta b_+(t)\frac{w(t+1)}{b(t+1)} - \frac{\Gamma(1-\alpha)k_2 w^2(t+1)}{b(t+1)r(t+1)} - b(t)p(t)k_1. \tag{6}$$

Let

$$X = \sqrt{\frac{\Gamma(1-\alpha)k_2}{b(t+1)r(t+1)}}\,w(t+1) \quad \text{and} \quad Y = \frac{\Delta b_+(t)}{2\sqrt{\dfrac{b(t+1)\Gamma(1-\alpha)k_2}{r(t+1)}}}.$$

Using Lemma 2.2 (Hardy type inequality) and setting $m = 2$, we obtain

$$2\sqrt{\frac{\Gamma(1-\alpha)k_2}{b(t+1)r(t+1)}}\,w(t+1)\quad \frac{\Delta b_+(t)}{2\sqrt{\dfrac{\Gamma(1-\alpha)k_2 b(t+1)}{r(t+1)}}} - \frac{\Gamma(1-\alpha)k_2}{b(t+1)r(t+1)}w^2(t+1)$$
$$\le \frac{(\Delta b_+(t))^2 r(t+1)}{4b(t+1)k_2\Gamma(1-\alpha)},$$

which implies that

$$\Delta w(t) \le \frac{(\Delta b_+(t))^2 r(t+1)}{4b(t+1)k_2\Gamma(1-\alpha)} - b(t)p(t)k_1.$$

Summing the above inequality from $t_1$ to $t-1$, we get

$$\sum_{s=t_1}^{t-1} \Delta w(t) \le \sum_{s=t_1}^{t-1} \left( \frac{(\Delta b_+(s))^2 r(s+1)}{4b(s+1)k_2\Gamma(1-\alpha)} - b(s)p(s)k_1 \right),$$

or

$$w(t) - w(t_1) \le \sum_{s=t_1}^{t-1} \left( \frac{(\Delta b_+(s))^2 r(s+1)}{4b(s+1)k_2\Gamma(1-\alpha)} - b(s)p(s)k_1 \right),$$

i.e.,

$$\sum_{s=t_1}^{t-1} \left( b(s)p(s)k_1 - \frac{(\Delta b_+(s))^2 r(s+1)}{4b(s+1)k_2\Gamma(1-\alpha)} \right) \le w(t_1) - w(t) \le w(t_1).$$

Letting $t \to \infty$, we have

$$\limsup_{t\to\infty} \sum_{s=t_1}^{t-1} \left( b(s)p(s)k_1 - \frac{(\Delta b_+(s))^2 r(s+1)}{4b(s+1)k_2\Gamma(1-\alpha)} \right) \le w(t_1) - w(t) \le w(t_1) < \infty,$$

which contradicts (5). The proof of the theorem is complete. □

**Theorem 3.2.** *Suppose that* (H1) *and* (H2) *hold and*

$$\sum_{s=t_1}^{\infty} g\big(1/r(s)\big) = \infty. \tag{7}$$

*Furthermore, assume that there exists a positive sequence $b(t)$ and a double positive sequence $H(t,s)$ such that*

$$H(t,t) = 0 \qquad \textit{for} \quad t \ge t_0,$$
$$H(t,s) > 0 \qquad \textit{for} \quad t > s \ge t_0$$

*and*

$$\Delta_s H(t,s) = H(t,s+1) - H(t,s) \le 0 \qquad \textit{for} \quad t > s \ge t_0.$$

*If*

$$\limsup_{t\to\infty} \sum_{s=t_1}^{t-1} \left( H(t,s)b(s)p(s) - \frac{h_+^2(t,s)b(s+1)r(s+1)}{4k_1k_2H(t,s)\Gamma(1-\alpha)} \right) = \infty, \tag{8}$$

*where*

$$h_+(t,s) = \frac{H(t,s)\Delta b_+(s)}{b(s+1)} + \Delta_s H(t,s) \quad \textit{and} \quad \Delta b_+(s) = \max\big\{\Delta b(s), 0\big\},$$

