CORRIGENDUM TO A CATEGORY ANALOGUE OF THE GENERALIZATION OF LEBESGUE DENSITY TOPOLOGY

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ABSTRACT. The notion of $\mathcal{A}_I$ -density point introduced in Wojdowski, W. *A topology stronger than the Lebesgue density topology*, in: Real Functions, Density Topology and Related Topics, Łódź Univ. Press, 2011, pp. 73–80 [WO1]. leads to the operator $\Phi_{\mathcal{A}_I}(A)$ which is not a lower density operator. We present a counterexample giving a corrected definition which should be used in [WO1] to keep all results valid.

In [WO] we introduced a notion of an $\mathcal{A}_I$ -density point of a set with the Baire property in the following way. Let S be the σ -algebra of sets having the Baire property on the real line $\mathbb{R}$ and $I \subset S$ the σ -ideal of sets of first category.

Let $\mathcal{A}_I$ be the family of subsets of interval $[-1, 1]$ that are from S and have 0 as its I -density point.

DEFINITION 1. We shall say that $x \in \mathbb{R}$ is an $\mathcal{A}_I$ -density point of $A \in S$, if for any sequence of real numbers $\{t_n\}_{n \in \mathbb{N}}$, decreasing to zero, there is a subsequence $\{t_{n_m}\}_{m \in \mathbb{N}}$ and a set $B \in \mathcal{A}_I$ such that the sequence $\{\chi_{\frac{1}{t_{n_m}} \cdot (A-x) \cap [-1, 1]}\}_{m \in \mathbb{N}}$ of characteristic functions converges I -almost everywhere on $[-1, 1]$ to χ_B .

In contrast to what was incorrectly claimed in [WO], the density operator $\Phi_{\mathcal{A}_I}(A)$ defined as a set of all $\mathcal{A}_I$ -density points of A is not monotonic and thus, it is not a lower density. We shall present a counterexample and show how to modify the definition of an $\mathcal{A}_I$ -density point so that the operator $\Phi_{\mathcal{A}_I}(A)$ is a lower density.

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A counterexample

Let $D = (0, \frac{1}{2})$. Then, D is an open set such that $(0, 1) \setminus D \in S \setminus I$. Let $\{c_n\}_{n \in \mathbb{N}}$ be an arbitrary sequence of real numbers decreasing to 0, such that $c_1 < 1$, $\frac{c_{n+1}}{c_n} < \frac{1}{2}$ and $\lim_{n \rightarrow \infty} \frac{c_{n+1}}{c_n} = 0$. We define a measurable set U as

$$U = \bigcup_{n=1}^{\infty} [(c_n \cdot D) \cap (c_{n+1}, c_n)].$$

Let $A = -U \cup U$.

By Proposition 3 of [WO], 0 is an $\mathcal{A}_I$ -density point of A according to Definition 1. In [WO] it is also shown that 0 fails to be an I -density point of A .

Now, let

$$D_1 = \left[0, \frac{1}{2}\right) \cup \left(\frac{3}{4}, \frac{4}{4}\right),$$

$$D_2 = \left[0, \frac{1}{2}\right) \cup \left(\frac{5}{8}, \frac{6}{8}\right) \cup \left(\frac{7}{8}, \frac{8}{8}\right),$$

and consecutively,

$$D_n = \left[0, \frac{1}{2}\right) \cup \left(\frac{1}{2} + \bigcup_{k=1}^{2^{n-1}} \left(\frac{2k-1}{2^{n+1}}, \frac{2k}{2^{n+1}}\right)\right).$$

Let $\{c_n\}_{n \in \mathbb{N}}$ be defined as above. We define a set $E \in S$ as

$$E = (-\infty, 0) \cup \bigcup_{n=1}^{\infty} [(c_n \cdot D_n) \cap (c_{n+1}, c_n)].$$

Clearly, E is a superset of A .

We shall show now that 0 is not an $\mathcal{A}_I$ -density point of E : Consider the sequence $\{c_n\}_{n \in \mathbb{N}}$. Suppose that for some $B \in \mathcal{A}_I$, and for some subsequence $\{c_{n_m}\}$,

$$\chi\left(\left(\frac{1}{c_{n_m}}E\right) \cap \left[\frac{1}{2}, 1\right]\right) \xrightarrow{a.e.} \chi_B \quad \text{on} \quad \left[\frac{1}{2}, 1\right].$$

We shall consider two cases:

a) The set $B \cap [\frac{1}{2}, 1] \in S \setminus I$.

By definition, for every $n \in \mathbb{N}$, the set D_n is periodic within interval $(\frac{1}{2}, 1)$ with period $\frac{1}{2^n}$. We have

$$\frac{1}{c_{n_m}}E \cap \left(\frac{1}{2}, 1\right) = D_{n_m} \cap \left(\frac{1}{2}, 1\right) \quad \text{for every } m \in \mathbb{N}.$$

Thus, for every $\delta > 0$, there is an $m_0 \in \mathbb{N}$ such that $\frac{1}{c_{n_m}}E \cap (\frac{1}{2}, 1)$ is periodic with period $\frac{1}{2^{n_m}} < \delta$, for every $m > m_0$. Thus, B has to be periodic within interval $(\frac{1}{2}, 1)$ with period $\frac{1}{2^n}$, $n \in \mathbb{N}$. This implies that

the set of $\mathcal{A}_I$ -density points of B is dense in $(\frac{1}{2}, 1)$ and thus B is residual in $(\frac{1}{2}, 1)$. Since

$$-\left(D_{n_m} \cap \left(\frac{1}{2}, 1\right)\right) = \left(D_{n_m}^c \cap \left(\frac{1}{2}, 1\right)\right) - \frac{3}{2},$$

we may repeat the above argument replacing sets D_n on $(\frac{1}{2}, 1)$ with their complements and we similarly obtain that the complement of B is residual in $(\frac{1}{2}, 1)$, a contradiction.

b) The set $B \cap [0, 1] \in I$.

