EXISTENCE OF POSITIVE BOUNDED SOLUTIONS
OF SYSTEM OF THREE DYNAMIC EQUATIONS
WITH NEUTRAL TERM ON TIME SCALES

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ABSTRACT. In this paper the system of three dynamic equations with neutral
term in the following form

\[
\begin{align*}
(x(t) + p(t) x(u_1(t)))^\Delta &= a(t) f(y(u_2(t))), \\
y^\Delta(t) &= b(t) g(z(u_3(t))), \\
z^\Delta(t) &= c(t) h(x(u_4(t)))
\end{align*}
\]

on time scales is considered. The aim of this paper is to present sufficient condi-
tions for the existence of positive bounded solutions of the considered system for
\(0 < p(t) \leq \text{const} < 1\). The main tool of the proof of presented here result is Kras-
koselskiï’s fixed point theorem. Also, the useful generalization of the Arzelá-Ascoli
theorem on time scales to the three-dimensional case is proved.

1. Introduction

Consider a nonlinear dynamic system of three equations of the form

\[
\begin{align*}
(x(t) + p(t) x(u_1(t)))^\Delta &= a(t) f(y(u_2(t))), \\
y^\Delta(t) &= b(t) g(z(u_3(t))), \\
z^\Delta(t) &= c(t) h(x(u_4(t)))
\end{align*}
\]  

(1)
on a time scale \(T\).

Throughout this paper \(x, y, z: T \rightarrow \mathbb{R}\) are unknown functions and \(p, a, b, c: T \rightarrow \mathbb{R}, f, g, h: \mathbb{R} \rightarrow \mathbb{R}\). Moreover, \(u_i: T \rightarrow T\) is such that \(\lim_{t \rightarrow \infty} u_i(t) = \infty\) for
\(i = 1, 2, 3, 4\). Here \(\mathbb{R}\) is the set of real numbers and \(T\) is an arbitrary time scale.

The time scale theory was found promising because it demonstrates the interplay between the theories of continuous-time and discrete-time systems.
Particularly, the system of dynamic equations on time scales was studied by many authors. For example, Taousser, Defoort and Djemai in [10] deal with the stability analysis of a class of uncertain switched systems on non-uniform time domains. The classical results on stabilization of nonlinear continuous-time and discrete-time systems are extended to systems on arbitrary time scales with bounded graininess function in [1] by Bartosiewicz and Piotrowska. A necessary and sufficient condition for the exponential stability of time-invariant linear systems on time scales in terms of the eigenvalues of the system matrix is found by Pötzsche, Siegmund and Wirth in [8]. In [12], Zhu and Wang considered the existence of nonoscillatory solutions to neutral dynamic equations on time scales. In the discrete case, the existence of a bounded nonoscillatory solution of nonlinear neutral type difference systems has been studied in [9] and [11]. In this paper the authors improved and generalised for arbitrary time scales some results obtained by Migda, Schmeidel and Zdanowicz in [7] in discrete case.

Let us recall some basic definitions and facts related to time scales.

**Definition 1 ([2]).** A time scale \( T \) is an arbitrary nonempty closed subset of the real numbers.

The mapping \( \sigma: T \to T \), defined by \( \sigma(t) = \inf \{ s \in T : s > t \} \) with \( \inf \emptyset = \sup T \) is called the forward jump operator. Similarly, we define the backward jump operator \( \rho: T \to T \), given by \( \rho(t) = \sup \{ s \in T : s < t \} \) with \( \sup \emptyset = \inf T \). The following classification of points is used within the theory: a point \( t \in T \) is called right-dense, right-scattered, left-dense and left-scattered if \( \sigma(t) = t \) (for \( t < \sup T \)), \( \sigma(t) > t \), \( \rho(t) = t \) (for \( t > \inf T \)) and \( \rho(t) < t \), respectively. We say that \( t \) is isolated if \( \rho(t) < t < \sigma(t) \), and that \( t \) is dense if \( \rho(t) = t = \sigma(t) \). The function \( \mu: T \to [0, \infty) \) defined by \( \mu(t) = \sigma(t) - t \) is called the graininess function. The delta (or Hilger) derivative of \( f: T \to \mathbb{R} \) at a point \( t \in T^\kappa \), where

\[
T^\kappa := \begin{cases} 
T \setminus (\rho(\sup T), \sup T] & \text{if } \sup T < \infty, \\
T & \text{if } \sup T = \infty
\end{cases}
\]

is defined in the following way.

