$\mathcal{J}$ -APPROXIMATELY CONTINUOUS FUNCTIONS

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ABSTRACT. In this paper, density-like points and density-like topology connected with a sequence $\mathcal{J} = \{J_n\}_{n \in \mathbb{N}}$ of closed intervals tending to 0 will be considered. We introduce the notion of an $\mathcal{J}$ -approximately continuous function associated with this kind of density points. Moreover, we present some properties of these functions and we demonstrate their connection with continuous functions with respect to this kind of density topology.

1. Introduction and preliminaries

We start with basic notation and definitions. We will use symbols $\mathbb{R}$ and $\mathbb{N}$ to denote real numbers and positive integers, respectively. The Lebesgue measure will be denoted by m . By $\mathcal{L}$ we will denote the σ -algebra of Lebesgue measurable sets of $\mathbb{R}$ and by $\mathbb{L}$ – the σ -ideal of Lebesgue null sets. The notation $A \sim B$ means that $A \triangle B \in \mathbb{L}$, where $A \triangle B = (A \setminus B) \cup (B \setminus A)$. Moreover, $\text{diam}(A)$ will denote a diameter of the set $A \subset \mathbb{R}$. The length of the interval J will be denoted by $|J|$. The symbol $\mathcal{T}_0$ will stand for the natural topology in $\mathbb{R}$.

In this paper, a special role will be played by sequences of closed intervals (in particular, sequences of intervals convergent to zero). Let $\mathcal{J} = \{J_n\}_{n \in \mathbb{N}}$ be a sequence of closed intervals. We shall say that the sequence $\mathcal{J}$ is tending to zero (denoted $J_n \xrightarrow{n \rightarrow \infty} 0$) if and only if $\lim_{n \rightarrow \infty} \text{diam}(J_n \cup \{0\}) = 0$. The family of all sequences of closed intervals tending to zero will be denoted by $\mathfrak{S}$. Moreover, if J is an interval then $x + J = \{x + a : a \in J\}$.

In the paper [5], one can find the following definition of $\mathcal{J}$ -density point.

DEFINITION 1.1. Let $A \in \mathcal{L}$, $\mathcal{J} \in \mathfrak{S}$ and $x_0 \in \mathbb{R}$. We shall say that a point x_0 is a $\mathcal{J}$ -density point of the set A if

$$\lim_{n \rightarrow \infty} \frac{m(A \cap (x_0 + J_n))}{|J_n|} = 1. \quad (1)$$

It is worthless that the above definition is an extension of the definition of an $\langle s \rangle$ -density point introduced in [3]. Moreover, condition (1) is equivalent to the following condition

$$\lim_{n \rightarrow \infty} \frac{m((\mathbb{R} \setminus A) \cap (x_0 + J_n))}{|J_n|} = 0. \quad (2)$$

Let us introduce one more notation. We will denote by $\mathfrak{S}_\alpha$ the family of all sequences $\mathcal{J} = \{J_n\}_{n \in \mathbb{N}}$ from $\mathfrak{S}$ such that $\alpha(\mathcal{J}) < \infty$, where

$$\alpha(\mathcal{J}) = \limsup_{n \rightarrow \infty} \frac{\text{diam}(J_n \cup \{0\})}{|J_n|}.$$

Let $A \in \mathcal{L}$ and $\mathcal{J} \in \mathfrak{S}$. Set

$$\Phi_{\mathcal{J}}(A) = \{x \in \mathbb{R} : x \text{ is a } \mathcal{J} - \text{density point of the set } A\}.$$

Some properties of operator $\Phi_{\mathcal{J}}: \mathcal{L} \rightarrow \mathcal{L}$ are given in [5]. Some of these properties, useful in this paper, are presented in the next theorem.

THEOREM 1.2 ([5]). *Let $A, B \in \mathcal{L}$ and $\mathcal{J} \in \mathfrak{S}$.*

- (1) $\Phi_{\mathcal{J}}(\emptyset) = \emptyset$ and $\Phi_{\mathcal{J}}(\mathbb{R}) = \mathbb{R}$,
- (2) $\Phi_{\mathcal{J}}(A \cap B) = \Phi_{\mathcal{J}}(A) \cap \Phi_{\mathcal{J}}(B)$,
- (3) if $A \subset B$, then $\Phi_{\mathcal{J}}(A) \subset \Phi_{\mathcal{J}}(B)$,
- (4) if $A \sim B$, then $\Phi_{\mathcal{J}}(A) = \Phi_{\mathcal{J}}(B)$,
- (5) if $\mathcal{J} \in \mathfrak{S}_\alpha$, then $\Phi_d(A) \subset \Phi_{\mathcal{J}}(A)$,
- (6) if $\mathcal{J} \in \mathfrak{S}_\alpha$, then $\Phi_{\mathcal{J}}(A) \sim A$.

