ON SOME OPERATIONS ON GRAPH CONTINUOUS AND GRAPH QUASI-CONTINUOUS FUNCTIONS

ANETA MIKUCKA

ABSTRACT. The notion of graph continuity and graph quasi-continuity was studied by Z. Grande, K. Sakálová, Przemski and A. Mikucka. In this paper, we investigate some operations on graph continuous and graph quasi-continuous function.

Let X be a topological space, $(Y, \mathcal{P})$ —a uniform one, and the uniformity in Y be induced by a family of pseudometrics $\mathcal{P}$. In this paper, we denote the graph of a function $f : X \rightarrow (Y, \mathcal{P})$ by $G(f)$, and the symbol $\overline{G(f)}$ stands for the closure of $G(f)$. The letter $\mathcal{R}$ stands for the set of real numbers. A set $A \subset X$ is said to be :

- *semi-open* if $A \subset \overline{\text{Int } A}$, [7];
- *semi-closed*, if $X \setminus A$ is semi-open, [2];
- *semi-neighborhood* of x_0 if A is semi-open and $x_0 \in A$.

LEMMA 1 ([8]).

- (a) *Each nowhere dense set is semi-closed.*
- (b) *If a set W is semi-open and dense, and a set A is dense, then the set $W \cap A$ is dense.*

DEFINITION 1. A function $f : X \rightarrow Y$ is said to be :

- *quasi-continuous* at $x_0 \in X$ if, for each $\varrho \in \mathcal{P}$, $\varepsilon > 0$ and for each neighborhood U of x_0 , there exists a nonempty open set $U_1 \subset U$ such that $\varrho(f(x), f(x_0)) < \varepsilon$ for each $x \in U_1$, [10];
- *cliquish* at x_0 if, for each $\varrho \in \mathcal{P}$, $\varepsilon > 0$ and for each neighborhood $U \subset X$ of x_0 , there exists a nonempty open set $U_1 \subset U$ such that $\varrho(f(x_1), f(x_2)) < \varepsilon$ for each $x_1, x_2 \in U_1$, [5]. A function f is called *quasi-continuous (cliquish)* if it has this property at each point of X .

DEFINITION 2. A function $f: X \rightarrow Y$ is said to be:

- *graph continuous* if there exists a continuous function $g: X \rightarrow Y$ such that $G(g) \subset \overline{G(f)}$, [6];
- *graph quasi-continuous* if there exists a quasi-continuous function $g: X \rightarrow Y$ such that $G(g) \subset \overline{G(f)}$, [8];

We will use the following notations:

$$\begin{aligned} A(f, g, \varrho, \varepsilon) &= \left\{ x \in X; \varrho(f(x), g(x)) < \varepsilon \right\}; \\ B(f, g, \varrho, \varepsilon) &= \left\{ x \in X; \varrho(f(x), g(x)) \geq \varepsilon \right\}; \\ K(f(x), \varrho, \varepsilon) &= \left\{ g(x); \varrho(f(x), g(x)) < \varepsilon \right\} \quad \text{for } \varrho \in \mathcal{P} \text{ and } \varepsilon > 0. \end{aligned}$$

Now, we will give the characterization of the graph quasi-continuity.

THEOREM 1 ([9]). *Let X be a topological space, $(Y, \mathcal{P})$ —a uniform one, and $f, g: X \rightarrow Y$.*

If g is quasi-continuous, then the following conditions are equivalent:

- (a) $G(g) \subset \overline{G(f)}$;
- (b) *the set $A(f, g, \varrho, \varepsilon)$ is dense in X for each $\varrho \in \mathcal{P}$ and for each $\varepsilon > 0$.*

THEOREM 2 ([9]). *Let X be a topological space, $(Y, \mathcal{P})$ —a uniform one and $f, g: X \rightarrow Y$.*

If g is quasi-continuous, then the following conditions are equivalent:

- (a) $G(g) \subset \overline{G(f)}$ and f is cliquish;
- (b) *the set $B(f, g, \varrho, \varepsilon)$ is nowhere dense in X for each $\varrho \in \mathcal{P}$ and for each $\varepsilon > 0$.*

Z. Grande in [6] proved that each real function from $[0, 1]$ is the sum of two graph continuous functions. Taking into consideration that the graph continuity implies the graph quasi-continuity, for the graph quasi-continuity, we have the similar situations as it is shown in [8].

Now, we consider functions with values in a locally convex topological vector space.