**Definition 2 ([2]).** The delta derivative of function \( f \) at a point \( t \), denoted by \( f^\Delta(t) \), is the number (provided it exists) with the property that given any \( \varepsilon > 0 \), there is a neighborhood \( U \) of \( t \) (i.e., \( U = (t - \delta; t + \delta) \cap T \) for some \( \delta > 0 \) such that

\[
\left| \left( f(\sigma(t)) - f(s) \right) - f^\Delta(t)(\sigma(t) - s) \right| \leq \varepsilon |\sigma(t) - s| \quad \text{for all } s \in U.
\]

(see [2], [3]), and from twenty years attracting attention of many researchers.
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We say that a function $f$ is delta (or Hilger) differentiable on $\mathbb{T}^\kappa$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$. The function $f^\Delta: \mathbb{T}^\kappa \to \mathbb{R}$ is then called the (delta) derivative of $f$ on $\mathbb{T}^\kappa$.

**Definition 3** ([2]). A function $f: \mathbb{T} \to \mathbb{R}$ is called regulated provided its right-sided limits exist (finite) at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. A function $f: \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-side limits exist (finite) at left-dense points in $\mathbb{T}$.

**Definition 4** ([2]). Assume $f: \mathbb{T} \to \mathbb{R}$ is a regulated function. We define the indefinite integral of a regulated function $f$ by \( \int f(t)\Delta t = F(t) + C \), where $C$ is an arbitrary constant and $F$ is a pre-derivative of $f$. We define the Cauchy integral by \( \int_a^b f(t)\Delta t = F(b) - F(a) \) for all $a, b \in \mathbb{T}$.

We are interested in the nonoscillatory behaviour of system (1), that is why the general assumption on the time scale $\mathbb{T}$ is the following $\inf \mathbb{T} = T_0$ and $\sup \mathbb{T} = \infty$.

By a solution of (1), we mean a sequence $(X(t)) = [x(t), y(t), z(t)]^T$ of delta differentiable functions which are defined on $\mathbb{T}$ and satisfy (1) for $t \geq T_1 \geq T_0$. A function $\varphi$ is called eventually positive (or eventually negative) if there exists $T \in \mathbb{T}$ such that $\varphi(t) > 0$ (or $\varphi(t) < 0$) for all $t \geq T$ in $\mathbb{T}$. If the function $\varphi$ is either eventually positive or eventually negative we call it nonoscillatory. A solution $X$ of system (1) is called nonoscillatory if all its components, i.e., $x, y, z$ are nonoscillatory.

**2. Preliminaries**

For $T_1, T_2 \in \mathbb{T}$, let \( [T_1, \infty)_\mathbb{T} = \{ t \in \mathbb{T}: t \geq T_1 \} \) and \( [T_1, T_2]_\mathbb{T} = \{ t \in \mathbb{T}: T_1 \leq t \leq T_2 \} \).

By $C(A, B)$, $C_{rd}(A, B)$ we denote the set of all continuous functions mapping $A$ to $B$, the set of all rd-continuous functions mapping $A$ to $B$, respectively.

For elements of $\mathbb{R}^3$ the symbol $| \cdot |$ stands for the maximum norm. By $B(\mathbb{T})$ we denote the Banach space of all triples of bounded and continuous functions with the supremum norm defined on time scale $\mathbb{T}$, i.e.,

\[
B(\mathbb{T}) = \left\{ X: X \in C([T_0, \infty)_\mathbb{T}, \mathbb{R}^3), \text{ and } \|X\| = \sup_{t \in \mathbb{T}} |X(t)| < \infty \right\}. \tag{2}
\]

Let $\Omega$ be a subset of $B(\mathbb{T})$. 

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**Definition 5.** \( \Omega \) is called uniformly Cauchy if for every \( \varepsilon > 0 \) exists \( T_1 \in [T_0, \infty)_T \) such that for any \( X \in \Omega \)

\[
|X(t_1) - X(t_2)| < \varepsilon \quad \text{for all} \quad t_1, t_2 \in [T_1, \infty)_T.
\]

**Definition 6.** \( \Omega \) is called equi-continuous on \( [a, b)_T \) if for every \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that for any \( X \in \Omega \) and \( t_1, t_2 \in [a, b)_T \) with \( |t_1 - t_2| < \delta \),

\[
|X(t_1) - X(t_2)| < \varepsilon.
\]

The analogue of the Arzelá-Ascoli theorem on time scales was proved by Zhu and Wang in \([12]\). We need the generalization of this theorem to the three-dimensional case.

**Lemma 1.** Assume that \( \Omega \subset B(\mathbb{T}) \) is bounded and uniformly Cauchy. Moreover, assume that \( \Omega \) is equi-continuous on \([T_0, T_1]_T \) for any \( T_1 \in [T_0, \infty)_T \). Then \( \Omega \) is relatively compact.