One can see that if $\mathcal{J} \in \mathfrak{S}_\alpha$, then $\Phi_{\mathcal{J}}: \mathcal{L} \rightarrow \mathcal{L}$ is the lower density operator. In [2], one can find the construction of the sequence $\mathcal{J} \in \mathfrak{S} \setminus \mathfrak{S}_\alpha$ and a set $A \in \mathcal{L}$ such that $\Phi_{\mathcal{J}}(A) \triangle A \notin \mathbb{L}$. Nevertheless, one can prove the following fact

THEOREM 1.3 ([5, Th. 10]). *Let $\mathcal{J} \in \mathfrak{S}$. The family*

$$\mathcal{T}_{\mathcal{J}} = \{A \in \mathcal{L} : A \subset \Phi_{\mathcal{J}}(A)\}$$

is a topology stronger than the natural topology.

The topology described in the above theorem is called $\mathcal{J}$ -density topology. We end this section with two technical lemmas used in the paper later.

LEMMA 1.4. *Let $A \in \mathcal{L}$ and $\mathcal{J} = \{J_n\}_{n \in \mathbb{N}} \in \mathfrak{S}$. The following conditions are equivalent:*

- (1) $x_0 \in \Phi_{\mathcal{J}}(A)$,
- (2) $\lim_{k \rightarrow \infty} \frac{m(A \cap (x_0 + J_{n_k}))}{|J_{n_k}|} = 1$ for any subsequence $\{J_{n_k}\}_{k \in \mathbb{N}}$ of sequence $\mathcal{J}$ such that the sequence $\{\text{diam}(J_{n_k} \cup \{0\})\}_{k \in \mathbb{N}}$ is decreasing.

Proof. The implication “(1) $\Rightarrow$ (2)” is obvious. Now, let assume that condition (2) is satisfied and suppose that $x_0 \notin \Phi_{\mathcal{J}}(A)$. Thus,

$$\liminf_{n \rightarrow \infty} \frac{m(A \cap (x_0 + J_n))}{|J_n|} < 1,$$

so there exists a subsequence $\{J_{n_k}\}_{k \in \mathbb{N}}$ of the sequence $\mathcal{J}$ such that

$$\lim_{k \rightarrow \infty} \frac{m(A \cap (x_0 + J_{n_k}))}{|J_{n_k}|} < 1.$$

Of course, there exists a subsequence $\{J_{n_{k_l}}\}_{l \in \mathbb{N}}$ of the sequence $\{J_{n_k}\}_{k \in \mathbb{N}}$ (so, a subsequence of $\mathcal{J}$) such that $\{\text{diam}(J_{n_{k_l}} \cup \{0\})\}_{l \in \mathbb{N}}$ is decreasing and

$$\lim_{l \rightarrow \infty} \frac{m(A \cap (x_0 + J_{n_{k_l}}))}{|J_{n_{k_l}}|} < 1.$$

This contradicts our assumption. $\square$

LEMMA 1.5. *Let $\mathcal{J} = \{J_n\}_{n \in \mathbb{N}} \in \mathfrak{S}$ and $p: \mathbb{N} \rightarrow \mathbb{N}$ be any permutation of the set $\mathbb{N}$. The sequence $\mathcal{J}^p = \{J_n^p\}_{n \in \mathbb{N}}$, where $J_n^p = J_{p(n)}$ for $n \in \mathbb{N}$, belongs to $\mathfrak{S}$ and $\Phi_{\mathcal{J}}(A) = \Phi_{\mathcal{J}^p}(A)$ for any $A \in \mathcal{L}$.*

Proof. We first proof that $\mathcal{J}^p \in \mathfrak{S}$. Obviously, $\mathcal{J}^p$ consists of closed intervals. Let $\varepsilon > 0$. There exists $n_0 \in \mathbb{N}$ such that $\text{diam}(J_n \cup \{0\}) < \varepsilon$ for any $n > n_0$. Putting

$$n_1 = \min\left\{n \in \mathbb{N}: \{1, \dots, n_0\} \subset \{p(1), \dots, p(n)\}\right\},$$

we have $p(n) > n_0$ for any $n > n_1$, so $\text{diam}(J_n^p \cup \{0\}) = \text{diam}(J_{p(n)} \cup \{0\}) < \varepsilon$ and, in consequence, $\mathcal{J}^p \in \mathfrak{S}$.

