REAL FUNCTIONS AND THE EXTENSION OF GENERALIZED PROBABILITY MEASURES II

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ABSTRACT. We continue our study of the extensions of generalized probability measures. First, we describe some extensions of generalized random events (represented by classes of functions with values in $[0,1]$) to which generalized probability measures can be extended. Second, we study products of domains of probability and describe states on such products. Third, we show that the events in IF-probability, introduced by B. Riečan, form a suitable category isomorphic to a subcategory of the category of fuzzy random events. Consequently, IF-probability can be interpreted within fuzzy probability theory. We put forward some problems related to the extensions of probability domains and hint some applications.

1. Introduction

Following J. Novák [22]–[24], using some recent results on generalized random events and states [2], [3], [20], [21], [30] and basic categorical methods [1], [15], in [19] we have studied the process of extending generalized probability measures.

In the second section, we present more results related to the extension process. The last two sections are devoted to the relationships between IF-probability, introduced by B. Riečan in [31], and fuzzy probability [8], [18], [34].

Relevant notions concerning generalized random events and generalized probability measures (states) are recalled in the text. Further, if $\mathbf{A}$ is a σ -field of sets, then $\mathcal{M}(\mathbf{A})$ denotes the class of all $\mathbf{A}$ -measurable functions whose values are in $[0,1]$. As usually, $\exists!$ means “there exists a unique”.

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D-posets have been introduced in [20] in order to model events in quantum probability. They generalize Boolean algebras, MV-algebras and other probability domains, and provide a category in which observables and states become morphisms [2], [11]. Recall that a D-poset is a partially ordered set with the greatest element 1_X , the least element 0_X , and a partial binary operation called **difference**, such that $a \ominus b$ is defined iff $b \leq a$, and the following axioms are assumed:

(D1) $a \ominus 0_X = a$ for each $a \in X$;

(D2) If $c \leq b \leq a$, then $a \ominus b \leq a \ominus c$ and $(a \ominus c) \ominus (a \ominus b) = b \ominus c$.

A canonical example is the interval $[0, 1] = I$ (linearly ordered, $a \ominus b = a - b$ whenever $b \leq a$). D-posets of fuzzy sets are fundamental to applications [16], [17], i.e., systems $\mathcal{X} \subseteq I^X$ carrying the coordinatewise partial order, coordinatewise convergence of sequences, containing the top and bottom elements of I^X , and closed with respect to the partial operation difference defined coordinatewise. We always assume that $\mathcal{X}$ is **reduced**, i.e., for $x, y \in X$, $x \neq y$, there exists $u \in \mathcal{X}$ such that $u(x) \neq u(y)$. Denote **ID** the category having (reduced) D-posets of fuzzy sets as objects and having sequentially continuous D-homomorphisms (preserving constants, order, and the difference) as morphisms. Objects of **ID** are subobjects of the powers I^X .

2. Extensions

Extensions of D-posets of fuzzy sets have been studied in [9], [11]–[13], [19], [26]. Essentially, there are two types of extensions. First, we can add functions, and then, we speak of an **ID-extension** (cf. [19]). Second, we can extend the domain of functions and such extensions have stochastic applications, e.g., when studying the duality between generalized random variables and observables [5], [7], [8], [10], [14]–[16], [26].

Let $\mathcal{X} \subseteq I^X$ be a D-poset of fuzzy sets. Denote $\mathcal{S}(\mathcal{X})$ the set of all sequentially continuous D-homomorphisms of $\mathcal{X}$ into I ; the elements of $\mathcal{S}(\mathcal{X})$ are called **states**. In what follows, each $x \in X$ will be considered as the **evaluation state** on $\mathcal{X}$: $x(u) = u(x)$, $u \in \mathcal{X}$. If $X = \mathcal{S}(\mathcal{X})$, then $\mathcal{X}$ is said to be **sober** (cf. [26]). Let Y be a set of states such that $X \subseteq Y \subseteq \mathcal{S}(\mathcal{X})$. For $u \in \mathcal{X}$, put $ev_Y(u) = \{y(u); y \in Y\} \in I^Y$ and by ev_Y denote the corresponding map of $\mathcal{X}$ into I^Y . Put $\mathcal{X}_Y = \{ev_Y(u); u \in \mathcal{X}\}$. A trivial proof of the next lemma is omitted.

LEMMA 2.1.

- (i) $\mathcal{X}_Y$ is a D-poset of fuzzy sets (with respect to the D-poset structure inherited from I^Y),
- (ii) ev_Y is an isomorphism.

DEFINITION 2.2. Let $\mathcal{X} \subseteq I^X$ be a D-poset of fuzzy sets and let Y be a set of states such that $X \subseteq Y \subseteq \mathcal{S}(\mathcal{X})$. Then, $\mathcal{X}_Y$ is said to be a **domain extension** of $\mathcal{X}$. If $Y = \mathcal{S}(\mathcal{X})$, then $\mathcal{X}_Y$ is said to be the **sobrification** of $\mathcal{X}$.

In accordance with [26], for $\mathcal{X} \subseteq I^X$ and $Y = \mathcal{S}(\mathcal{X})$, the sobrification $\mathcal{X}_Y$ of $\mathcal{X}$ will be denoted by $\mathcal{X}^*$. Denote **SID** the (full) subcategory of **ID** consisting of sober D-posets of fuzzy sets.

LEMMA 2.3.