**Proof.** Since \( \Omega \) is uniformly Cauchy we have that for any \( \varepsilon > 0 \) there exists \( T_1 \in [T_0, \infty)_T \) such that for any \( X \in \Omega \)

\[
|X(t_1) - X(t_2)| < \frac{\varepsilon}{3}, \quad t_1, t_2 \in [T_1, \infty)_T.
\] (3)

By the assumption of boundness there exists \( \alpha > 0 \) such that \( \|X\| \leq \alpha \) for every \( X \in \Omega \). We can choose the sequence of \( N_1 + 1 \) real numbers \( \beta_0, \beta_1, \ldots, \beta_{N_1} \) such that \(-\alpha = \beta_0 < \beta_1 < \cdots < \beta_{N_1} = \alpha \) and

\[
|\beta_{i+1} - \beta_i| < \frac{\varepsilon}{3}, \quad i = 0, 1, \ldots, N_1 - 1.
\] (4)

The equi-continuity of \( \Omega \) on \([T_0, T_1]_T \) implies that for the chosen \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for any \( X \in \Omega \)

\[
|X(t) - X(s)| < \frac{\varepsilon}{3} \quad \text{if} \quad |t - s| \leq \delta, \quad s, t \in [T_0, T_1]_T.
\] (5)

Obviously, we can choose \( N_2 \) real numbers from the interval \([T_0, T_1]\) so that \( T_0 = t_1 < t_2 < \cdots < t_{N_2} = T_1 \) and

\[
|t_{i+1} - t_i| \leq \delta, \quad i = 1, 2, \ldots, N_2 - 1.
\] (6)

Now, we construct a continuous mapping class \( \mathcal{U} \subset C([T_0, \infty)_T, \mathbb{R}^3) \). For each \( k \in \{1, 2, 3\}, i \in \{1, 2, \ldots, N_2 - 1\} \) and \( j \in \{1, 2, \ldots, N_1 - 1\} \) we define a function \( u_{i,j}^{(k)} \) on \([t_i, t_{i+1}]\) as follows

\[
u_{i,j}^{(k)}(t) = \beta_j + \frac{\beta_{j+1} - \beta_j}{t_{i+1} - t_i} (t - t_i), \quad t \in [t_i, t_{i+1}]
\]
or

\[
u_{i,j}^{(k)}(t) = \beta_{j+1} + \frac{\beta_j - \beta_{j+1}}{t_{i+1} - t_i} (t - t_i), \quad t \in [t_i, t_{i+1}].
\]
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Observe that function \( u_{ij}^{(k)} \) connects with the line points \((t_i, \beta_j)\) and \((t_{i+1}, \beta_{j+1})\) or \((t_i, \beta_{j+1})\) and \((t_{i+1}, \beta_j)\) being opposite vertices of the rectangle domain: \(t_i \leq t \leq t_{i+1}\) and \(\beta_j \leq \beta \leq \beta_{j+1}\). Let \( \mathcal{U}_k \) be the set of all continuous functions \( u^{(k)} \) on \([T_0, T_1]\) connecting \(u_{ij}^{(k)}\) as above from \([t_1, t_2]\) to \([t_{N_2-1}, t_{N_2}]\).

Note that each \( \mathcal{U}_k \) is a finite set for any fixed numbers \( N_1 \) and \( N_2 \). Every function \( u^{(k)} \in \mathcal{U}_k \) we extend to the function \( \bar{u}^{(k)} \) defined on the whole \([T_0, \infty)_T\) in the following way

\[
\bar{u}^{(k)}(t) = \begin{cases} u^{(k)}(t), & t \in [T_0, T_1)_T, \\ u^{(k)}(T_1), & t \in [T_1, \infty)_T. \end{cases}
\]

Let \( \mathcal{U} \) be the set of all possible triples \( U(t) = [\bar{u}^{(1)}(t), \bar{u}^{(2)}(t), \bar{u}^{(3)}(t)] \). Clearly, \( \mathcal{U} \) is finite since \( \mathcal{U}_k \) is a finite set for \( k = 1, 2, 3 \). We will show that \( \mathcal{U} \) is a finite \( \varepsilon \)-net for \( \Omega \). Since inequalities (4) and (5) and the definition of functions \( \bar{u}^{(k)} \) for any \( X = [x, y, z]^T \in \Omega \) we can choose \( U = [\bar{u}^{(1)}, \bar{u}^{(2)}, \bar{u}^{(3)}]^T \in \mathcal{U} \) such that

\[
|\bar{u}^{(1)}(t) - x(t)| < \frac{\varepsilon}{3}, \quad |\bar{u}^{(2)}(t) - y(t)| < \frac{\varepsilon}{3}, \quad |\bar{u}^{(3)}(t) - z(t)| < \frac{\varepsilon}{3}
\]

for any \( t \in [T_0, T_1)_T \), so on the interval \([T_0, T_1)_T\) we have

\[
|U(t) - X(t)| < \frac{\varepsilon}{3}. \tag{7}
\]