Now, let $A \in \mathcal{L}$, $x_0 \in \Phi_{\mathcal{J}}(A)$ and $\varepsilon > 0$. There exists $n_* \in \mathbb{N}$ such that

$$\left| \frac{m(A \cap (x_0 + J_n))}{|J_n|} - 1 \right| < \varepsilon \quad \text{for any } n > n_*.$$

Put

$$n_2 = \min\left\{n \in \mathbb{N}: \{1, \dots, n_*\} \subset \{p(1), \dots, p(n)\}\right\}.$$

An analysis similar to the above one shows that

$$\left| \frac{m(A \cap (x_0 + J_n^p))}{|J_n^p|} - 1 \right| < \varepsilon \quad \text{for } n \geq n_2, \quad \text{thus } x_0 \in \Phi_{\mathcal{J}^p}(A).$$

If $x_0 \in \Phi_{\mathcal{J}^p}(A)$, then it is sufficient to consider a permutation $p': \mathbb{N} \rightarrow \mathbb{N}$ such that $(J^p)^{p'} = \{J_{p'(p(n))}\}_{n \in \mathbb{N}} = \{J_n\}_{n \in \mathbb{N}}$. The rest of the proof runs as before. $\square$

Taking the above lemma into account, we can assume that $\{\text{diam}(J_n \cup \{0\})\}_{n \in \mathbb{N}}$ is not increasing. From now on, $\mathfrak{S}$ will denote the family of all sequences $\{J_n\}_{n \in \mathbb{N}}$ of closed intervals tending to zero, such that $\{\text{diam}(J_n \cup \{0\})\}_{n \in \mathbb{N}}$ is not increasing. Moreover, $\mathfrak{S}_\alpha$ will stand for the family of all sequences $\{J_n\}_{n \in \mathbb{N}} \in \mathfrak{S}$ such that $\{\text{diam}(J_n \cup \{0\})\}_{n \in \mathbb{N}}$ is not increasing and $\alpha(\mathcal{J}) < \infty$.

2. $\mathcal{J}$ -approximately continuous functions

The notion of approximately continuous function, connected with the notion of density point, was introduced at the beginning of XX century. This kind of function plays an important role in the real functions theory. It is well-known that a function $f: \mathbb{R} \rightarrow \mathbb{R}$ is Lebesgue measurable if and only if it is approximately continuous almost everywhere [1]. If $\langle s \rangle$ -density points instead of density points are considered in the definition of approximately continuous function, then we obtain the concept of $\langle s \rangle$ -approximately continuous function. These functions were investigated, for example, in [4], [6]. In the case of $\mathcal{J}$ -density points, we have the following definition

DEFINITION 2.1. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ and $\mathcal{J} \in \mathfrak{S}$. We shall say that f is $\mathcal{J}$ -approximately continuous at a point $x_0 \in \mathbb{R}$ if there exists a set $A_{x_0} \in \mathcal{L}$ such that $x_0 \in \Phi_{\mathcal{J}}(A_{x_0})$ and $f(x_0) = \lim_{x \rightarrow x_0, x \in A_{x_0}} f(x)$.

Obviously, we shall say that $f: \mathbb{R} \rightarrow \mathbb{R}$ is a $\mathcal{J}$ -approximately continuous function if it is $\mathcal{J}$ -approximately continuous at each point $x \in \mathbb{R}$. Now, we present some elementary properties of these functions. These properties follow immediately from the properties of $\mathcal{J}$ -density topology and $\mathcal{J}$ -density points presented in [5].

Remark 2.2.

- (I) If $f: \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, then it is $\mathcal{J}$ -approximately continuous for any sequence $\mathcal{J} \in \mathfrak{S}$.
- (II) If $f: \mathbb{R} \rightarrow \mathbb{R}$ is an approximately continuous function, then it is $\mathcal{J}$ -approximately continuous for any sequence $\mathcal{J} \in \mathfrak{S}_\alpha$.

Moreover, we have

Remark 2.3.

- (i) For any $\mathcal{J} \in \mathfrak{S} \setminus \mathfrak{S}_\alpha$, there exists an approximately continuous function $f: \mathbb{R} \rightarrow \mathbb{R}$ which is not $\mathcal{J}$ -approximately continuous.
- (ii) There exists a sequence $\mathcal{J} \in \mathfrak{S} \setminus \mathfrak{S}_\alpha$ and an $\mathcal{J}$ -approximately continuous function which is not approximately continuous.

THEOREM 2.4. Let $\mathcal{J} \in \mathfrak{S}$ and $f, g: \mathbb{R} \rightarrow \mathbb{R}$ be $\mathcal{J}$ -approximately continuous at a point $x_0 \in \mathbb{R}$. The functions $f + g$, $f \cdot g$ and $a \cdot f$ for any $a \in \mathbb{R}$ are $\mathcal{J}$ -approximately continuous at the point x_0 . Moreover, the function $\frac{1}{g}$ is $\mathcal{J}$ -approximately continuous at the point x_0 , whenever there exists $\delta > 0$ such that $g(x) \neq 0$ for any $x \in (x_0 - \delta, x_0 + \delta)$.