LEMMA 2 ([1]). *In each locally convex topological space $(Y, \mathcal{T})$ there exists a family seminorm $\mathcal{S}$ with the property:*

- (a) *for all $p_1, p_2 \in \mathcal{S}$ there exists $p_3 \in \mathcal{S}$ such that $\max\{p_1, p_2\} \leq p_3$,*
- (b) *for all $p_1 \in \mathcal{S}$ there exists $p_2 \in \mathcal{S}$ with $p_1(x) \leq \frac{1}{2}p_2(x)$,*

and such that the family $\{x; p(x) \leq 1\}$ for $p \in \mathcal{S}$ is a neighbourhood base of the null. Moreover, the topology induced by the family $\mathcal{S}$ coincides with the topology $\mathcal{T}$.

For $p \in \mathcal{S}$, we consider the function $\varsigma_p: Y \times Y \rightarrow [0, \infty)$ defined by

$$\varsigma_p(x, y) = p(x - y) \quad \text{for } x, y \in Y.$$

The function ς_p is a pseudometric for all $p \in \mathcal{S}$. We will use the symbol $\mathcal{P}_{\mathcal{S}}$ to denote the family of pseudometrics ς_p . Let

$$V(\varsigma_p, \varepsilon) = \{(x, y); p(x - y) < \varepsilon\} \quad \text{for } p \in \mathcal{S}, \varepsilon > 0.$$

Then, the family

$$\{V(\varsigma_p, \varepsilon); p \in \mathcal{S}, \varepsilon > 0\}$$

is a base of a uniform structure on Y . This uniform space will be denoted by $(Y, \mathcal{P}_{\mathcal{S}})$. A topology induced by the family $\mathcal{P}_{\mathcal{S}}$ coincides with a given topology on Y . So, we can consider the locally convex topological vector space as a uniform space $(Y, \mathcal{P}_{\mathcal{S}})$.

THEOREM 3. *Let X be a topological space, $(Y, \mathcal{P}_{\mathcal{S}})$ —a locally convex topological vector space. If $f_1: X \rightarrow Y$ is graph quasi-continuous (graph continuous), $f_2: X \rightarrow Y$ is graph continuous, and if at least one of them is cliquish, then $f_1 + f_2: X \rightarrow Y$ is graph quasi-continuous (graph continuous).*

Proof. First, assume that the function f_1 is cliquish. Let us choose any seminorm $p \in \mathcal{S}$ and $\varepsilon > 0$. Let $U \subset X$ be any nonempty open set. Since f_1 is graph quasi-continuous (graph continuous), there exists a quasi-continuous (continuous) function g_1 such that $G(g_1) \subset \overline{G(f_1)}$. From Theorem 2, we have that the set $B(f_1, g_1, p, \varepsilon)$ is nowhere dense in X . In the consequence, by Lemma 1(a) and Theorem 1, the set $A(f_1, g_1, p, \varepsilon)$ is semi-open dense in X . From assumptions about the function f_2 , there exists a continuous function g_2 such that $G(g_2) \subset \overline{G(f_2)}$. Hence, by Theorem 1, the set $A(f_2, g_2, p, \varepsilon)$ is dense in X . Then, applying Lemma 1(b), we conclude that the set $A(f_1, g_1, p, \varepsilon) \cap A(f_2, g_2, p, \varepsilon)$ is dense in X . Moreover, $A(f_1, g_1, p, \varepsilon) \cap A(f_2, g_2, p, \varepsilon) \subset A(f_1 + f_2, g_1 + g_2, p, 2\varepsilon)$. Therefore, the set $A(f_1 + f_2, g_1 + g_2, p, 2\varepsilon)$ is dense in X . Since the function $g_1 + g_2$ is quasi-continuous (continuous), by Theorem 1, we have

$$G(g_1 + g_2) \subset \overline{G(f_1 + f_2)}.$$

It means that the function $f_1 + f_2$ is graph quasi-continuous (graph continuous). $\square$

If we take $Y = \mathcal{R}$ in the last theorem, then we get results obtained by Z. Grande in [6] and by A. Mikucka in [8].

THEOREM 4. *Let X be a topological space, $(Y, \mathcal{P}_{\mathcal{S}})$ —a space with F -norm. If a function $f_1: X \rightarrow Y$ is graph quasi-continuous (graph continuous), a function $f_2: X \rightarrow Y$ is graph continuous, and if at least one of them is cliquish, then $f_1 + f_2: X \rightarrow Y$ is graph quasi-continuous (graph continuous).*

The last theorem is true when we assume that Y is a space with F -norm (observe that the notion of uniformity and the F -norm are independent).