In case when \( t \in [T_1, \infty)_T \), from (3) and (7), we obtain

\[
|\bar{u}^{(1)}(t) - x(t)| = |u^{(1)}(T_1) - x(t)| \leq |x(T_1) - x(t)| + |u^{(1)}(T_1) - x(T_1)| < \frac{2\varepsilon}{3},
\]

and the same arguments give us

\[
|\bar{u}^{(2)}(t) - y(t)| < \frac{2\varepsilon}{3} \quad \text{and} \quad |\bar{u}^{(3)}(t) - z(t)| < \frac{2\varepsilon}{3}.
\]

This means that for \( t \in [T_1, \infty)_T \) we have

\[
|U(t) - X(t)| < \frac{2\varepsilon}{3}. \tag{9}
\]

Finally, since (8) and (9) we conclude that

\[
\|U - X\| = \sup_{t \in [T_0, \infty)_T} |U(t) - X(t)| \leq \frac{2\varepsilon}{3}.
\]

Thus \( \mathcal{U} \) is a finite \( \varepsilon \)-net for \( \Omega \) and this completes the proof of relative compactness of \( \Omega \).}

We also recall Krasnoselskii’s fixed point theorem which will be crucial to establish the existence of nonoscillatory solutions for (11).

**Theorem 1** ([5]). Let \( B \) be a Banach space, let \( \Omega \) be a bounded, convex and closed subset of \( B \) and let \( F, G \) be maps of \( \Omega \) into \( B \) such that

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(i) $FX + GY \in \Omega$ for any $X, Y \in \Omega$,
(ii) $F$ is a contraction,
(iii) $G$ is completely continuous.

Then the equation $FX + GX = X$ has a solution in $\Omega$.

3. Main results

We will assume in (1) that

(A1) $f, g, h \in C(\mathbb{R}, \mathbb{R})$,
(A2) $a, b, c \in C_{rd}(T, \mathbb{R})$ and
$$
\int_{T_1}^\infty |a(s)|\Delta s < \infty, \quad \int_{T_1}^\infty |b(s)|\Delta s < \infty, \quad \text{and} \quad \int_{T_1}^\infty |c(s)|\Delta s < \infty,
$$
(A3) $u_i \in C_{rd}(T, T)$ and $\lim_{t \to \infty} u_i(t) = \infty$ for $i = 1, 2, 3, 4$,
(A4) $p: T \to \mathbb{R}$ is delta differentiable.

Using Krasnoselskii’s fixed point theorem we will prove the following

**Theorem 2.** Assume that conditions (A1)–(A4) hold. If there exists a positive real number $c_p$ such that

(A5) $0 < p(t) \leq c_p < 1$ for any $t \in T$,

then system (1) has a positive bounded solution.

**Proof.** For the fixed positive real number $r$ we define set

$$
\Omega = \{X \in B(T): x(t), y(t), z(t) \in I, \ t \in T\},
$$

where $I = \left[\frac{1}{3}(1 - c_p)r, r\right]$. $\Omega$ is bounded closed convex subset of the Banach space $B(T)$. Since condition (A1) is satisfied, we can set

$$
M = \max \{|f(x)|, |g(x)|, |h(x)|: x \in I\}.
$$

From (A2), there exists $T_1 \in T$ such that

$$
\int_{T_1}^\infty |a(s)|\Delta s \leq \frac{(1 - c_p)r}{3M}, \quad \int_{T_1}^\infty |b(s)|\Delta s \leq \frac{(1 - c_p)r}{3M}, \quad \int_{T_1}^\infty |c(s)|\Delta s \leq \frac{(1 - c_p)r}{3M}.
$$

Next, we define the maps $F, G: \Omega \to B(T)$ where

$$
F = \begin{bmatrix}
F_1 \\
F_2 \\
F_3
\end{bmatrix}, \quad G = \begin{bmatrix}
G_1 \\
G_2 \\
G_3
\end{bmatrix},
$$