Proof. We will only show that $f \cdot g$ is $\mathcal{J}$ -approximately continuous at the point x_0 . The details of the proof in the other cases are left to the reader.

Let $\varepsilon > 0$. Since f and g are $\mathcal{J}$ -approximately continuous at the point x_0 , there exist sets $A_{x_0}, B_{x_0} \in \mathcal{L}$ such that

$$x_0 \in \Phi_{\mathcal{J}}(A_{x_0}) \cap \Phi_{\mathcal{J}}(B_{x_0}) \quad \text{and} \quad f(x_0) = \lim_{\substack{x \rightarrow x_0, \\ x \in A_{x_0}}} f(x) \quad \text{and} \quad g(x_0) = \lim_{\substack{x \rightarrow x_0, \\ x \in B_{x_0}}} g(x).$$

Putting $C_{x_0} = A_{x_0} \cap B_{x_0}$, we obtain that $x_0 \in \Phi_{\mathcal{J}}(C_{x_0})$. Of course, there exists $M > 0$ and $\delta > 0$ such that

$$|g(x)| < M, |g(x) - g(x_0)| < \frac{\varepsilon}{2(|f(x_0)| + 1)} \quad \text{and} \quad |f(x) - f(x_0)| < \frac{\varepsilon}{2M}$$

whenever $x \in C_{x_0} \cap (x_0 - \delta, x_0 + \delta)$. Therefore,

$$\begin{aligned} |(f \cdot g)(x) - (f \cdot g)(x_0)| &= |f(x)g(x) - f(x_0)g(x) + f(x_0)g(x) - f(x_0)g(x_0)| \\ &\leq |f(x_0)||g(x) - g(x_0)| + |g(x)||f(x) - f(x_0)| < \varepsilon \end{aligned}$$

for any $x \in C_{x_0} \cap (x_0 - \delta, x_0 + \delta)$, and, in consequence, $f \cdot g$ is $\mathcal{J}$ -approximately continuous at the point x_0 . $\square$

COROLLARY 2.5. *Let $\mathcal{J} \in \mathfrak{S}$ and $f, g: \mathbb{R} \rightarrow \mathbb{R}$ be $\mathcal{J}$ -approximately continuous functions. The functions $f + g$, $f \cdot g$ and $a \cdot f$ for any $a \in \mathbb{R}$ are $\mathcal{J}$ -approximately continuous. Moreover, the function $\frac{1}{g}$ is $\mathcal{J}$ -approximately continuous, whenever $g(x) \neq 0$ for any $x \in \mathbb{R}$.*

Now, we concentrate on sequences $\mathcal{J}$ belonging to $\mathfrak{S}_\alpha$. The proof of the next lemma is based on the idea presented in papers [6] and [8]. Moreover, as in the case of approximately continuous functions, the condition presented in this lemma will be called the condition (J_2) of J. M. J ę d r z e j e w s k i.

LEMMA 2.6. *Let $\mathcal{J} \in \mathfrak{S}_\alpha$. For any decreasing sequence of sets $\{E_n\}_{n \in \mathbb{N}} \subset \mathcal{L}$ and any $x_0 \in \bigcap_{n \in \mathbb{N}} \Phi_{\mathcal{J}}(E_n)$, there exists a decreasing sequence $\{r_n\}_{n \in \mathbb{N}}$ of positive numbers tending to 0 such that x_0 is a $\mathcal{J}$ -density point of the set*

$$A_{x_0} = \bigcup_{n=1}^{\infty} (E_n \setminus (x_0 - r_n, x_0 + r_n)).$$

P r o o f. Let $\mathcal{J} = \{J_k\}_{k \in \mathbb{N}} \in \mathfrak{S}_\alpha$ and $\{E_n\}_{n \in \mathbb{N}}$ be a decreasing sequence of Lebesgue measurable sets. Moreover, let $x_0 \in \mathbb{R}$ be such that $x_0 \in \Phi_{\mathcal{J}}(E_n)$ for any $n \in \mathbb{N}$ and $\{\varepsilon_n\}_{n \in \mathbb{N}} \subset (0, 1)$ be a strictly decreasing sequence of positive numbers tending to 0. By assumption, for any $n \in \mathbb{N}$, we have

$$\lim_{k \rightarrow \infty} \frac{m(E_n \cap (x_0 + J_k))}{|J_k|} = 1.$$

Thus, for any $n \in \mathbb{N}$ there exists $k_n \in \mathbb{N}$ such that for any $k \geq k_n$ we have

$$\frac{m(E_n \cap (x_0 + J_k))}{|J_k|} > 1 - \varepsilon_n.$$

Obviously, we can choose k_n (for $n \in \mathbb{N}$) such that the sequence $\{k_n\}_{n \in \mathbb{N}}$ is increasing and the sequence $\text{diam}\{\{0\} \cup J_{k_n}\}_{n \in \mathbb{N}}$ is decreasing. Consider a subsequence $\{J_{k_n}\}_{n \in \mathbb{N}}$ of the sequence $\{J_k\}_{k \in \mathbb{N}}$ and put $r_n = \varepsilon_n \cdot \text{diam}\{\{0\} \cup J_{k_{n+1}}\}$ for $n \in \mathbb{N}$. Clearly, $\lim_{n \rightarrow \infty} r_n = 0$ and the sequence $\{r_n\}_{n \in \mathbb{N}}$ is strictly decreasing.