- (i) *Let $\mathcal{X} \subseteq I^X$ be a D-poset of fuzzy sets and let h be a sequentially continuous D-homomorphism of $\mathcal{X}$ into a sober D-poset of fuzzy sets $\mathcal{Z}$. Then, there exists a unique sequentially continuous D-homomorphism h^* of $\mathcal{X}^* = \mathcal{X}_{\mathcal{S}(\mathcal{X})}$ into $\mathcal{Z}$ such that $h = h^* \circ ev_{\mathcal{S}(\mathcal{X})}$.*
- (ii) *Let h be a sequentially continuous D-homomorphism of a D-poset of fuzzy sets $\mathcal{X}$ into a D-poset of fuzzy sets $\mathcal{Y}$. Then, there exists a unique sequentially continuous D-homomorphism h^* of $\mathcal{X}^*$ into $\mathcal{Y}^*$ such that $h^* \circ ev_{\mathcal{S}(\mathcal{X})} = h \circ ev_{\mathcal{S}(\mathcal{Y})}$.*

Proof. (i) By the previous lemma, $\mathcal{X}$ and $\mathcal{X}^*$ are isomorphic. Define h^* by putting $h^*(u^*) = h(u)$, $u \in \mathcal{X}$. Clearly, h^* is a sequentially continuous D-homomorphism h^* of $\mathcal{X}^*$ into $\mathcal{Z}$. We have to prove the uniqueness of h^* . So, let g be a sequentially continuous D-homomorphism of $\mathcal{X}^*$ into $\mathcal{Z}$ such that $h = g \circ ev_{\mathcal{S}(\mathcal{X})}$. Then

$$g(u^*) = h(u) = h^*(u^*) \quad \text{for all } u^* \in \mathcal{X}^*.$$

Hence $g = h^*$ and (i) follows.

(ii) Follows from (i). □

THEOREM 2.4. *The subcategory **SID** of **ID** is epireflective in **ID**.*

Proof. The assertion follows from the pre previous lemma. □

Note that the sobrification sending a D-poset of fuzzy sets $\mathcal{X}$ to $\mathcal{X}^*$ yields a functor (epireflector) sending a morphism $h: \mathcal{X} \rightarrow \mathcal{Y}$ to $h^*: \mathcal{X}^* \rightarrow \mathcal{Y}^*$.

EXAMPLE 2.5. For $X = \{0, 1\}$, let $\mathcal{X} \subseteq I^X$ be the σ -field of all subsets of X . The set $\mathcal{S}(\mathcal{X})$ of all states of $\mathcal{X}$ is the same as the set of all probability measures on the σ -field in question [26]. Each $s \in \mathcal{S}(\mathcal{X})$ can be visualized as an element a_s of the closed unit interval $[0, 1]$, where the number a_s is equal to the corresponding probability of the singleton $\{0\}$ and $(1 - a_s)$ is equal to the corresponding probability of the singleton $\{1\}$; we identify s and the corresponding number a_s and, consequently, we identify $\mathcal{S}(\mathcal{X})$ and $[0, 1]$. Further, the sobrification $\mathcal{X}^*$ can be visualized as the D-poset $\mathcal{X}_{\mathcal{S}(\mathcal{X})} \subseteq I^{[0,1]}$ consisting of the following four functions: constant functions $0_{[0,1]}$ and $1_{[0,1]}$, functions x and $1 - x$, $x \in [0, 1]$.

Now, we turn to ID-extensions. Even though the embedding of a field of sets $\mathbf{A}$ into the generated σ -field $\sigma(\mathbf{A})$ is well-known, there are still some unanswered related questions. We present some motivating examples and, in order to describe additional properties of extensions, we introduce some new notions.

Let $\mathcal{X} \subseteq I^X$ be a D-poset of fuzzy sets and let $\mathcal{Y} \subseteq I^X$ be an ID-extension of $\mathcal{X}$. Recall (cf. [19]) that if each state on $\mathcal{X}$ can be extended to a state on $\mathcal{Y}$, i.e., $(\forall s \in \mathcal{S}(\mathcal{X}))(\exists t \in \mathcal{S}(\mathcal{Y}))[t \upharpoonright \mathcal{X} = s]$, then $\mathcal{X}$ is said to be **$\mathcal{S}(\mathcal{X})$ -embedded** in $\mathcal{Y}$.

Let $\mathcal{X} \subseteq I^X$ be a D-poset of fuzzy sets and let $\mathcal{Y} \subseteq I^X$ be an ID-extension of $\mathcal{X}$. Let t be a state on $\mathcal{Y}$ and let $t \upharpoonright \mathcal{X}$ be the restriction of t to $\mathcal{X}$. Since $t \upharpoonright \mathcal{X} \in \mathcal{S}(\mathcal{X})$, the restriction yields a **restriction map** r of $\mathcal{S}(\mathcal{Y})$ into $\mathcal{S}(\mathcal{X})$ sending t to $r(t) = t \upharpoonright \mathcal{X}$.