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\[
(FX)(t) = \begin{cases} 
-p(t)x(u_1(t)) + \frac{(2 + c_p)r}{3} & \text{for } t \geq T_1, \\
\frac{2(1 - c_p)r}{3} & \text{for } T_0 \leq t < T_1,
\end{cases}
\]

\[ (FX)(t) = (FX)(T_1) \quad \text{for } T_0 \leq t < T_1, \]

\[
(GX)(t) = \begin{cases} 
- \int_t^\infty a(s) f(y(u_2(s))) \Delta s \\
- \int_t^\infty b(s) g(z(u_3(s))) \Delta s \\
- \int_t^\infty c(s) h(x(u_4(s))) \Delta s
\end{cases} \quad \text{for } t \geq T_1 \quad (10)
\]

\[
(GX)(t) = (GX)(T_1) \quad \text{for } T_0 \leq t < T_1.
\]

We will show that \( F \) and \( G \) satisfy the conditions of the Theorem 1. First we show that for any \( X, \bar{X} \in \Omega \) we have that \( FX + G\bar{X} \in \Omega \). For \( t \geq T_1 \) we get the following upper and lower estimations

\[
(F_1X)(t) + (G_1\bar{X})(t) = -p(t)x(u_1(t)) + \frac{(2 + c_p)r}{3} - \int_t^\infty a(s) f(\bar{y}(u_2(s))) \Delta s
\]

\[
\leq \frac{(2 + c_p)r}{3} + \int_t^\infty |a(s)| \left| f(\bar{y}(u_2(s))) \right| \Delta s
\]

\[
\leq \frac{(2 + c_p)r}{3} + M \int_t^\infty |a(s)| \Delta s
\]

\[
\leq \frac{2}{3} r + \frac{1}{3} c_p r + M \cdot \frac{(1 - c_p)r}{3M} = r,
\]

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\[(F_1X)(t) + (G_1\bar{X})(t) = -p(t)x(u_1(t)) + \frac{(2 + c_p)r}{3} - \int_{t}^{\infty} a(s) f(\bar{y}(u_2(s))) \Delta s \geq \frac{(2 + c_p)r}{3} - \int_{t}^{\infty} |a(s)| f(\bar{y}(u_2(s))) | \Delta s - p(t)x(u_1(t)) \]

\[\geq \frac{2}{3}r + \frac{1}{3}c_p r - M \cdot \frac{(1 - c_p)r}{3M} - c_p r \]

\[= \frac{2}{3}r + \frac{1}{3}c_p r - \frac{1}{3}r + \frac{1}{3}c_p r - c_p r \]

\[= \frac{1}{3}(1 - c_p)r.\]

Therefore \((F_1X)(t) + (G_1\bar{X})(t) \in I\) for all \(t \in \mathbb{T}\) and any \(X, \bar{X} \in \Omega\).

Below we present reasoning for maps \(F_2\) and \(G_2\), but the same conclusions can be drawn for \(F_3\) and \(G_3\).

\[(F_2X)(t) + (G_2\bar{X})(t) = \frac{2(1 - c_p)r}{3} - \int_{t}^{\infty} b(s) g(\bar{z}(u_3(s))) \Delta s \leq \frac{2(1 - c_p)r}{3} + \int_{t}^{\infty} |b(s)| | g(\bar{z}(u_3(s))) | \Delta s \leq \frac{2}{3}r - \frac{2}{3}c_p r + M \cdot \frac{(1 - c_p)r}{3M} = (1 - c_p)r \leq r,\]

\[(F_2X)(n) + (T_2\bar{X})(n) = \frac{2(1 - c_p)r}{3} - \int_{t}^{\infty} b(s) g(\bar{z}(u_3(s))) \Delta s \geq \frac{2(1 - c_p)r}{3} - \int_{t}^{\infty} |b(s)| | g(\bar{z}(u_3(s))) | \Delta s \geq \frac{2}{3}r - \frac{2}{3}c_p r - M \cdot \frac{(1 - c_p)r}{3M} = \frac{1}{3}(1 - c_p)r.\]

Hence for any \(X, \bar{X} \in \Omega\) we have that \(FX + G\bar{X} \in \Omega\).
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The next step is to prove that $F$ is a contraction mapping. It is easy to see that
\[
|(F_1 X)(t) - (F_1 \bar{X})(t)| \leq p(t) \left| x(u_1(t)) - \bar{x}(u_1(t)) \right|
\leq c_p \left| x(u_1(t)) - \bar{x}(u_1(t)) \right|
\leq c_p \sup_{t \in \mathbb{T}} \left| x(u_1(t)) - \bar{x}(u_1(t)) \right|
\leq c_p \sup_{t \in \mathbb{T}} \left| x(t) - \bar{x}(t) \right|,
\]
\[
|(F_2 X)(t) - (F_2 \bar{X})(t)| = 0,
\]
\[
|(F_3 X)(t) - (F_3 \bar{X})(t)| = 0, \quad \text{for } X, \bar{X} \in \Omega \text{ and } t \geq T_1.
\]
Hence
\[
\|F X - F \bar{X}\| \leq c_p \|X - \bar{X}\|,
\]
where, by (A5), there is $0 < c_p < 1$.