Now, let X be a topological space, $(Y, \mathcal{P}_S)$ —a locally convex topological vector space. For the functions $\phi : \mathcal{R} \rightarrow \mathcal{R}$ and $f : X \rightarrow Y$, we define the function $f_* = \phi f : \mathcal{R} \times X \rightarrow Y$ by

$$f_*(\lambda, x) = \phi(\lambda)f(x) \text{ for each } \lambda \in \mathcal{R}, \quad x \in X.$$

LEMMA 3. *Let X be a topological space, $(Y, \mathcal{P}_S)$ —a locally convex topological vector space. If $f : X \rightarrow Y$ is quasi-continuous (continuous) and $\phi : \mathcal{R} \rightarrow \mathcal{R}$ is continuous, then the function $f_* = \phi f$ is quasi-continuous (continuous).*

Proof. Let us assume that the function f is graph quasi-continuous. Let us choose any seminorm $p \in \mathcal{S}$, $\varepsilon > 0$, $r > 0$ and any point $(\lambda_0, x_0) \in \mathcal{R} \times X$. Let $U \subset X$ be any nonempty open set and $x_0 \in U$. Then, the set

$$(\lambda_0 - r, \lambda_0 + r) \times U \text{ is a neighbourhood of } (\lambda_0, x_0).$$

First, we consider that $p(f(x_0)) \neq 0$. Because the function ϕ is continuous at λ_0 , we can find $\delta \in (0, r)$ such that

$$|\phi(\lambda_0) - \phi(\lambda)| < \frac{\varepsilon}{2p(f(x_0))} \quad \text{for each } \lambda \in (\lambda_0 - \delta, \lambda_0 + \delta).$$

Therefore,

$$|\phi(\lambda)| < \frac{\varepsilon}{2p(f(x_0))} + |\phi(\lambda_0)| \quad \text{for each } \lambda \in (\lambda_0 - \delta, \lambda_0 + \delta).$$

We put $M = \frac{\varepsilon}{2p(f(x_0))} + |\phi(\lambda_0)|$. Hence, $M > 0$. Since f is quasi-continuous at x_0 , there exists a nonempty open set $U_1 \subset U$ such that $p(f(x) - f(x_0)) < \frac{\varepsilon}{2M}$ for each $x \in U_1$. Let us select a point

$$(\lambda, x) \in (\lambda_0 - \delta, \lambda_0 + \delta) \times U_1.$$

Hence, we have

$$\begin{aligned} p(f_*(\lambda, x) - f_*(\lambda_0, x_0)) &= p(\phi(\lambda)f(x) - \phi(\lambda_0)f(x_0)) \\ &\leq p(\phi(\lambda)f(x) - \phi(\lambda)f(x_0))p(\phi(\lambda)f(x_0) - \phi(\lambda_0)f(x_0)) \\ &= |\phi(\lambda)|p(f(x) - f(x_0)) + |\phi(\lambda) - \phi(\lambda_0)|p(f(x_0)) \\ &< M\frac{\varepsilon}{2M} + \frac{\varepsilon}{2p(f(x_0))}p(f(x_0)) = \varepsilon. \end{aligned}$$

Now, we focus on $p(f(x_0)) = 0$. Since ϕ is continuous at $\lambda_0 \in \mathcal{R}$, there exists $\delta \in (0, r)$ such that $|\phi(\lambda_0) - \phi(\lambda)| < \varepsilon$ for each $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$. Then, $|\phi(\lambda)| < \varepsilon + |\phi(\lambda_0)|$ for each $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$. We put $M = \varepsilon + |\phi(\lambda_0)|$. Then, $M > 0$. As the function f is quasi-continuous at $x_0 \in U$, we can find a nonempty

open set $U_1 \subset U$ such that $p(f(x) - f(x_0)) < \frac{\varepsilon}{M}$ for each $x \in U_1$. We choose $(\lambda, x) \in (\lambda_0 - \delta, \lambda_0 + \delta) \times U_1$. Summarizing the previous, we have

$$\begin{aligned} p(f_*(\lambda, x) - f_*(\lambda_0, x_0)) &= p(\phi(\lambda)f(x) - \phi(\lambda_0)f(x_0)) \\ &\leq p(\phi(\lambda)f(x) - \phi(\lambda)f(x_0)) + p(\phi(\lambda)f(x_0) - \phi(\lambda_0)f(x_0)) \\ &= |\phi(\lambda)|p(f(x) - f(x_0)) + |\phi(\lambda) - \phi(\lambda_0)|p(f(x_0)) \\ &< M\frac{\varepsilon}{M} = \varepsilon. \end{aligned}$$