DEFINITION 2.6. Let $\mathcal{X} \subseteq I^X$ be a D-poset of fuzzy sets and let $\mathcal{Y} \subseteq I^X$ be an ID-extension of $\mathcal{X}$. If the restriction map $r : \mathcal{S}(\mathcal{Y}) \rightarrow \mathcal{S}(\mathcal{X})$ is one-to-one and onto, then $\mathcal{Y}$ is said to be a **state extension** of $\mathcal{X}$. A state extension $\mathcal{Y}$ of $\mathcal{X}$ is said to be **maximal** if there is no proper state extension $\mathcal{Z} \subseteq I^X$ of $\mathcal{Y}$. Let $\mathcal{C}$ be a class of state extensions of $\mathcal{X}$ and let $\mathcal{Y} \in \mathcal{C}$. If there is no proper extension of $\mathcal{X}$ in $\mathcal{C}$, then $\mathcal{Y}$ is said to be **maximal in $\mathcal{C}$** . If $\mathcal{Y}$ is a state extension of $\mathcal{X}$ and $\mathcal{Y}$ is an ID-extension of each state extension $\mathcal{Z}$ of $\mathcal{X}$, then $\mathcal{Y}$ is said to be the **absolute state extension** of $\mathcal{X}$. If $\mathcal{Y} \in \mathcal{C}$ and $\mathcal{Y}$ is an ID-extension of each state extension $\mathcal{Z}$ of $\mathcal{X}$ in $\mathcal{C}$, then $\mathcal{Y}$ is said to be the **absolute state extension of $\mathcal{X}$ in $\mathcal{C}$** .

LEMMA 2.7. Let $\mathcal{X} \subseteq I^X$ be a D-poset of fuzzy sets and let $\mathcal{Y} \subseteq I^X$ be an ID-extension of $\mathcal{X}$. The following are equivalent:

- (i) $\mathcal{Y}$ is a state extension of $\mathcal{X}$;
- (ii) $(\forall s \in \mathcal{S}(\mathcal{X}))(\exists ! t \in \mathcal{S}(\mathcal{Y}))[t \upharpoonright \mathcal{X} = s]$;
- (iii) $(\forall t \in \mathcal{S}(\mathcal{Y}))(\exists ! s \in \mathcal{S}(\mathcal{X}))[t \upharpoonright \mathcal{X} = s]$.

Proof. The proof is straightforward and is left out. □

EXAMPLE 2.8. Consider $\mathcal{X}$ and its sobrification $\mathcal{X}^* \subseteq I^{[0,1]}$ in the previous example. Let $\mathbf{B}$ be the σ -field of all Borel measurable subsets of $[0, 1]$ and let $\mathcal{M}(\mathbf{B})$ be the set of all $\mathbf{B}$ -measurable functions with values in I . Then, $\mathcal{M}(\mathbf{B})$ can be considered as an ID-extension of $\mathcal{X}^*$. Clearly, $\mathcal{X}^*$ is $\mathcal{S}(\mathcal{X}^*)$ -embedded in $\mathcal{S}(\mathcal{M}(\mathbf{B}))$. On the other hand, the restriction map r of $\mathcal{S}(\mathcal{M}(\mathbf{B}))$ into $\mathcal{S}(\mathcal{X}^*)$ is far from being one-to-one. Hence, $\mathcal{M}(\mathbf{B})$ fails to be a state extension of $\mathcal{X}^*$.

Let X be a set, let $\mathbf{A}$ be a field of subsets of X , let $\sigma(\mathbf{A})$ be the generated σ -field, and let $\mathbf{M}_{\mathbf{A}}$ be the σ -field of all absolutely $\mathbf{A}$ -measurable subsets of X ($M \subseteq X$ belongs to $\mathbf{M}_{\mathbf{A}}$ if and only if it is p -measurable for all $p \in \mathcal{P}(\mathbf{A})$, see [19]). Both $\sigma(\mathbf{A})$ and $\mathbf{M}_{\mathbf{A}}$ can be considered as ID-extensions of $\mathbf{A}$. A natural question arises what can be said about such extensions.

EXAMPLE 2.9. Let $\mathbf{A}$ be a field of subsets of X , let $\sigma(\mathbf{A})$ be the generated σ -field, and let $\mathbf{P}(X)$ be the σ -field of all subsets of X . For a system $\mathbf{B}$ of subsets of X , define $\text{cl}(\mathbf{B}) \subseteq \mathbf{P}(X)$ as the set of all sequential limits of sequences of sets in $\mathbf{B}$ and, inductively, for each ordinal number $\alpha \leq \omega_1$ define $\text{cl}^0(\mathbf{B}) = \mathbf{B}$, $\text{cl}^\alpha(\mathbf{B}) = \text{cl}(\text{cl}^{\alpha-1}(\mathbf{B}))$ for an isolated ordinal number, $\text{cl}^\alpha(\mathbf{B}) = \text{cl}(\bigcup_{\beta < \alpha} \text{cl}^\beta(\mathbf{B}))$ for a limit ordinal number; it is known (cf. [25]) that each cl^α is a closure operator (need not be idempotent), cl^{ω_1} is a topological closure operator (topology), each $\text{cl}^\alpha(\mathbf{A})$ is a field of subsets of X , and $\text{cl}^{\omega_1}(\mathbf{A}) = \sigma(\mathbf{A})$. If $\text{cl}^\alpha(\mathbf{B}) = \mathbf{C}$ for some ordinal number α , then $\mathbf{B}$ is said to be sequentially dense in $\mathbf{C}$. For each ordinal number $\alpha \leq \omega_1$, put $\mathbf{A}_\alpha = \text{cl}^\alpha(\mathbf{A})$. Then (see in [19, Example 4.2]), each $\mathbf{A}_\alpha$ is a state extension of $\mathbf{A}$. For $\mathcal{C} = \{\mathbf{A}_\alpha; \alpha \leq \omega_1\}$, $\mathbf{A}_{\omega_1} = \sigma(\mathbf{A})$ is the absolute state extension of $\mathbf{A}$ in $\mathcal{C}$.