It remains to show that $G$ is a completely continuous mapping. We start it showing that $G$ is continuous. Consider sequence $X_n = [x_n, y_n, z_n]^T \in \Omega$ for any $n \in \mathbb{N}$ such that $\|X_n - X\| \to 0$ as $n \to \infty$, then $X \in \Omega$ and for any $t \in \mathbb{T}$ we have that $|x_n(t) - x(t)| \to 0$, $|y_n(t) - y(t)| \to 0$, $|z_n(t) - z(t)| \to 0$ as $n \to \infty$. Because $f$ is continuous, then for any $t \in \mathbb{T}$ we have
\[
|a(t)| \left| f\left(y_n(u_2(t))\right) - f\left(y(u_2(t))\right) \right| \to 0 \quad \text{as } n \to \infty. \quad (11)
\]
On the other hand, for $y_n(t), y(t) \in I$ we get that
\[
|a(t)| \left| f\left(y_n(u_2(t))\right) - f\left(y(u_2(t))\right) \right| \leq 2M|a(t)|. \quad (12)
\]
From (10) we obtain
\[
|(G_1 X_n)(t) - (G_1 X)(t)| \leq \int_t^\infty |a(s)| \left| f\left(y_n(u_2(s))\right) - f\left(y(u_2(s))\right) \right| \Delta s
\]
for any $t \geq T_1$, and
\[
|(G_1 X_n)(t) - (G_1 X)(t)| \leq \int_{T_1}^\infty |a(s)| \left| f\left(y_n(u_2(s))\right) - f\left(y(u_2(s))\right) \right| \Delta s
\]
for $T_0 \leq t < T_1$. Therefore, we can conclude that
\[
\sup_{t \in \mathbb{T}} \left| (G_1 X_n)(t) - (G_1 X)(t) \right| \leq \int_{T_1}^\infty |a(s)| \left| f\left(y_n(u_2(s))\right) - f\left(y(u_2(s))\right) \right| \Delta s. \quad (13)
\]
Since (11), (12) and (13) applying the Lebesgue dominated convergence theorem on time scales for the integral on time scales [4], [6] we obtain
\[
\sup_{t \in \mathbb{T}} \left| (G_1 X_n)(t) - (G_1 X)(t) \right| \to 0 \quad \text{if } n \to \infty.
\]
Analogously, if \( n \to \infty \), then
\[
\sup_{t \in \mathbb{T}} |(G_2 X_n)(t) - (G_2 X)(t)| \to 0 \quad \text{and} \quad \sup_{t \in \mathbb{T}} |(G_3 X_n)(t) - (G_3 X)(t)| \to 0.
\]

Hence,
\[
\|G X_n - G X\| \to 0 \quad \text{if} \quad n \to \infty.
\]

It means that \( G \) is a continuous mapping on \( \Omega \).

To prove that \( G \Omega \) is relatively compact it is sufficient to verify that \( G \Omega \) satisfies all conditions in the Lemma 1. Obviously \( G \Omega \) is a bounded set. Now we will show that it is uniformly Cauchy. Let \( X \in \Omega \). Observe that for any given \( \varepsilon > 0 \) by assumptions (A1) and (A2) there exists \( T_2 > T_1 \) such that for all \( t \geq T_2 \) the following inequality holds
\[
\int_{t_1}^{\infty} |a(s)| \left| f \left( y(u_2(s)) \right) \Delta s < \frac{\varepsilon}{2}. \]

Hence by definition of \( G \),
\[
|(G_1 X)(t_1) - (G_1 X)(t_2)| = \left| \int_{t_1}^{t_2} a(s) f \left( y(u_2(s)) \right) \Delta s \right| < \varepsilon
\]
for arbitrary \( t_1, t_2 \in [T_2, \infty)_{\mathbb{T}} \). Since similar arguments can be apply to \( G_2 \) and \( G_3 \) we conclude that \( G \Omega \) is uniformly Cauchy.