We have shown that for each seminorm $p \in \mathcal{S}$, $\varepsilon > 0$, $r > 0$, for any point $(\lambda_0, x_0) \in \mathcal{R} \times X$ and for any neighbourhood $(\lambda_0 - r, \lambda_0 + r) \times U$ of (λ_0, x_0) , there exists a nonempty open set $(\lambda_0 - \delta, \lambda_0 + \delta) \times U_1 \subset (\lambda_0 - r, \lambda_0 + r) \times U$ such that $p(f_*(\lambda, x) - f_*(\lambda_0, x_0)) < \varepsilon$ for each $(\lambda, x) \in (\lambda_0 - \delta, \lambda_0 + \delta) \times U_1$. It means that f_* is quasi-continuous in (λ_0, x_0) . In case when the function f is continuous, using similar arguments, we can prove the continuity of f_* . $\square$

LEMMA 4. *Let X be a topological space, $(Y, \mathcal{P}_S)$ —a locally convex topological vector space. For a quasi-continuous function $g: X \rightarrow Y$ and for a continuous function $\phi: \mathcal{R} \rightarrow \mathcal{R}$, the following conditions are equivalent:*

- (a) $G(g_*) \subset \overline{G(f_*)}$;
- (b) *the set $A(f_*, g_*, p, \varepsilon)$ is dense in $\mathcal{R} \times X$ for each seminorm $p \in \mathcal{S}$ and for each $\varepsilon > 0$.*

Proof. (a) $\Rightarrow$ (b) Let us take a seminorm $p \in \mathcal{S}$, $\varepsilon > 0$, a nonempty open set $(a, b) \times U \subset \mathcal{R} \times X$, and let us choose a point $(\lambda_0, x_0) \in (a, b) \times U$. According to Lemma 3, the function g_* is quasi-continuous at (λ_0, x_0) . So, there exists a nonempty open set $(a_1, b_1) \times U_1 \subset (a, b) \times U$ such that $p(g_*(\lambda, x) - g_*(\lambda_0, x_0)) < \frac{\varepsilon}{3}$ for each $(\lambda, x) \in (a_1, b_1) \times U_1$.

Now, let us select a point $(\lambda_1, x_1) \in (a_1, b_1) \times U_1$. The set

$$[(a_1, b_1) \times U_1] \times K\left(g_*(\lambda_1, x_1), p, \frac{\varepsilon}{3}\right)$$

is a neighbourhood of $((\lambda_1, x_1), g_*(\lambda_1, x_1))$. Because $G(g_*) \subset \overline{G(f_*)}$, we have

$$\left[((a_1, b_1) \times U_1) \times K\left(g_*(\lambda_1, x_1), p, \frac{\varepsilon}{3}\right) \right] \cap G(f_*) \neq \emptyset.$$

We can find

$$((\lambda_2, x_2), f_*(\lambda_2, x_2)) \in G(f_*)$$

such that

$$(\lambda_2, x_2) \in (a_1, b_1) \times U_1 \quad \text{and} \quad f_*(\lambda_2, x_2) \in K\left(g_*(\lambda_1, x_1), p, \frac{\varepsilon}{3}\right).$$

Finally, we obtain

$$\begin{aligned} p(g_*(\lambda_2, x_2) - f_*(\lambda_2, x_2)) &\leq p(g_*(\lambda_2, x_2) - g_*(\lambda_0, x_0)) \\ &\quad + p(g_*(\lambda_0, x_0) - g_*(\lambda_1, x_1)) \\ &\quad + p(g_*(\lambda_1, x_1) - f_*(\lambda_2, x_2)) < \varepsilon. \end{aligned}$$

It implies that

$$(\lambda_2, x_2) \in [(a, b) \times U] \cap A(f_*, g_*, p, \varepsilon),$$

so $A(f_*, g_*, p, \varepsilon)$ is dense in $\mathcal{R} \times X$.

(b) $\Rightarrow$ (a) Let $((\lambda_0, x_0), g_*(\lambda_0, x_0)) \in G(g_*)$ and let $[(\lambda_0 - r, \lambda_0 + r) \times U] \times K(g_*(\lambda_0, x_0), p, \varepsilon)$ be a neighbourhood of $((\lambda_0, x_0), g_*(\lambda_0, x_0))$. Since g_* is quasi-continuous at (λ_0, x_0) , there exists a nonempty open set

$$(a, b) \times U_1 \subset (\lambda_0 - r, \lambda_0 + r) \times U \quad \text{with} \quad p(g_*(\lambda_0, x_0) - g_*(\lambda, x)) < \frac{\varepsilon}{2} \\ \text{for } (\lambda, x) \in (a, b) \times U_1.$$