Similarly as in a), we may show that also $B^c \cap [0, 1] \in I$ a contradiction. Apparently, 0 cannot be an $\mathcal{A}_I$ -density point of E in the sense of $\mathcal{A}_I$ -density point as defined in [WO].

Finally, we have $A \subset E$ but $0 \in \Phi_{\mathcal{A}_I}(A) \setminus \Phi_{\mathcal{A}_I}(E)$, i.e., $\Phi_{\mathcal{A}_I}(E)$ is not monotonic. In particular, part (4) of Theorem 1 in [WO] is false.

A new definition

Following the ideas from [WO1], we replace the Definition 1 in [WO] with

DEFINITION 2. We shall say that $x \in \mathbb{R}$ is an $\mathcal{A}_I$ -density point of $A \in S$, if for any sequence of real numbers $\{t_n\}_{n \in \mathbb{N}}$, decreasing to zero, there is a subsequence $\{t_{n_m}\}_{m \in \mathbb{N}}$ and a set $B \in \mathcal{A}_I$ such that the sequence $\left\{\chi_{\frac{1}{t_{n_m}} \cdot (A-x) \cap [-1, 1]}\right\}_{m \in \mathbb{N}}$ of characteristic functions converges I -almost everywhere on B to 1.

The part (4) of Theorem 1 in [WO] can be now proved as follows

THEOREM 1. *The mapping $\Phi_{\mathcal{A}_I} : S \rightarrow 2^{\mathbb{R}}$ has the following properties:*

- (0) For each $A \in S$, $\Phi_{\mathcal{A}_I}(A) \in S$.
- (1) For each $A \in S$, $A \sim \Phi_{\mathcal{A}_I}(A)$.
- (2) For each $A, B \in S$, if $A \sim B$ then $\Phi_{\mathcal{A}_I}(A) = \Phi_{\mathcal{A}_I}(B)$.
- (3) $\Phi_{\mathcal{A}_I}(\emptyset) = \emptyset$, $\Phi_{\mathcal{A}_I}(\mathbb{R}) = \mathbb{R}$.
- (4) For each $A, B \in S$, $\Phi_{\mathcal{A}_I}(A \cap B) = \Phi_{\mathcal{A}_I}(A) \cap \Phi_{\mathcal{A}_I}(B)$.

Proof. (4) Observe first that if $A \subset B$, $A, B \in S$, then $\Phi_{\mathcal{A}_I}(A) \subset \Phi_{\mathcal{A}_I}(B)$, so

$$\Phi_{\mathcal{A}_I}(A \cap B) \subset \Phi_{\mathcal{A}_I}(A) \cap \Phi_{\mathcal{A}_I}(B).$$

To prove the opposite inclusion, assume $x \in \Phi_{\mathcal{A}_I}(A) \cap \Phi_{\mathcal{A}_I}(B)$. Let $\{t_n\}_{n \in \mathbb{N}}$ be an arbitrary sequence of real numbers decreasing to zero. From $x \in \Phi_{\mathcal{A}_I}(A)$, by definition, there is its subsequence $\{t_{n_m}\}_{m \in \mathbb{N}}$ and a set $A_1 \in \mathcal{A}_I$ such that the sequence

$$\left\{\chi_{\frac{1}{t_{n_m}} \cdot (A-x) \cap [-1, 1]}\right\}_{m \in \mathbb{N}}$$

of characteristic functions converges I -almost everywhere on A_1 to 1. Similarly, for $\{t_{n_m}\}_{m \in N}$ from $x \in \Phi_{\mathcal{A}_I}(B)$, by definition, there is a subsequence $\{t_{n_{m_k}}\}_{k \in N}$ and a set $B_1 \in \mathcal{A}_I$ such that the sequence

$$\left\{ \chi_{\frac{1}{t_{n_{m_k}}}} \cdot (A-x) \cap [-1,1] \right\}_{k \in N}$$

of characteristic functions converges I -almost everywhere on B_1 to 1. It is clear that the sequence

$$\left\{ \chi_{\frac{1}{t_{n_{m_k}}}} \cdot ((A \cap B) - x) \cap [-1,1] \right\}_{k \in N}$$

converges I -almost everywhere on $A_1 \cap B_1$ to 1, i.e., x is a $\Phi_{\mathcal{A}_I}$ -density point of $A \cap B$. $\square$

With the Definition 2, all results of [WO] stay valid. Since we do not require any convergence of the sequence

$$\left\{ \chi_{\frac{1}{t_{n_m}}} \cdot (A-x) \cap [-1,1] \right\}_{m \in N}$$

on the set $[-1,1] \setminus B$, some proofs may be even shorter, for example, we may omit case $a < 1$ in part a) in proof of Proposition 3 in [WO].

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