Finally, it remains to prove the equi-continuity on \([T_0, T_2]_{\mathbb{T}} \) for any \( T_2 \in [T_0, \infty)_{\mathbb{T}} \). Observe that for any \( X \in \Omega \) and \( t_1, t_2 \in [T_0, T_1]_{\mathbb{T}} \)
\[
|(G X)(t_1) - (G X)(t_2)| \equiv 0,
\]
that is why we can assume that \( T_2 > T_1 \). Taking now \( t_1, t_2 \in [T_1, T_2]_{\mathbb{T}} \) we obtain the following estimation
\[
|(G_1 X)(t_1) - (G_1 X)(t_2)| = \left| \int_{t_1}^{t_2} a(s) f \left( y(u_2(s)) \right) \Delta s \right|
\]
\[
\leq M \left| \int_{t_1}^{t_2} a(s) \Delta s \right|.
\]

Hence, for any \( \varepsilon > 0 \), there exists
\[
\delta_1 = \frac{\varepsilon}{M \cdot \max_{t \in [T_1, T_2]_{\mathbb{T}}} |a(t)|}
\]
such that, when \( t_1, t_2 \in [T_1, T_2]_{\mathbb{T}} \) and \(|t_1 - t_2| < \delta_1 \), we get that
\[
|(G_1 X)(t_1) - (G_1 X)(t_2)| < \varepsilon.
\]
Since
\[
|(GX)(t_1) - (GX)(t_2)| \leq \max \left\{ M \left| \int_{t_1}^{t_2} a(s) \Delta s \right|, M \left| \int_{t_1}^{t_2} b(s) \Delta s \right|, M \left| \int_{t_1}^{t_2} c(s) \Delta s \right| \right\},
\]
we obtain the equi-continuity of \( G \Omega \) with \( \delta = \min\{\delta_1, \delta_2, \delta_3\} \), where \( \delta_2 = \frac{\varepsilon}{M \cdot \max_{\varepsilon \in [T_1, T_2] \cap \mathbb{T}} |b(t)|} \) and \( \delta_3 = \frac{\varepsilon}{M \cdot \max_{\varepsilon \in [T_1, T_2] \cap \mathbb{T}} |c(t)|} \). Lemma 1 implies that \( G \Omega \) is relatively Cauchy. From the above we obtain that \( G \) is a completely continuous mapping.

By the Theorem 1 there exists \( X^* \) such that \( FX^* + GX^* = X^* \). We will verify that \( X^*(t) \) satisfies system (1) for \( t \geq T_1 \). Since \( (F_1X^*)(t) + (G_1X^*)(t) = x^*(t) \) we have
\[
-p(t)x^*(u_1(t)) + \left( \frac{2 + c_p}{3} \right) - \int_{t}^{\infty} a(s)f\left(y^*(u_2(s))\right) \Delta s = x^*(t). \tag{14}
\]
After moving the term \(-p(t)x^*(u_1(t))\) to the right-hand side of the equation and then applying to its both sides delta differentiation we get
\[
\left( x^*(t) + p(t)x^*(u_1(t)) \right)^\Delta = \left[ -\int_{t}^{\infty} a(s)f\left(y^*(u_2(s))\right) \Delta s \right]^\Delta.
\]
Thus
\[
\left( x^*(t) + p(t)x^*(u_1(t)) \right)^\Delta = a(t)f\left(y^*(u_2(t))\right), \tag{15}
\]
since \( a(t)f(y^*(u_2(t))) \), by assumption (A3), as rd-continuous function has its antiderivative.

Let us notice that the Theorem 1 guaranties the equality (14) and by the assumptions (A1)–(A5) we arrive to (15). Hence, we see that \( x^*(t) \) is not only rd-continuous but, moreover, rd-continuously delta differentiable. Similarly, from equation \( (F_2X^*)(t) + (G_2X^*)(t) = y^*(t) \) we get that
\[
(y^*)^\Delta(t) = \left[ \int_{t}^{\infty} b(s)g\left(z^*(u_3(s))\right) \Delta s \right]^\Delta.
\]
Again, by assumption (A3), we know that \( b(t)g(z^*(u_3(t))) \) is rd-continuous function and in consequence
\[
(y^*)^\Delta(t) = b(t)g\left(z^*(u_3(t))\right).
\]
In the same manner we verify that \( (F_3X^*)(t) + (G_3X^*)(t) = z^*(t) \) implies the third equation of (11). Hence \( X^* \) is the solution of system (11). The proof is complete.
Finally, the above theorem is illustrated with examples in which four different time scales are employed.