The set $A(f_*, g_*, p, \frac{\varepsilon}{2})$ is dense, so, $[(a, b) \times U_1] \cap A(f_*, g_*, p, \frac{\varepsilon}{2}) \neq \emptyset$. Then, there exists a point $(\lambda_1, x_1) \in (a, b) \times U_1$ such that

$$p(g_*(\lambda_1, x_1) - f_*(\lambda_1, x_1)) < \frac{\varepsilon}{2}.$$

Therefore, we obtain

$$\begin{aligned} p(g_*(\lambda_0, x_0) - f_*(\lambda_1, x_1)) &\leq p(g_*(\lambda_0, x_0) - g_*(\lambda_1, x_1)) \\ &\quad + p(g_*(\lambda_1, x_1) - f_*(\lambda_1, x_1)) < \varepsilon. \end{aligned}$$

It implies that

$$f_*(\lambda_1, x_1) \in K(g_*(\lambda_0, x_0), p, \varepsilon).$$

Because

$$(\lambda_1, x_1) \in (\lambda_0 - r, \lambda_0 + r) \times U$$

and

$$((\lambda_1, x_1), f_*(\lambda_1, x_1)) \in [(\lambda_0 - r, \lambda_0 + r) \times U] \times K(g_*(\lambda_0, x_0), p, \varepsilon),$$

we have $[(\lambda_0 - r, \lambda_0 + r) \times U] \cap G(f_*) \neq \emptyset$. This means that

$$((\lambda_0, x_0), g_*(\lambda_0, x_0)) \in \overline{G(f_*)}.$$

Hence, we have $G(g_*) \subset \overline{G(f_*)}$, which finishes the proof. $\square$

THEOREM 5. *Let X be a topological space, $(Y, \mathcal{P}_S)$ —a locally convex topological vector space. If a function $f: X \rightarrow Y$ is graph quasi-continuous (graph continuous), $\phi: \mathcal{R} \rightarrow \mathcal{R}$ is continuous, then $f_*: \mathcal{R} \times X \rightarrow Y$ is graph quasi-continuous (graph continuous).*

Proof. Let us assume that f is graph quasi-continuous. Let us choose a seminorm $p \in \mathcal{S}$ and $\varepsilon > 0$. Let $U \subset X$ be a nonempty open set. Since a function ϕ is continuous at $\lambda_0 \in \mathcal{R}$, there exists $\delta > 0$ such that $|\phi(\lambda_0) - \phi(\lambda)| < \varepsilon$ for each $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$. Let $\lambda_1 \in (\lambda_0 - \delta, \lambda_0 + \delta)$. Then, $|\phi(\lambda_1)| < |\phi(\lambda_0)| + \varepsilon$. We put $M = |\phi(\lambda_0)| + \varepsilon$. Then, $M > 0$. Since the function f is graph quasi-continuous, there exists a quasi-continuous function $g: X \rightarrow Y$ such that the set $A(f, g, p, \frac{\varepsilon}{M})$ is dense in X . So, $U \cap A(f, g, p, \frac{\varepsilon}{M}) \neq \emptyset$. Hence, there exists $x_1 \in U$ such that $p(f(x_1) - g(x_1)) < \frac{\varepsilon}{M}$. In a consequence, we have

$$p(\phi(\lambda_1)f(x_1) - \phi(\lambda_1)g(x_1)) \leq |\phi(\lambda_1)| \cdot p(f(x_1) - g(x_1)) < M \cdot \frac{\varepsilon}{M} = \varepsilon.$$

It means that $(\lambda_1, x_1) \in A(\phi f, \phi g, p, \varepsilon)$. Therefore, $[(\lambda_0 - \delta, \lambda_0 + \delta) \times U] \cap A(\phi f, \phi g, p, \varepsilon) \neq \emptyset$. In a consequence, the set $A(\phi f, \phi g, p, \varepsilon)$ is dense in $\mathcal{R} \times X$. From Lemma 3, we have that $G(g_*) \subset \overline{G(f_*)}$ and the function f_* is graph quasi-continuous. If the function f is graph continuous, then the proof is analogous. $\square$

COROLLARY 1. *Let X be a topological space, $(Y, \mathcal{P}_S)$ —locally convex topological vector space. If a function $f: X \rightarrow Y$ is graph quasi-continuous (graph continuous) and $c \in \mathcal{R}$, then $cf: \mathcal{R} \times X \rightarrow Y$ is graph quasi-continuous (graph continuous, respectively).*

For a topological space $(X, \mathcal{T})$, let $\mathcal{T}_\alpha = \{U \setminus H; U \in \mathcal{T}, H \text{ is nowhere dense}\}$, see [11].