*then all solutions of* (1) *are oscillatory.*

P r o o f. Assume, for the sake of contradiction, that $x(t)$ is a nonoscillatory solution of (1). Without loss of generality, we can assume that $x(t)$ is an eventually positive sequence of (1). Proceeding as in Theorem 3.1, we arrive at equation (6). Multiplying (6) by $H(t,s)$ and summing from $t_1$ to $t-1$, we get

$$\begin{aligned}&\sum_{s=t_1}^{t-1} H(t,s)k_1 b(s)p(s)\\&\le \sum_{s=t_1}^{t-1}\left(\frac{H(t,s)\Delta b_+(s)w(s+1)}{b(s+1)} - \frac{H(t,s)\Gamma(1-\alpha)k_2 w^2(s+1)}{b(s+1)r(s+1)}\right)\\&\quad - \sum_{s=t_1}^{t-1} H(t,s)\Delta w(s).\end{aligned}$$

Using the summation by parts formula, we have that

$$\begin{aligned}-\sum_{s=t_1}^{t-1} H(t,s)\Delta w(s) &= -\big[H(t,s)w(s)\big]_{t_1}^{t} + \sum_{s=t_1}^{t-1} w(s+1)\Delta_s H(t,s)\\&= H(t,t_1)w(t_1) + \sum_{s=t_1}^{t-1} w(s+1)\Delta_s H(t,s),\end{aligned}$$

which implies that

$$\begin{aligned}&\sum_{s=t_1}^{t-1} H(t,s)k_1 b(s)p(s)\\&\le \sum_{s=t_1}^{t-1}\left(\frac{H(t,s)\Delta b_+(s)w(s+1)}{b(s+1)} - \frac{H(t,s)\Gamma(1-\alpha)k_2 w^2(s+1)}{b(s+1)r(s+1)}\right)\\&\quad + H(t,t_1)w(t_1) + \sum_{s=t_1}^{t-1} w(s+1)\Delta_s H(t,s)\\&= \sum_{s=t_1}^{t-1}\left(\frac{H(t,s)\Delta b_+(s)}{b(s+1)} + \Delta_s H(t,s)\right) w(s+1)\\&\quad - \sum_{s=t_1}^{t-1}\frac{H(t,s)\Gamma(1-\alpha)k_2 w^2(s+1)}{b(s+1)r(s+1)} + H(t,t_1)w(t_1)\\&= \sum_{s=t_1}^{t-1}\left(h_+(t,s)w(s+1) - \frac{H(t,s)\Gamma(1-\alpha)k_2 w^2(s+1)}{b(s+1)r(s+1)}\right)\\&\quad + H(t,t_1)w(t_1).\end{aligned}$$

Set

$$X = \sqrt{\frac{H(t,s)\Gamma(1-\alpha)k_2}{b(s+1)r(s+1)}}w(s+1) \quad \text{and} \quad Y = \frac{h_+(t,s)}{2\sqrt{\frac{H(t,s)\Gamma(1-\alpha)k_2}{b(s+1)r(s+1)}}}.$$

Using Lemma 2.2 (Hardy type inequality) with $m = 2$, we have that

$$2\sqrt{\frac{H(t,s)\Gamma(1-\alpha)k_2}{b(s+1)r(s+1)}}w(s+1)\ \frac{h_+(t,s)}{2\sqrt{\frac{H(t,s)\Gamma(1-\alpha)k_2}{b(s+1)r(s+1)}}} - \frac{H(t,s)\Gamma(1-\alpha)k_2 w^2(s+1)}{b(s+1)r(s+1)} \le \frac{(h_+(t,s))^2 b(s+1)r(s+1)}{4H(t,s)\Gamma(1-\alpha)k_2},$$

or

$$\sum_{s=t_1}^{t-1} H(t,s)k_1 b(s)p(s) \le \sum_{s=t_1}^{t-1}\left(\frac{(h_+(t,s))^2 b(s+1)r(s+1)}{4H(t,s)\Gamma(1-\alpha)k_2}\right) + H(t,t_1)w(t_1),$$

i.e.,

$$\begin{aligned}\sum_{s=t_1}^{t-1}\left(H(t,s)b(s)p(s) - \frac{k_1^{-1}(h_+(t,s))^2 b(s+1)r(s+1)}{4H(t,s)\Gamma(1-\alpha)k_2}\right) &\le k_1^{-1}H(t,t_1)w(t_1)\\ &\le k_1^{-1}H(t,t_0)w(t_1).\end{aligned}$$