Fix $\varepsilon > 0$. There is no loss of generality in assuming that $\text{diam}\{\{0\} \cup J_n\}_{n \in \mathbb{N}}$ is decreasing (otherwise, it is sufficient to consider the sequence $\{l_k\}_{k \in \mathbb{N}}$ of positive numbers such that $\text{diam}\{\{0\} \cup J_{l_k}\}$ is decreasing and to use Lemma 1.4). Since $\alpha(J) < \infty$, it follows that there exist $M > 0$ and $l_0 \in \mathbb{N}$ such that

$$\frac{\text{diam}(\{0\} \cup J_k)}{|J_k|} < M \quad \text{for } k > l_0. \quad (3)$$

Obviously, one can find $n_0 \in \mathbb{N}$ such that $(1 + 2M)\varepsilon_n < \varepsilon$ for $n > n_0$. Moreover, there exists $l_1 > l_0$ such that $\text{diam}\{\{0\} \cup J_k\} < \text{diam}\{\{0\} \cup J_{k_{n_0+1}}\}$ for $k > l_1$.

Fix $k > l_1$. Clearly, there exists $n_1 > n_0$ such that

$$\text{diam}(\{0\} \cup J_k) \in \left[\text{diam}(\{0\} \cup J_{k_{n_1+1}}), \text{diam}(\{0\} \cup J_{k_{n_1}}) \right). \quad (4)$$

Furthermore, the fact that the sequence $\{\text{diam}(\{0\} \cup J_k)\}_{k \in \mathbb{N}}$ is decreasing implies that $k > k_{n_1}$. Thus

$$\begin{aligned} \frac{m(A_{x_0} \cap (x_0 + J_k))}{|J_k|} &\geq \frac{m((E_{n_1} \setminus (x_0 - r_{n_1}, x_0 + r_{n_1})) \cap (x_0 + J_k))}{|J_k|} \\ &= \frac{m(E_{n_1} \cap (x_0 + J_k) \cap (x_0 - r_{n_1}, x_0 + r_{n_1})')}{|J_k|} \\ &\geq \frac{m(E_{n_1} \cap (x_0 + J_k)) - 2r_{n_1}}{|J_k|} \\ &= \frac{m(E_{n_1} \cap (x_0 + J_k))}{|J_k|} - \frac{2r_{n_1}}{|J_k|} \\ &> 1 - \varepsilon_{n_1} - \frac{2r_{n_1}}{|J_k|} \\ &= 1 - \varepsilon_{n_1} - \frac{2\varepsilon_{n_1} \cdot \text{diam}(\{0\} \cup J_{k_{n_1+1}})}{|J_k|} = *. \end{aligned}$$

From conditions (4), $k > l_1 > l_0$, and (3), we obtain that

$$* \geq 1 - \varepsilon_{n_1} - \frac{2\varepsilon_{n_1} \cdot \text{diam}(\{0\} \cup J_k)}{|J_k|} \geq 1 - \varepsilon_{n_1} - 2\varepsilon_{n_1}M = 1 - (1 + 2M)\varepsilon_{n_1} > 1 - \varepsilon.$$

Since $k > l_1$ and $\varepsilon > 0$ were fixed arbitrarily, we have that

$$\lim_{k \rightarrow \infty} \frac{m(A_{x_0} \cap (x_0 + J_k))}{|J_k|} = 1,$$

so x_0 is a $\mathcal{J}$ -density point of the set $A_{x_0} = \bigcup_{n=1}^{\infty} (E_n \setminus (x_0 - r_n, x_0 + r_n))$. $\square$

Now, we can prove the following theorem.

THEOREM 2.7. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a Lebesgue measurable function and let $\mathcal{J} \in \mathfrak{S}_\alpha$. The function f is $\mathcal{J}$ -approximately continuous at a point $x_0 \in \mathbb{R}$ if and only if the following condition is fulfilled:*

$$\forall_{\varepsilon > 0} \quad x_0 \in \Phi_{\mathcal{J}}(\{x \in \mathbb{R} : |f(x) - f(x_0)| < \varepsilon\}). \quad (5)$$

The proof of the above theorem is similar to the proof in [6, Theorem 3] (see also [8]), so the details of the proof are omitted.