LEMMA 5 ([9]). *Let $(X, \mathcal{T})$ be a topological space, $(Y, \mathcal{P})$ —a uniform space. A function $f: (X, \mathcal{T}) \rightarrow Y$ is graph quasi-continuous (graph continuous) if and only if a function $f: (X, \mathcal{T}_\alpha) \rightarrow Y$ is graph quasi-continuous (graph continuous, respectively).*

DEFINITION 3 ([3]). Let $(X, \mathcal{T}), (Y, \tau)$ be topological spaces. A function $f: (X, \mathcal{T}) \rightarrow (Y, \tau)$ is a semi-homeomorphism if:

- (a) the function f is bijective,
- (b) $f(W)$ is semi-open for each semi-open set $W \subset X$,
- (c) $f^{-1}(V)$ is semi-open for each semi-open set $V \subset Y$.

LEMMA 6 ([3]). *Let $(X, \mathcal{T}), (Y, \tau)$ be topological spaces. If a function $f: (X, \mathcal{T}) \rightarrow (Y, \tau)$ is a semi-homeomorphism, then $f: (X, \mathcal{T}_\alpha) \rightarrow (Y, \tau_\alpha)$ is a homeomorphism.*

THEOREM 6. *Let $(X, \mathcal{T}), (Y, \tau)$ be topological spaces, $(Z, \mathcal{P})$ —a uniform space. Then:*

- (a) *If a function $h: (X, \mathcal{T}) \rightarrow (Y, \tau)$ is a semi-homeomorphism and $f: (Y, \mathcal{T}) \rightarrow (Z, \mathcal{P})$ is graph quasi-continuous, then $f \circ h: (X, \mathcal{T}) \rightarrow (Z, \mathcal{P})$ is graph quasi-continuous.*

- (b) If a function $h: (X, \mathcal{T}) \rightarrow (Y, \tau)$ is a semi-homeomorphism and $f: (Y, \tau) \rightarrow (Z, \mathcal{P})$ is graph continuous, then $f \circ h: (X, \mathcal{T}) \rightarrow (Z, \mathcal{P})$ is graph continuous.

Proof. (a) Let the function $h: (X, \mathcal{T}) \rightarrow (Y, \tau)$ is a semi-homeomorphism. From Lemma 6, we have that the function $h: (X, \mathcal{T}_\alpha) \rightarrow (Y, \tau_\alpha)$ is a homeomorphism. From the assumption and from Lemma 5, we have that the function $f: (Y, \tau_\alpha) \rightarrow (Z, \mathcal{P})$ is graph quasi-continuous. So, there exists a quasi-continuous function $g: (Y, \tau_\alpha) \rightarrow (Z, \mathcal{P})$ such that the set $A(f, g, \varrho, \varepsilon)$ is τ_α dense in (Y, τ_α) . Let $A(f, g, \varrho, \varepsilon) = W$ and let us take any open set $U \subset X$. So, the set $h(U)$ is open in (Y, τ_α) . Then, $W \cap h(U) \neq \emptyset$. So, there exists $y \in h(U) \cap W$. Hence,

$$h^{-1}(y) \in U \quad \text{and} \quad h^{-1}(y) \in h^{-1}(W), \quad \text{and} \quad h^{-1}(W) \cap U \neq \emptyset.$$

It means that the set $h^{-1}(W)$ is dense in (X, τ_α) . Let $x \in h^{-1}(W)$. Then,

$$h(x) \in W = A(f, g, \varrho, \varepsilon) \quad \text{and} \quad \varrho((f \circ h)(x), (g \circ h)(x)) < \varepsilon.$$

So, $x \in A(f, g, \varrho, \varepsilon)$. In this case, we have that $h^{-1}(W) \subset A(f \circ h, g \circ h, \varrho, \varepsilon)$. Because the set $h^{-1}(W)$ is dense in $(X, \mathcal{T}_\alpha)$, so the set $A(f \circ h, g \circ h, \varrho, \varepsilon)$ is dense in (X, τ_α) .

Now, we prove that the function $g \circ h$ is quasi-continuous. Let us take any nonempty open set $U \subset Z$. The function $g: (Y, \tau_\alpha) \rightarrow (Z, \mathcal{P})$ is quasi-continuous, so the set $g^{-1}(U)$ is semi-open in (Y, τ_α) . There exists any nonempty open in (Y, τ_α) set V such that $V \subset g^{-1}(U) \subset \overline{V}$. Since the function $h: (X, \mathcal{T}_\alpha) \rightarrow (Y, \tau_\alpha)$ is homeomorphism, $h^{-1}(\overline{V}) \subset \overline{h^{-1}(V)}$, in a consequence, we have

$$h^{-1}(V) \subset (g \circ h)^{-1}(U) \subset h^{-1}(\overline{V}) \subset \overline{h^{-1}(V)}.$$