EXAMPLE 2.10. Let $\mathbf{A}$ be a field of subsets of X and let $\mathbf{M}_\mathbf{A}$ be the σ -field of all absolutely $\mathbf{A}$ -measurable subsets of X . In [19, Example 4.3], we have erroneously claimed that “ $\mathbf{M}_\mathbf{A}$ is the largest σ -field of subsets of X in which $\mathbf{A}$ is $\mathcal{P}(\mathbf{A})$ -embedded”. For sure, $\mathbf{M}_\mathbf{A}$ is the largest σ -field of subsets of X to which each probability measure on $\mathbf{A}$ has a measurable extension. Hence, $\mathcal{P}(\mathbf{A})$ -embeddedness has to be replaced by a more restrictive assumption.

PROBLEM 1. Let $\mathbf{A}$ be a field of subsets of X and let $\mathbf{M}_\mathbf{A}$ be the σ -field of all absolutely $\mathbf{A}$ -measurable subsets of X . Is $\mathbf{M}_\mathbf{A}$ a state extension of $\mathbf{A}$?

Remark 2.11. Observe that a positive solution to Problem 1 implies that the extension $\mathbf{A} \hookrightarrow \mathbf{M}_\mathbf{A}$ yields an epireflection. Indeed, if each probability measure on $\mathbf{A}$ can be uniquely extended to a probability measure on $\mathbf{M}_\mathbf{A}$, then the corresponding epireflector can be constructed by standard categorical methods.

PROBLEM 2. Characterize fields of sets $\mathbf{A}$ for which $\mathbf{M}_\mathbf{A}$ is a state extension.

PROBLEM 3. Let $\mathcal{X} \subseteq I^X$ be a D-poset of fuzzy sets. Can $\mathcal{X}$ have different maximal state extensions?

PROBLEM 4. Characterize D-posets of fuzzy sets having state extensions (absolute state extensions, or absolute state extensions in a nontrivial class of state extensions).

Since, in generalized probability theory, bold algebras provide a natural model of generalized random events, we will study ID-extensions which are bold algebras.

Recall [3] that a **bold algebra** is a system $\mathcal{X} \subseteq [0, 1]^X$ containing the constant functions $0_X, 1_X$ and closed with respect to the usual (Łukasiewicz) operations: for $u, v \in \mathcal{X}$ put

$$(u \oplus v)(x) = u(x) \oplus v(x) = \min\{1, u(x) + v(x)\}, \quad u^*(x) = 1 - u(x), \quad x \in X.$$

Bold algebras are MV-algebras representable as $[0, 1]$ -valued functions, MV-algebras generalize Boolean algebras and bold algebras generalize in a natural way fields of sets (viewed as indicator functions). More information concerning MV-algebras and probability on MV-algebras can be found in [29]. If a bold algebra $\mathcal{X} \subseteq [0, 1]^X$ is sequentially closed in $[0, 1]^X$ (with respect to the coordinatewise sequential convergence), then $\mathcal{X}$ is a **Lukasiewicz tribe** ($\mathcal{X}$ is closed not only with respect to finite, but also with respect to countable Lukasiewicz sums, cf. [6, Corollary 2.8]). It is known (cf. [3]) that if $\mathcal{X} \subseteq [0, 1]^X$ is a Lukasiewicz tribe, then there exists a unique σ -field $\mathbf{A}_{\mathcal{X}}$ of subsets of X such that $\mathbf{A}_{\mathcal{X}} \subseteq \mathcal{X} \subseteq \mathcal{M}(\mathbf{A}_{\mathcal{X}})$ and $\mathcal{X} = \mathcal{M}(\mathbf{A}_{\mathcal{X}})$ if and only if $\mathcal{X}$ contains all constant functions r_X , $r \in [0, 1]$. Let $\mathcal{X} \subseteq [0, 1]^X$ be a bold algebra. Then, $[0, 1]^X$ is a Lukasiewicz tribe containing $\mathcal{X}$ and the intersection of all Lukasiewicz tribes $\mathcal{Y} \subseteq [0, 1]^X$ such that $\mathcal{X} \subseteq \mathcal{Y}$ is a Lukasiewicz tribe; it will be denoted by $\sigma(\mathcal{X})$. Lukasiewicz tribes of the form $\mathcal{M}(\mathbf{A}_{\sigma(\mathcal{X})})$ are called **generated**. **Each bold algebra $\mathcal{X} \subseteq [0, 1]^X$ is a lattice** where, for $u, v \in \mathcal{X}$, we have $(u \vee v)(x) = u(x) \vee v(x)$, $(u \wedge v)(x) = u(x) \wedge v(x)$, $x \in X$, and **each ID-poset $\mathcal{X} \subseteq [0, 1]^X$ which is a lattice (with respect to the coordinatewise order) can be reorganized into a bold algebra** (the Lukasiewicz operations can be redefined via the difference and order, cf. [16], [28]). Finally, each bold algebra can be considered as an object of **ID**. The corresponding subcategory is denoted by **BID** and **CGBID** denotes the subcategory of generated Lukasiewicz tribes.

Let $\mathcal{X} \subseteq I^X$ be a D-poset of fuzzy sets. Clearly, the intersection of all bold algebras $\mathcal{Y} \subseteq I^X$ such that $\mathcal{Y}$ is an ID-extension of $\mathcal{X}$ is the smallest ID-extension of $\mathcal{X}$ which is a bold algebra; denote it by $b(\mathcal{X})$. Below we also give an inductive construction of $b(\mathcal{X})$.