Since $0 < H(t,s) \le H(t,t_0)$ for $t > s \ge t_0$, then we have $0 < \frac{H(t,s)}{H(t,t_0)} \le 1$ for $t > s \ge t_0$. Hence, it follows that

$$\begin{aligned}&\frac{1}{H(t,t_0)}\sum_{s=t_0}^{t-1}\left(H(t,s)b(s)p(s) - \frac{k_1^{-1}(h_+(t,s))^2 b(s+1)r(s+1)}{4H(t,s)\Gamma(1-\alpha)k_2}\right)\\
&= \frac{1}{H(t,t_0)}\sum_{s=t_0}^{t_1-1}\left(H(t,s)b(s)p(s) - \frac{k_1^{-1}(h_+(t,s))^2 b(s+1)r(s+1)}{4H(t,s)\Gamma(1-\alpha)k_2}\right)\\
&\quad + \frac{1}{H(t,t_0)}\sum_{s=t_1}^{t-1}\left(H(t,s)b(s)p(s) - \frac{k_1^{-1}(h_+(t,s))^2 b(s+1)r(s+1)}{4H(t,s)\Gamma(1-\alpha)k_2}\right)\\
&\le \frac{1}{H(t,t_0)}\sum_{s=t_0}^{t_1-1}\left(H(t,s)b(s)p(s) - \frac{k_1^{-1}(h_+(t,s))^2 b(s+1)r(s+1)}{4H(t,s)\Gamma(1-\alpha)k_2}\right) + k_1^{-1}w(t_1)\\
&\le \frac{1}{H(t,t_0)}\sum_{s=t_0}^{t_1-1}\big(H(t,s)b(s)p(s)\big) + k_1^{-1}w(t_1)\\
&\le \sum_{s=t_0}^{t_1-1}\big(b(s)p(s)\big) + k_1^{-1}w(t_1).\end{aligned}$$

Letting $t \to \infty$, we have

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \sum_{s=t_0}^{t-1} \left( H(t,s)b(s)p(s) - \frac{(h_+(t,s))^2 b(s+1)r(s+1)}{4k_1k_2H(t,s)\Gamma(1-\alpha)} \right)$$
$$\le \sum_{s=t_0}^{t_1-1} \big(b(s)p(s)\big) + k_1^{-1}w(t_1) < \infty,$$

which contradicts (8). The proof of the theorem is complete. □

## 4. Examples

Example 4.1. Consider the fractional difference equation

$$\Delta\Big(tg\big(\Delta^{0.5}x(t)\big)\Big) + tf\left(\sum_{s=t_0}^{t-0.5}(t-s-1)^{(-0.5)}x(s)\right) = 0 \qquad \text{for} \quad t \in \mathbb{N}_{t_0+0.5}, \quad (9)$$

where $\alpha = 0.5$, $r(t) = t$, $p(t) = t$, $b(t) = \frac{1}{t^2}$, $t_0 = 2$, $k_1 = 1$, $k_2 = 1$, $g(x) = x$ and $f(x) = x$. It is easy to see that (H1) and (H2) hold. Then we have

$$\sum_{s=2}^{\infty} g\left(\frac{1}{r(s)}\right) = \sum_{s=2}^{\infty}\frac{1}{r(s)} = \sum_{s=2}^{\infty}\frac{1}{s} = \infty.$$

Now

$$\limsup_{t\to\infty}\sum_{s=t_1}^{t-1}\left[k_1b(s)p(s) - \frac{1}{k_2}R(s)\right] =$$
$$\limsup_{t\to\infty}\sum_{s=t_1}^{t-1}\left(\frac{1}{s} - \frac{(\Delta b_+(s))^2(s+1)^3}{4\sqrt{\pi}}\right).$$

Since $\Delta b(s) < 0$, therefore we can choose $\Delta b_+(s) = 0$. Then we have

$$\limsup_{t\to\infty}\sum_{s=t_1}^{t-1}\left[k_1b(s)p(s) - \frac{1}{k_2}R(s)\right] =$$
$$\limsup_{t\to\infty}\sum_{s=t_1}^{t-1}\frac{1}{s} = \infty,$$

that is condition (5) of Theorem 3.1 is satisfied. Therefore, all solutions of (9) are oscillatory.