Similarly as in the case of $\langle s \rangle$ -approximately continuous functions (see [6]), it is possible to show the relationship between $\mathcal{J}$ -approximate continuity of the function $f : \mathbb{R} \rightarrow \mathbb{R}$ and its Lebesgue measurability.

THEOREM 2.8. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$. The following conditions are equivalent:*

- (a) *f is a Lebesgue measurable function,*
- (b) *there exists $B \in \mathbb{L}$ such that for any sequence $\mathcal{J} \in \mathfrak{S}_\alpha$ and any $x \in \mathbb{R} \setminus B$ the function f is $\mathcal{J}$ -approximately continuous at a point x ,*
- (c) *there exists a sequence $\mathcal{J} \in \mathfrak{S}_\alpha$ and there exists $B_{\mathcal{J}} \in \mathbb{L}$ such that the function f is $\mathcal{J}$ -approximately continuous at each point $x \in \mathbb{R} \setminus B_{\mathcal{J}}$.*

Proof. We immediately see that (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c).

Now, let us assume that there exists a sequence $\mathcal{J} \in \mathfrak{S}_\alpha$ and there exists $B_{\mathcal{J}} \in \mathbb{L}$ such that the function f is $\mathcal{J}$ -approximately continuous at each point $x \in \mathbb{R} \setminus B_{\mathcal{J}}$.

Fix $\gamma \in \mathbb{R}$ and put $E_\gamma = \{x \in \mathbb{R} : f(x) < \gamma\}$. We will show that $E_\gamma \in \mathcal{L}$. Clearly,

$$E_\gamma = (E_\gamma \setminus B_{\mathcal{J}}) \cup (E_\gamma \cap B_{\mathcal{J}}) \quad \text{and} \quad E_\gamma \cap B_{\mathcal{J}} \in \mathbb{L}.$$

Moreover, for any $x \in E_\gamma \setminus B_{\mathcal{J}}$, there exists $E_x \in \mathcal{L}$ such that

$$E_x \subset E_\gamma \setminus B_{\mathcal{J}} \quad \text{and} \quad x \in \Phi_{\mathcal{J}}(E_x). \quad (6)$$

Indeed, if $x \in E_\gamma \setminus B_{\mathcal{J}}$, then $f(x) < \gamma$, and there exists $A_x \in \mathcal{L}$ such that

$$x \in \Phi_{\mathcal{J}}(A_x) \quad \text{and} \quad f(x) = \lim_{\substack{t \rightarrow x, \\ t \in A_x}} f(t).$$

Thus, one can find $\delta > 0$ such that $f(t) < \gamma$ for any $t \in A_x \cap (x - \delta, x + \delta)$. Putting $E_x = A_x \cap (x - \delta, x + \delta) \cap (\mathbb{R} \setminus B_{\mathcal{J}})$, we obtain (6).

Now, let B be a Lebesgue measurable kernel of the set $E_\gamma \setminus B_\mathcal{J}$. From (6), it follows that for any $x \in E_\gamma \setminus B_\mathcal{J}$ there exists $E_x \in \mathcal{L}$ such that $E_x \setminus B \in \mathbb{L}$, $E_x \subset (E_x \setminus B) \cup B$ and $x \in \Phi_\mathcal{J}(E_x)$. Since $(E_x \setminus B) \cup B \sim B$, we have that $\Phi_\mathcal{J}(E_x) \subset \Phi_\mathcal{J}(B)$. Therefore,

$$B \subset E_\gamma \setminus B_\mathcal{J} \subset \bigcup_{x \in E_\gamma \setminus B_\mathcal{J}} \Phi_\mathcal{J}(E_x) \subset \Phi_\mathcal{J}(B).$$

Moreover, Theorem 1.2 (6) implies that $B \sim \Phi_\mathcal{J}(B)$. Consequently, $E_\gamma \setminus B_\mathcal{J} \in \mathcal{L}$, and finally, we obtain that $E_\gamma \in \mathcal{L}$. This gives that f is a Lebesgue measurable function. $\square$

Analysis similar to that in the proof in [6, Theorem 5] shows

THEOREM 2.9. *Let $f: \mathbb{R} \rightarrow \mathbb{R}$ and $\mathcal{J} \in \mathfrak{S}_\alpha$. The function f is $\mathcal{J}$ -approximately continuous if and only if, for any $a \in \mathbb{R}$, the sets $E_a = \{x \in \mathbb{R}: f(x) < a\}$ and $E^a = \{x \in \mathbb{R}: f(x) > a\}$ belong to the topology $\mathcal{T}_\mathcal{J}$.*

Let us denote by $\mathcal{C}_{\mathcal{T}_0}^\mathcal{J}$ the family of all continuous functions $f: (\mathbb{R}, \mathcal{T}_\mathcal{J}) \rightarrow (\mathbb{R}, \mathcal{T}_0)$. Theorem 2.9 immediately implies

THEOREM 2.10. *Let $f: \mathbb{R} \rightarrow \mathbb{R}$ and $\mathcal{J} \in \mathfrak{S}_\alpha$. The function f is $\mathcal{J}$ -approximately continuous if and only if $f \in \mathcal{C}_{\mathcal{T}_0}^\mathcal{J}$.*

In view of the above theorem, $\mathcal{J}$ -approximately continuous functions will be called $\mathcal{T}_\mathcal{J}$ -continuous.