It means that the set $(g \circ h)^{-1}(U)$ is semi-open in $(X, \mathcal{T}_\alpha)$. In this way, we conclude that the set $(g \circ h)^{-1}(U)$ is semi-open in $(X, \mathcal{T}_\alpha)$ for each open in $(X, \mathcal{T}_\alpha)$ set U . It follows that the function $g \circ h: (X, \mathcal{T}_\alpha) \rightarrow (Z, \mathcal{P})$ is quasi-continuous and $G(g \circ h) \subset \overline{G(f \circ h)}$. So, we have that the function $f \circ h: (X, \mathcal{T}_\alpha) \rightarrow (Z, \mathcal{P})$ is graph quasi-continuous. By the Lemma 5, we conclude that $f \circ h: (X, \mathcal{T}) \rightarrow (Z, \mathcal{P})$ is graph quasi-continuous.

(b) If the function $f: (Y, \tau) \rightarrow (Z, \mathcal{P})$ is graph continuous, the proof is analogous. $\square$

LEMMA 7 ([3]). *Let $(X, \mathcal{T}), (Y, \tau)$ be topological spaces. If a function $f: X \rightarrow Y$ is a homeomorphism, then $f: X \rightarrow Y$ is a semi-homeomorphism.*

COROLLARY 2. *Let $(X, \mathcal{T}), (Y, \tau)$ be topological spaces, $(Z, \mathcal{P})$ —a uniform space. Then:*

- (a) *If a function $h: (X, \mathcal{T}) \rightarrow Y$ is a homeomorphism and $f: (Y, \tau) \rightarrow Z$ is graph continuous, then $f \circ h: (X, \mathcal{T}) \rightarrow Z$ is graph continuous.*
- (b) *If a function $h: (X, \mathcal{T}) \rightarrow Y$ is a homeomorphism and $f: (Y, \tau) \rightarrow Z$ is graph quasi-continuous, then $f \circ h: (X, \mathcal{T}) \rightarrow Z$ is graph quasi-continuous.*

THEOREM 7. *Let $(X, \mathcal{T})$ be a topological space, $(Y, \mathcal{P})$ —a uniform space. Let a function $f: (X, \mathcal{T}) \rightarrow Y$ be graph quasi-continuous and cliquish. Then:*

- (a) *The function $f|_M$ is graph quasi-continuous for each dense set $M \subset X$.*
- (b) *The function $f|_U$ is graph quasi-continuous for each open set $U \subset X$.*

PROOF. (a) Let us take any $\varrho \in \mathcal{P}$ and $\varepsilon > 0$. Since the function f is graph quasi-continuous and cliquish, according to Theorem 2, there exists a quasi-continuous function g such that the set $B(f, g, \varrho, \varepsilon)$ is nowhere dense in X . By a Lemma 1(a), the set $A(f, g, \varrho, \varepsilon)$ is semi-open. This fact and Lemma 1(b) give that the set $A(f, g, \varrho, \varepsilon) \cap M = B(f|_M, g|_M, \varrho, \varepsilon)$ is dense in X . Since $g|_M$ is quasi-continuous, in virtue of Theorem 1, the function $f|_M$ is graph quasi-continuous.

(b) Let $x_0 \in U$. Since f is graph quasi-continuous, there exists a quasi-continuous g such that $G(g) \subset \overline{G(f)}$. Let us take $(x_0, g(x_0)) \in G(g)$ and a neighbourhood $U_1 \times V$ of $(x_0, g(x_0))$, where $U_1 \subset U$. Then, $(U_1 \times V) \cap G(f) \neq \emptyset$. We have $G(g|_U) \subset \overline{G(f|_U)}$. Because the function $g|_U$ is quasi-continuous, the function $f|_U$ is graph quasi-continuous. The proof is completed. $\square$

We can prove the following theorem in a similar way:

THEOREM 8. *Let $(X, \mathcal{T})$ be a topological space and let $(Y, \mathcal{P})$ —a uniform space. Let a function $f: (X, \mathcal{T}) \rightarrow Y$ be graph continuous and cliquish. Then:*

- (a) *The function $f|_M$ is graph continuous for each dense set $M \subset X$.*
- (b) *The function $f|_U$ is graph continuous for each open set $U \subset X$.*