Example 4.2. Consider the fractional difference equation

$$\Delta\Big(tg\big(\Delta^{0.5}x(t)\big)\Big)+t^{1/2}f\left(\sum_{s=t_0}^{t-0.5}(t-s-1)^{(-0.5)}x(s)\right)=0 \quad \text{for } t\in\mathbb{N}_{t_0+0.5}, \tag{10}$$

where $\alpha = 0.5$, $r(t) = t$, $p(t) = t^{1/2}$, $b(t) = \frac{1}{t^{3/2}}$, $t_0 = 2$, $k_1 = 1$, $k_2 = 1$, $g(x) = x$ and $f(x) = x$. Clearly conditions (H1) and H(2) hold. We apply Theorem 3.1 with $\Delta b_+(s) = 0$, we obtain

$$\sum_{s=2}^{\infty} g\left(\frac{1}{r(s)}\right) = \sum_{s=2}^{\infty}\frac{1}{r(s)} = \sum_{s=2}^{\infty}\frac{1}{s} = \infty.$$

Hence,

$$\limsup_{t\to\infty}\sum_{s=t_1}^{t-1}\left[k_1 b(s)p(s) - \frac{1}{k_2}R(s)\right] =$$

$$\limsup_{t\to\infty}\sum_{s=t_1}^{t-1}\left(\frac{s^{1/2}}{s^{3/2}} - \frac{(\Delta b_+(s))^2\sqrt{s+1})}{4\sqrt{\pi}}\right)$$

and therefore

$$\limsup_{t\to\infty}\sum_{s=t_1}^{t-1}\left[k_1 b(s)p(s) - \frac{1}{k_2}R(s)\right] =$$

$$\limsup_{t\to\infty}\sum_{s=t_1}^{t-1}\frac{1}{s} = \infty.$$

That is, condition (5) of Theorem 3.1 is satisfied. Therefore all solutions of (10) are oscillatory.

Example 4.3. Consider the fractional difference equation

$$\Delta\Big(tg\big(\Delta^{0.5}x(t)\big)\Big)+tf\left(\sum_{s=t_0}^{t-0.5}(t-s-1)^{(-0.5)}x(s)\right)=0 \qquad \text{for } t\in\mathbb{N}_{t_0+0.5}, \tag{11}$$

where $\alpha = 0.5$, $r(t) = t$, $p(t) = t$, $b(t) = \frac{1}{t^3}$, $t_0 = 2$, $H(t,s) = t+s$, $\Delta_s H(t,s) = -1$, $h_+(t,s) = -1$, $\Delta b_+(s) = 0$, $k_1 = 1$, $k_2 = 1$, $g(x) = x$ and $f(x) = x$. Clearly, conditions (H1) and H(2) hold. Then we have

$$\sum_{s=2}^{\infty} g\left(\frac{1}{r(s)}\right) = \sum_{s=2}^{\infty}\frac{1}{r(s)} = \sum_{s=2}^{\infty}\frac{1}{s} = \infty.$$

Thus, the condition (8) reads

$$\limsup_{t\to\infty} \sum_{s=t_1}^{t-1} \left[ H(t,s)b(s)p(s) - \frac{h_+^2(t,s)b(s+1)r(s+1)}{4k_1k_2H(t,s)\Gamma(1-\alpha)} \right] =$$

$$\limsup_{t\to\infty} \sum_{s=t_1}^{t-1} \left[ \frac{t}{s^2} + \frac{1}{s} - \frac{1}{4\sqrt{\pi}(t-s)(s+1)^2} \right] = \infty.$$

That is, all the conditions of the Theorem 3.2 are satisfied. Therefore, all solutions of (11) are oscillatory.

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