For $u, v \in I^X$, define pointwise binary operations $u \vee_p v$ and $u \wedge_p v$ as follows: $(u \vee_p v)(x) = u(x) \vee v(x)$, $(u \wedge_p v)(x) = u(x) \wedge v(x)$, $x \in X$. For $u, v \in I^X$, $v \leq u$, define pointwise a partial operation $u - v$: $(u - v)(x) = u(x) - v(x)$, $x \in X$.

For $n = 0, 1, 2, \dots$, define $b_0(\mathcal{X}) = \mathcal{X}$, $b_n(\mathcal{X}) = \{u \vee_p v; u, v \in b_{n-1}(\mathcal{X})\} \cup \{u \wedge_p v; u, v \in b_{n-1}(\mathcal{X})\}$ for n odd, $b_n(\mathcal{X}) = \{u - v; u, v \in b_{n-1}(\mathcal{X}), v \leq u\}$ for n even.

LEMMA 2.12. $b(\mathcal{X}) = \bigcup_{n=0}^{\infty} b_n(\mathcal{X})$.

Proof. It is easy to see that $\bigcup_{n=0}^{\infty} b_n(\mathcal{X}) \subseteq I^X$ contains the constant functions $0_X, 1_X$ and it is closed with respect to $u \vee_p v$, $u \wedge_p v$ and the partial difference. Indeed, if $u, v \in b(\mathcal{X})$, then there exists a natural number n such that $u, v \in b_n(\mathcal{X})$. Hence, $(u \vee_p v), (u \wedge_p v) \in b_{n+2}(\mathcal{X})$. If $v \leq u$, then also $(u - v) \in b_{n+2}(\mathcal{X})$. Thus, $\bigcup_{n=0}^{\infty} b_n(\mathcal{X})$ is a bold algebra. Let $\mathcal{Y} \subseteq I^X$ be an ID-extension of $\mathcal{X}$ and assume that $\mathcal{Y}$ is a bold algebra. Then, $\mathcal{Y}$ contains the constant functions $0_X, 1_X$ and it is closed with respect to $u \vee_p v$, $u \wedge_p v$ and the partial difference. Thus, $\bigcup_{n=0}^{\infty} b_n(\mathcal{X}) \subseteq \mathcal{Y}$ and the proof is complete. $\square$

LEMMA 2.13. *Let $\mathcal{X} \subseteq I^X$ be a sober D-poset of fuzzy sets. Then, $\mathcal{X}$ is $\mathcal{S}(\mathcal{X})$ -embedded in $b(\mathcal{X})$.*

Proof. Let s be a state on $\mathcal{X}$. Since $\mathcal{X}$ is sober, there exists a unique $x \in X$ such that s is the evaluation state s_x . Then the x th projection pr_x of I^X to the x th factor I_x restricted to $\mathcal{X}$ is equal to s and the restriction $pr_x \upharpoonright b(\mathcal{X})$ is a state on $b(\mathcal{X})$ which extends s . $\square$

3. Products

The random events in IF-probability [31] are pairs (u, v) of fuzzy random events, i.e., elements of probability domains of the form $\mathcal{M}(\mathbf{A}) \times \mathcal{M}(\mathbf{A})$ such that $u \leq v$, and some relevant constructions in the IF-probability can be carried out (cf. [31], [32]) as the corresponding constructions in fuzzy probability theory (via $\mathcal{M}(\mathbf{A}) \times \mathcal{M}(\mathbf{A})$). This section is devoted to products of D-posets of fuzzy sets and some of the results will be used in the next section.

Let $\mathcal{X} \subseteq I^X$ and $\mathcal{Y} \subseteq I^Y$ be D-posets of fuzzy sets. Recall that (see [26], [27]), along with the two natural projections, the product $\mathcal{X} \times \mathcal{Y}$ consists of all pairs (u, v) , $u \in \mathcal{X}$, $v \in \mathcal{Y}$, where the ID-structure (partial order, difference, and convergence) is defined coordinatewise; observe that if Z is the disjoint union of X and Y (their coproduct in the category of sets and maps) then each (u, v) can be visualized as a function w on Z , where u and v are “disjointly glued” to form w . Let T be a (nonempty) set and let $\{\mathcal{X}_t; t \in T\}$ be an indexed family of D-posets of fuzzy sets. As before, their product $\prod_{t \in T} \mathcal{X}_t$, along with the indexed family $\{pr_t; t \in T\}$ of natural projections $pr_s : \prod_{t \in T} \mathcal{X}_t \rightarrow \mathcal{X}_s$, $s \in T$, consists of the indexed family $\{u_t; u_t \in \mathcal{X}_t, t \in T\}$, equipped with the coordinatewise ID-structure.

Further, if $\mathcal{X}_t$, $t \in T$, are bold algebras, then the Łukasiewicz operations $\oplus, \odot, (\cdot)^c$ on their product are defined pointwise and $\prod_{t \in T} \mathcal{X}_t$ becomes the product bold algebra.