Now, let X_1, X_2 be topological spaces and let $(Y_1, \mathcal{P}_1), (Y_2, \mathcal{P}_2)$ be uniform spaces. Then,

$$(Y_1 \times Y_2, \mathcal{P}_1 \times \mathcal{P}_2)$$

is a uniform space, where $\mathcal{P}_1 \times \mathcal{P}_2$ consists of seminorms ϱ such that $\varrho = \varrho_1 + \varrho_2$ for some $\varrho_1 \in \mathcal{P}_1, \varrho_2 \in \mathcal{P}_2$. Then,

$$\varrho((x_1, x_2), (y_1, y_2)) = \varrho_1(x_1, y_1) + \varrho_2(x_2, y_2)$$

for each $((x_1, x_2), (y_1, y_2)) \in \mathcal{P}_1 \times \mathcal{P}_2$. For the functions $f_1: X_1 \rightarrow Y_1$, and $f_2: X_2 \rightarrow Y_2$, we define the function $f_1 \times f_2: X_1 \times X_2 \rightarrow Y_1 \times Y_2$ as follows:

$$(f_1 \times f_2)(x_1, x_2) = (f_1(x_1), f_2(x_2))$$

for each $(x_1, x_2) \in X_1 \times X_2$, (see [14]).

THEOREM 9. *Let X_1, X_2 be topological spaces and let $(Y_1, \mathcal{P}_1), (Y_2, \mathcal{P}_2)$ be uniform spaces. If the functions $f_1: X_1 \rightarrow Y_1, f_2: X_2 \rightarrow Y_2$ are graph quasi-continuous (graph continuous), then $f_1 \times f_2$ is graph quasi-continuous (graph continuous, respectively).*

Proof. Let us take $\varrho_1 \in \mathcal{P}_1$, $\varrho_2 \in \mathcal{P}_2$ and $\varepsilon > 0$. From assumptions about functions f_1, f_2 , there exist quasi-continuous functions $g_1: X_1 \rightarrow Y_1$ and $g_2: X_2 \rightarrow Y_2$ such that

$$G(g_1) \subset \overline{G(f_1)} \quad \text{and} \quad G(g_2) \subset \overline{G(f_2)}.$$

In virtue of Theorem 1, we have that the sets

$$A(f_1, g_1, \varrho_1, \frac{\varepsilon}{2}) \quad \text{and} \quad A(f_2, g_2, \varrho_2, \frac{\varepsilon}{2})$$

are dense in X_1 and X_2 , respectively. On the other hand, we have

$$A(f_2, g_2, \varrho_2, \frac{\varepsilon}{2}) \times A(f_1, g_1, \varrho_1, \frac{\varepsilon}{2}) \subset A(f_1 \times f_2, g_1 \times g_2, \varrho_1, \varepsilon).$$

In the consequence, the set $A(f_1 \times f_2, g_1 \times g_2, \varrho_1, \varepsilon)$ is dense in $X_1 \times X_2$. It is known that the function $g_1 \times g_2$ is quasi-continuous. So, according to Theorem 1, we have that the function $f_1 \times f_2$ is graph quasi-continuous. $\square$

REFERENCES

- [1] ALEXIEWICZ, A.: *Analiza Funkcjonalna*. PWN, Warszawa, 1969.
- [2] CROSSLEY, S. G.—HILDEBRAND, S. K.: *Semi-closure*, Texas J. Sci **22** (1971), 99–112.
- [3] CROSSLEY, S. G.—HILDEBRAND, S. K.: *Semi topological properties*, Fund. Math. **74** (1972), 233–254.
- [4] ENGELKING, R.: *General Topology*. PWN, Warszawa, 1984.
- [5] EWERT, J.: *On quasi-continuous and cliquish maps with values in uniform spaces*, Bull. Polish Acad. Sci. Math. **32** (1984), 81–88.
- [6] GRANDE, Z.: *Sur les fonctions A-continues*, Demonstratio Math. **11** (1978), 519–526.
- [7] LEVINE, N.: *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly **70** (1963), 36–41.
- [8] MIKUCKA, A.: *Graph quasi-continuity*, Demonstratio Math. **36** (2003), 483–494.
- [9] MIKUCKA, A.: *On graph quasi-continuous functions*, Tatra Mt. Math. Publ. **34** (2006), 93–105.
- [10] NEUBRUNN, T.: *Quasi-continuity*, Real Anal. Exchange **14** (1988–1989), 259–306.
- [11] NJÅSTAD, O.: *On some classes of nearly open set*, Pacific J. Math. **15** (1965), 961–970.
- [12] SAKÁLOVÁ, K.: *On structural properties of graph continuous function*, Tatra Mt. Math. Publ. **6** (1996), 123–126.
- [13] SAKÁLOVÁ, K.: *On the maximal multiplicative family for the class of graph continuous functions*, Demonstratio Math. **32** (1999), 615–620.
- [14] SCHUBERT, H.: *Topology*. MacDonald Technical and Scientific, London, 1968.

Received December 4, 2013

*Institute of Mathematics
Academia Pomeraniensis
Arciszewskiego 22 d
PL-76-200 Słupsk
POLAND
E-mail: aneta.mikucka@wp.pl*