Now, we will focus on the relationship between $\mathcal{J}$ -approximately continuous functions and functions from the first class of Baire. The following, easy to prove, lemma will be useful in the proof of the next theorem.

LEMMA 2.11. *Let $\mathcal{J} \in \mathfrak{S}$. If $f: \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function and $g: \mathbb{R} \rightarrow \mathbb{R}$ is a $\mathcal{J}$ -approximately continuous function, then $f \circ g$ is $\mathcal{J}$ -approximately continuous.*

THEOREM 2.12. *Let $\mathcal{J} \in \mathfrak{S}_\alpha$. If $f: \mathbb{R} \rightarrow \mathbb{R}$ is a $\mathcal{J}$ -approximately continuous function, then f belongs to the first class of Baire.*

Proof. Let $\mathcal{J} = \{J_n\}_{n \in \mathbb{N}} \in \mathfrak{S}_\alpha$ and let a_n, b_n denote the left and the right ends of intervals J_n , respectively. Thus, $J_n = [a_n, b_n]$. Since $\mathcal{J} \in \mathfrak{S}_\alpha$ and $f: \mathbb{R} \rightarrow \mathbb{R}$ is a $\mathcal{J}$ -approximately continuous function, it follows, by Theorem 2.8, that f is a Lebesgue measurable function. Assume first that f is bounded. Thus there exists a positive number M such that $|f(x)| < M$ for $x \in \mathbb{R}$.

Let $a \in \mathbb{R}$ and $F(x) = \int_a^x f(t)dt$ for $x \in \mathbb{R}$. Obviously, $F: \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function. Consider a sequence $\{f_n\}_{n \in \mathbb{N}}$ of continuous functions defined in the following way

$$f_n(x) = \frac{1}{|J_n|} (F(x + b_n) - F(x + a_n)) \quad \text{for } n \in \mathbb{N}.$$

Now, we will show that

$$\lim_{n \rightarrow \infty} f_n(x) = f(x) \quad \text{for any } x \in \mathbb{R}. \quad (7)$$

Let $x_0 \in \mathbb{R}$ and $\varepsilon > 0$. Since f is a $\mathcal{J}$ -approximately continuous function, we obtain that there exists a set $A_{x_0} \in \mathcal{L}$ such that $x_0 \in \Phi_J(A_{x_0})$ and there exists $\delta > 0$ such that if $x \in A_{x_0} \cap (x_0 - \delta, x_0 + \delta)$, then $|f(x) - f(x_0)| < \frac{\varepsilon}{2}$. Moreover, since

$$\lim_{n \rightarrow \infty} \text{diam}(\{0\} \cup J_n) = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{m((\mathbb{R} \setminus A) \cap (x_0 + J_n))}{|J_n|} = 0,$$

we have that there exists $n_0 \in \mathbb{N}$ such that for any $n > n_0$

$$0 < \text{diam}(\{0\} \cup J_n) < \delta \quad \text{and} \quad m((x_0 + J_n) \setminus A_{x_0}) < \frac{\varepsilon}{4M} \cdot |J_n|.$$

Consequently, for any $n > n_0$, we have

$$\begin{aligned} |f_n(x_0) - f(x_0)| &= \left| \frac{1}{|J_n|} (F(x_0 + b_n) - F(x_0 + a_n)) - f(x_0) \right| \\ &= \left| \frac{1}{|J_n|} \left(\int_a^{x_0+b_n} f(t) dt - \int_a^{x_0+a_n} f(t) dt \right) - f(x_0) \right| \\ &= \left| \frac{1}{|J_n|} \left(\int_{x_0+a_n}^{x_0+b_n} f(t) dt \right) - \frac{1}{|J_n|} \left(\int_{x_0+a_n}^{x_0+b_n} f(x_0) dt \right) \right| \\ &\leq \frac{1}{|J_n|} \int_{x_0+a_n}^{x_0+b_n} |f(t) - f(x_0)| dt \\ &= \frac{1}{|J_n|} \int_{x_0+J_n} |f(t) - f(x_0)| dt \\ &= \frac{1}{|J_n|} \int_{(x_0+J_n) \cap A_{x_0}} |f(t) - f(x_0)| dt + \frac{1}{|J_n|} \int_{(x_0+J_n) \setminus A_{x_0}} |f(t) - f(x_0)| dt \\ &\leq \frac{1}{|J_n|} \cdot \frac{\varepsilon}{2} \cdot |J_n| + \frac{1}{|J_n|} \cdot \frac{\varepsilon}{4M} \cdot |J_n| \cdot 2M = \varepsilon, \end{aligned}$$

and the proof of (7) is finished. Finally, we have that f is a function of the first class of Baire.