Let $n > 1$ be a natural number. Denote $\mathbf{BID}_n$ the category of products bold algebras of the form $\prod_{i=1}^n \mathcal{X}_i$ as objects and n -tuples of sequentially continuous D-homomorphisms as morphisms, i.e., if $h_i : \mathcal{X}_i \rightarrow \mathcal{Y}_i$, $i = 1, 2, \dots, n$, are sequentially continuous D-homomorphisms, then $(h_1, h_2, \dots, h_n)$ is the corresponding morphism on $\prod_{i=1}^n \mathcal{X}_i$ into $\prod_{i=1}^n \mathcal{Y}_i$ in $\mathbf{BID}_n$ and the composition of two n -tuples $(h_1, h_2, \dots, h_n)$ and $(g_1, g_2, \dots, g_n)$ is defined coordinatewise, i.e., if the compositions $g_i \circ h_i$, $i = 1, 2, \dots, n$, are defined, then $(g_1, g_2, \dots, g_n) \circ (h_1, h_2, \dots, h_n) = (g_1 \circ h_1, g_2 \circ h_2, \dots, g_n \circ h_n)$. Recall that a generated Łukasiewicz tribe is a bold algebra $\mathcal{M}(\mathbf{A})$ of all measurable $[0, 1]$ -valued functions with respect to a σ -algebra $\mathbf{A}$ of subsets. Denote $\mathbf{CGBID}_n$ the (full) subcategory of $\mathbf{BID}_n$

the object of which are the products of generated Łukasiewicz tribes of the form $\prod_{i=1}^n \mathcal{M}(\mathbf{A}_i)$.

Let $n > 1$ be a natural number, let $\mathcal{X}_i$, $i = 1, 2, \dots, n$, be D-posets of fuzzy sets, and let $\mathcal{X} = \prod_{i=1}^n \mathcal{X}_i$ be their product. Recall that a sequentially continuous D-homomorphism on $\mathcal{X}$ into I is said to be a state. In this section, we generalize some results on states on products in [33].

Let $n > 1$ be a natural number, let $\mathcal{X}_i$, $i = 1, 2, \dots, n$, be D-posets of fuzzy sets, and let $\mathcal{X} = \prod_{i=1}^n \mathcal{X}_i$ be their product. Let h_i , $i = 1, 2, \dots, n$, be a sequentially continuous D-homomorphism of $\mathcal{X}_i$ into a generated Łukasiewicz tribe $\mathcal{M}(\mathbf{A}) \subseteq I^Y$, and let $w_i \in \mathcal{M}(\mathbf{A})$, $i = 1, 2, \dots, n$, be functions such that $\sum_{i=1}^n w_i(y) = 1$ for all $y \in Y$. For $u = (u_1, \dots, u_n) \in \mathcal{X}$, put $h(u_1, \dots, u_n) = \sum_{i=1}^n w_i h_i(u_i)$; clearly, $h(u_1, \dots, u_n) \in \mathcal{M}(\mathbf{A})$. Denote $h = \sum_{i=1}^n w_i h_i$ the resulting map of $\mathcal{X}$ into $\mathcal{M}(\mathbf{A})$.

The proof of the next lemma is straightforward and is omitted.

LEMMA 3.1. *The map h is a sequentially continuous D-homomorphism of $\mathcal{X}$ into $\mathcal{M}(\mathbf{A})$.*

DEFINITION 3.2. Let $n > 1$ be a natural number, let $\mathcal{X}_i$, $i = 1, 2, \dots, n$, be D-posets of fuzzy sets, and let $\mathcal{X}$ be their product. Let h_i , $i = 1, 2, \dots, n$, be a sequentially continuous D-homomorphism of $\mathcal{X}_i$ into a generated Łukasiewicz tribe $\mathcal{M}(\mathbf{A}) \subseteq I^Y$, and let $w_i \in \mathcal{M}(\mathbf{A})$, $i = 1, 2, \dots, n$, be functions such that $\sum_{i=1}^n w_i(y) = 1$ for all $y \in Y$. Then, $h = \sum_{i=1}^n w_i h_i$ is said to be a **convex combination** of h_i , $i = 1, 2, \dots, n$.

Let $\{a\}$ be a singleton and let $\mathbf{T} = \{\emptyset, \{a\}\}$ be the trivial σ -field of subsets of $\{a\}$. As usual, we identify I and $\mathcal{M}(\mathbf{T})$. Observe that the convex combination of sequentially continuous D-homomorphisms into $I = \mathcal{M}(\mathbf{T})$ is a state.

DEFINITION 3.3. Let $n > 1$ be a natural number, let $\mathcal{X}_i$, $i = 1, 2, \dots, n$, be D-posets of fuzzy sets, and let $\mathcal{X}$ be their product. Let h be a sequentially continuous D-homomorphism of $\mathcal{X}$ into a generated Łukasiewicz tribe $\mathcal{M}(\mathbf{A}) \subseteq I^Y$. For $i \in \{1, 2, \dots, n\}$, if $h(0_1, \dots, 0_{i-1}, 1_i, 0_{i+1}, \dots, 0_n) = w$, and $w(y) > 0$ for all $y \in Y$, then we say that h **strongly depends on the i th coordinate**.