**Example 1.** Let \( T = [5, \infty) \). Consider the following system

\[
\begin{align*}
(x(t) + \frac{1}{t^2} x(t - 1))^{\Delta} &= a(t) y(t - 2), \\
y^{\Delta}(t) &= b(t) (z(t - 1))^2, \\
z^{\Delta}(t) &= c(t) x(t)
\end{align*}
\]

with

\[
\begin{align*}
a(t) &= -\frac{(4t^2 - 6t + 3)(t - 2)}{2t^2(t - 1)^2(3t - 5)}, \\
b(t) &= -\frac{(t - 1)^4}{t^2(3t^2 - 6t + 4)^2}, \\
c(t) &= -\frac{2}{t^2(2t + 1)}.
\end{align*}
\]

It easy to check that all the conditions (A1)–(A5) are satisfied. Here \( c_p = \frac{1}{10} \). One of the bounded solutions of the above system is

\[
X(t) = \left[ 2 + \frac{1}{t}, 3 + \frac{1}{t}, 3 + \frac{1}{t^2} \right]^T.
\]

**Example 2.** Let \( T = \{ n : n \geq 3, n \in \mathbb{N} \} \). Consider the following system

\[
\begin{align*}
(x(t) + \frac{1}{t^2} x(t - 2))^{\Delta} &= a(t) [(y(t))^2 + 2], \\
y^{\Delta}(t) &= b(t) (z(t - 1))^3, \\
z^{\Delta}(t) &= c(t) (x(t - 2))^2
\end{align*}
\]

with

\[
\begin{align*}
a(t) &= -\frac{(7t^2 - 17t + 12)t}{3(t + 1)(t - 1)(t - 2)(6t^2 + 4t + 1)}, \\
b(t) &= -\frac{(t - 1)^6}{t^4(t + 1)(t - 2)^3}, \\
c(t) &= \frac{(2t + 1)(t - 2)^2}{t^4(t + 1)^2}.
\end{align*}
\]

Again it is easy to see that all the conditions (A1)–(A4) are satisfied. Also the condition (A5) of the Theorem is satisfied with \( c_p = \frac{1}{5} \). One of the bounded solutions of the above system is

\[
X(t) = \left[ 1 + \frac{2}{t}, 2 + \frac{1}{t}, 1 - \frac{1}{t^2} \right]^T.
\]
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Example 3. Let $\mathbb{T} = \{2^n : n \in \mathbb{N}_0\}$, where $\mathbb{N}_0$ is the set of nonnegative integers. Consider the following system

$$\begin{cases}
(x(t) + \frac{1}{2} x(\rho(t)))^\Delta = a(t) y(\rho(t)), \\
y^\Delta(t) = b(t) z(t), \\
z^\Delta(t) = c(t) (x(t))^3
\end{cases}$$

with

$$\begin{align*}
\rho(t) &= \frac{t}{2}, \\
a(t) &= \frac{1}{2t(t+1)}, \\
b(t) &= \frac{1}{2t(t+1)}, \\
c(t) &= \frac{t}{2(t+1)^3}.
\end{align*}$$

One can verify that all the conditions (A1)–(A5) are satisfied (condition (A5) with constant $c_p = \frac{1}{2}$). One of the bounded solutions of this system is

$$X(t) = \left[1 + \frac{1}{t}, 2 + \frac{1}{t}, 1 + \frac{1}{t}\right]^T.$$

Example 4. Let $\mathbb{T} = P_{1,1} = \bigcup_{k=0}^\infty [2k, 2k+1]$. Consider the following system

$$\begin{cases}
(x(t) + \frac{1}{3} x(t-2))^\Delta = a(t) y(t-2), \\
y^\Delta(t) = b(t) (z(t-2))^2, \\
z^\Delta(t) = c(t) (x(t))^2
\end{cases}$$

with

$$\begin{align*}
a(t) &= -\frac{4(t^2 + 7t + 13)}{3(2t + 7)(t + 3)(t + 5)^2}, \\
b(t) &= -\frac{(t + 1)^2}{(t + 5)^2(2t + 3)^2}, \\
c(t) &= -\frac{(t + 5)^2}{(t + 3)^2(4t + 21)^2},
\end{align*}$$

for $t \in \bigcup_{k=0}^\infty [2k, 2k+1]$ and
\[
\begin{align*}
  a(t) &= -\frac{2(2t^2 + 16t + 33)}{3(2t + 7)(t + 6)(t + 5)(t + 4)}, \\
  b(t) &= -\frac{(t + 1)^2}{(t + 5)(t + 6)(2t + 3)^2}, \\
  c(t) &= -\frac{(t + 5)^2}{(t + 3)(t + 4)(4t + 21)^2}
\end{align*}
\]

for \( t \in \bigcup_{k=0}^{\infty} \{2k + 1\} \). One of the bounded solutions of this system is

\[
  X(t) = \begin{bmatrix}
  4 + \frac{1}{t + 5}, 2 + \frac{1}{t + 5}, 2 + \frac{1}{t + 3}
  \end{bmatrix}^T.
\]

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