Now, assume that f is unbounded. Obviously, $f = h^{-1} \circ (h \circ f)$ where h is a homeomorphism from $\mathbb{R}$ to $(0, 1)$. From what has already been proved and from Lemma 2.11, it may be concluded that f is a function of the first class of Baire. $\square$

3. The separation axioms

We end this paper with some information about separation axioms for a space $(\mathbb{R}, \mathcal{T}_{\mathcal{J}})$, where $\mathcal{J} \in \mathfrak{S}$. Since $\mathcal{T}_0 \subset \mathcal{T}_{\mathcal{J}}$, for any sequence $\mathcal{J} \in \mathfrak{S}$, we obtain immediately the following property.

PROPERTY 3.13. *For any $\mathcal{J} \in \mathfrak{S}$, the space $(\mathbb{R}, \mathcal{T}_{\mathcal{J}})$ is a Hausdorff space.*

Moreover, we have

THEOREM 3.14. *For any $\mathcal{J} \in \mathfrak{S}_{\alpha}$, the space $(\mathbb{R}, \mathcal{T}_{\mathcal{J}})$ is regular.*

Proof. The proof is similar in spirit to the proof in [7, Theorem 4]. Let $\mathcal{J} = \{J_n\}_{n \in \mathbb{N}} \in \mathfrak{S}_{\alpha}$ and let $x \in \mathbb{R}$. Moreover, let $V \in \mathcal{T}_{\mathcal{J}}$ be such that $x \in V$. Obviously, $V \in \mathcal{L}$ and $x \in \Phi_{\mathcal{J}}(V)$. Without loss of generality, we can assume that, for any $n \in \mathbb{N}$, there exists $\mathcal{T}_0$ -closed set F_n such that $F_n \subset V \cap (x + J_n)$ and $m(F_n) > (1 - \text{diam}(\{0\} \cup J_n))m(V \cap (x + J_n))$. Putting $F = \{x\} \cup \bigcup_{n=1}^{\infty} F_n$, we obtain that $F \subset V$ and $x \in F$. Moreover, it is easy to see that the set F is closed in the natural topology.

Now, we will show that $x \in \Phi_{\mathcal{J}}(F)$. Clearly,

$$1 \geq \frac{m(F \cap (x + J_n))}{|J_n|} \geq \frac{(1 - \text{diam}(\{0\} \cup J_n))m(V \cap (x + J_n))}{|J_n|}.$$

Since

$$\lim_{n \rightarrow \infty} \frac{m(V \cap (x + J_n))}{|J_n|} = 1 \quad \text{and} \quad \lim_{n \rightarrow \infty} (1 - \text{diam}(\{0\} \cup J_n)) = 1,$$

we have

$$\lim_{n \rightarrow \infty} \frac{m(F \cap (x + J_n))}{|J_n|} = 1,$$

so $x \in \Phi_{\mathcal{J}}(F)$.

Moreover, it is easy to show that $F \cap \Phi_{\mathcal{J}}(F) \in \mathcal{T}_{\mathcal{J}}$. From this and the fact that $x \in F \cap \Phi_{\mathcal{J}}(F) \subset \text{cl}_{\mathcal{T}_0}(F \cap \Phi_{\mathcal{J}}(F)) \subset V$, we obtain that $(\mathbb{R}, \mathcal{T}_{\mathcal{J}})$ is a regular space. $\square$

Since for any sequence $\mathcal{J} \in \mathfrak{S}_{\alpha}$, a $\mathcal{J}$ -approximately continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ belongs to the first class of Baire, we can proceed analogously to the proof in [8, Theorem 3.11] to show the following result.

THEOREM 3.15. *For any $\mathcal{J} \in \mathfrak{S}_{\alpha}$ the space $(\mathbb{R}, \mathcal{T}_{\mathcal{J}})$ is not normal.*

The question whether for any sequence $\mathcal{J} \in \mathfrak{S}_{\alpha}$ the space $(\mathbb{R}, \mathcal{T}_{\mathcal{J}})$ is completely regular, is still open. Just like the question whether for any $\mathcal{J} \in \mathfrak{S} \setminus \mathfrak{S}_{\alpha}$ the space $(\mathbb{R}, \mathcal{T}_{\mathcal{J}})$ is regular, completely regular or normal.

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