THEOREM 3.4. *Let $n > 1$ be a natural number, let $\mathcal{X}_i$, $i = 1, 2, \dots, n$, be D-posets of fuzzy sets, let $\mathcal{X}$ be their product, and let h be a sequentially continuous D-homomorphism of $\mathcal{X}$ into a generated Łukasiewicz tribe $\mathcal{M}(\mathbf{A}) \subseteq I^Y$ which strongly depends on all coordinates. Then, there are unique sequentially continuous D-homomorphisms $h_i : \mathcal{X}_i \rightarrow \mathcal{M}(\mathbf{A})$, and unique functions $w_i \in \mathcal{M}(\mathbf{A})$ such that $\sum_{i=1}^n w_i(y) = 1$ for all $y \in Y$, $i = 1, 2, \dots, n$, and $h = \sum_{i=1}^n w_i h_i$.*

PROOF. For $i \in \{1, 2, \dots, n\}$, put $w_i = h(0_1, \dots, 0_{i-1}, 1_i, 0_{i+1}, \dots, 0_n)$. Then $0 < w_i(y)$ and $1 = h(1_1, 1_2, \dots, 1_n)(y) = \sum_{i=1}^n h(0_1, \dots, 0_{i-1}, 1_i, 0_{i+1}, \dots, 0_n)(y)$ for all $y \in Y$, $i \in \{1, 2, \dots, n\}$. Clearly, for each $i \in \{1, 2, \dots, n\}$,

$$h_i(u) = h(0_1, \dots, 0_{i-1}, u, 0_{i+1}, \dots, 0_n)/w_i, \quad u \in \mathcal{X}_i,$$

defines a sequentially continuous D-homomorphism $h_i: \mathcal{X}_i \rightarrow \mathcal{M}(\mathbf{A})$. Further, we have to prove

$$h(u_1, \dots, u_n) = \sum_{i=1}^n w_i h_i(u_i), \quad \text{for all } (u_1, \dots, u_n) \in \prod_{i=1}^n \mathcal{X}_i.$$

It is easy to see that

$$h(u_1, \dots, u_n) = \sum_{i=1}^n w_i h(0_1, \dots, 0_{i-1}, u_i, 0_{i+1}, \dots, 0_n)/w_i$$

and the assertion follows. An analogous argument shows that if $h = \sum_{i=1}^n v_i g_i$, then $w_i = v_i$, and $h_i = g_i$ for all $i \in \{1, 2, \dots, n\}$.

Observe that Theorem 3.4 generalizes Theorem 3.5 in [33]. Indeed, if $\mathbf{A}$ is the trivial σ -field of subsets of a singleton, then $\mathcal{M}(\mathbf{A}) = I$, the homomorphism h into $\mathcal{M}(\mathbf{A})$ becomes a state, and the strong dependence of h amounts to the dependence of h .

In [13], it is proved that **CGBID** is an epireflective subcategory of **BID**. This is a positive solution to Problem 2 posed in [19]. Let $\mathcal{X} \subseteq I^X$ be a bold algebra. Then, $\mathcal{M}(\mathbf{A}_{\sigma(\mathcal{X})})$ is its epireflection. In connection with products of probability domains, it is natural to ask whether the epireflection is productive. In general topology, the question whether an epireflector like the Čech-Stone compactification, the Hewitt realcompactification, etc. is productive, is rather hard. For sequential envelopes, it is proved [4] that the corresponding epireflector is productive. We have proved that the reflector sending $\mathcal{X}$ to $\mathcal{M}(\mathbf{A}_{\sigma(\mathcal{X})})$ is productive, too. $\square$

THEOREM 3.5. *Let $\mathcal{X} \subseteq I^X$, $\mathcal{Y} \subseteq I^Y$ be bold algebras and let $\mathcal{X} \times \mathcal{Y}$ be their product. Then, $\mathcal{M}(\mathbf{A}_{\sigma(\mathcal{X})}) \times \mathcal{M}(\mathbf{A}_{\sigma(\mathcal{Y})})$ is the epireflection of $\mathcal{X} \times \mathcal{Y}$ into the subcategory **CGBID** of **BID**.*

PROOF. Let $\mathbf{A}$ be a σ -field of sets and let h be a sequentially continuous D-homomorphism on $\mathcal{X} \times \mathcal{Y}$ into $\mathcal{M}(\mathbf{A})$. First, we have to prove that h can be uniquely extended to a sequentially continuous D-homomorphism on $\mathcal{M}(\mathbf{A}_{\sigma(\mathcal{X})}) \times \mathcal{M}(\mathbf{A}_{\sigma(\mathcal{Y})})$ into $\mathcal{M}(\mathbf{A})$. Let Z be the disjoint union of X , and Y and let $\mathcal{Z} \subseteq I^Z$ be the corresponding “disjoint union” of $\mathcal{X}$ and $\mathcal{Y}$. Then, we can identify $\mathcal{X} \times \mathcal{Y}$ and $\mathcal{Z}$, or more precisely, $\mathcal{M}(\mathbf{A}_{\sigma(\mathcal{X})}) \times \mathcal{M}(\mathbf{A}_{\sigma(\mathcal{Y})})$ and $\mathcal{M}(\mathbf{A}_{\sigma(\mathcal{Z})})$. Since $\mathcal{M}(\mathbf{A}_{\sigma(\mathcal{Z})})$ is the epireflection of $\mathcal{Z}$, the assertion follows. Second, the embedding of $\mathcal{X} \times \mathcal{Y}$ into $\mathcal{M}(\mathbf{A}_{\sigma(\mathcal{X})}) \times \mathcal{M}(\mathbf{A}_{\sigma(\mathcal{Y})})$ is an epimorphism, the proof is complete.

COROLLARY 3.6. *The epireflector sending objects of **BID** into **CGBID** is productive.*

COROLLARY 3.7. *Let $\mathcal{X} \subseteq I^X$, $\mathcal{Y} \subseteq I^Y$ be bold algebras and let $\mathcal{X} \times \mathcal{Y}$ be their product. Let h be a sequentially continuous D -homomorphism on $\mathcal{X} \times \mathcal{Y}$ into I . Then h can be uniquely extended to a sequentially continuous D -homomorphism on $\mathcal{M}(\mathbf{A}_{\sigma(\mathcal{X})}) \times \mathcal{M}(\mathbf{A}_{\sigma(\mathcal{Y})})$ into I .*

4. Applications

4.1. IF-probability

This subsection is devoted to relationships between IF-probability [31]–[33] and fuzzy probability [15]–[18], [28].

Let X be a set. Recall that an IF-subset of X is a pair $A = (\mu_A, \nu_A)$, where μ_A, ν_A are fuzzy subsets of X (called the membership and nonmembership functions of A , respectively) and $\mu_A + \nu_A \leq 1_X$, where 1_X denotes the constant function with value 1. Clearly, for $\nu_A = 1_X - \mu_A$, (μ_A, ν_A) can be considered as a fuzzy subset of X ($\nu_A = 1_X - \mu_A$ is a redundant information). Generalized random events in IF-probability are certain systems of IF-subsets.

Let $\mathbf{A}$ be a σ -field of subsets of a set X . Let $\mathcal{F}$ be the set of all measurable IF-subsets of X , i.e., pairs (μ_A, ν_A) where $\mu_A, \nu_A \in \mathcal{M}(\mathbf{A})$, $\mu_A + \nu_A \leq 1_X$; we will call them IF-events. Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be IF-events. B. Riečan (cf. [31]) defined two operations and a partial order on IF-events as follows:

$$\begin{aligned} A \oplus B &= ((\mu_A + \mu_B) \wedge 1_X, (\nu_A + \nu_B - 1_X) \vee 0_X); \\ A \odot B &= ((\mu_A + \mu_B - 1_X) \vee 0_X, (\nu_A + \nu_B) \wedge 1_X); \\ A \preceq B &\quad \text{if and only if } (\mu_A \leq \mu_B) \wedge (\nu_B \leq \nu_A). \end{aligned}$$

Observe that IF-events are closed with respect to $\oplus$ and $\odot$, but not with respect to the operation $\neg$ defined by $\neg(\mu_A, \nu_A) = (1_X - \mu_A, 1_X - \nu_A)$. In [31], Riečan defined states and observables for IF-events and succeeded in proving limit theorems for IF-probability (avoiding complement).

Denote

$$A(\mathcal{M}(\mathbf{A}) \times \mathcal{M}(\mathbf{A})) = \{(u, v) \in \mathcal{M}(\mathbf{A}) \times \mathcal{M}(\mathbf{A}); u \leq v\}.$$

Then, $A(\mathcal{M}(\mathbf{A}) \times \mathcal{M}(\mathbf{A}))$ is closed with respect to the binary Łukasiewicz operations, but it is not closed with respect to the complement $\neg$. Define a map h of $\mathcal{F}$, the set of all measurable IF-subsets of X , into $A(\mathcal{M}(\mathbf{A}) \times \mathcal{M}(\mathbf{A}))$ by $h((\mu_A, \nu_A)) = (\mu_A, 1_X - \nu_A)$. Then (cf. [31], [32]), h is a bijection preserving the order and operations $\oplus, \odot$ in $\mathcal{F}$ and the usual order and the binary Łukasiewicz operations in $A(\mathcal{M}(\mathbf{A}) \times \mathcal{M}(\mathbf{A}))$. Accordingly, IF-events $\mathcal{F}$ can be viewed as the corresponding part $A(\mathcal{M}(\mathbf{A}) \times \mathcal{M}(\mathbf{A}))$ of the fuzzy probability domain $\mathcal{M}(\mathbf{A}) \times \mathcal{M}(\mathbf{A})$. In fact, $A(\mathcal{M}(\mathbf{A}) \times \mathcal{M}(\mathbf{A}))$ forms a “complement

dense" part of $\mathcal{M}(\mathbf{A}) \times \mathcal{M}(\mathbf{A})$. Lemma 4.1 in [33] states that if B is a bold algebra such that $A(\mathcal{M}(\mathbf{A}) \times \mathcal{M}(\mathbf{A})) \subseteq B \subseteq \mathcal{M}(\mathbf{A}) \times \mathcal{M}(\mathbf{A})$, then $B = \mathcal{M}(\mathbf{A}) \times \mathcal{M}(\mathbf{A})$. As proved by Riečan in [31], the system $\mathcal{F}$ of IF-events can be extended to an MV-algebra $\mathcal{M}$ in such a way that each state on $\mathcal{F}$ is the restriction of a uniquely determined state on $\mathcal{M}$. Moreover, Riečan proved a representation theorem for states on $\mathcal{M}$ (and hence on $\mathcal{F}$). The embedding of $A(\mathcal{M}(\mathbf{A}) \times \mathcal{M}(\mathbf{A}))$ into $\mathcal{M}(\mathbf{A}) \times \mathcal{M}(\mathbf{A})$ is equivalent to the embedding of $\mathcal{F}$ into $\mathcal{M}$, but the representation theorem for states on $\mathcal{M}(\mathbf{A}) \times \mathcal{M}(\mathbf{A})$ is less technical and much more intuitive [32].

Let $\mathcal{X} \subseteq [0, 1]^X$ be a bold algebra and let $\mathcal{M}(\mathbf{A}_{\sigma(\mathcal{X})})$ be the corresponding generated Łukasiewicz tribe. Motivated by classical probability theory, we describe the transition from fuzzy events $(u, v) \in \mathcal{X} \times \mathcal{X}$ to fuzzy events in $\mathcal{M}(\mathbf{A}_{\sigma(\mathcal{X})}) \times \mathcal{M}(\mathbf{A}_{\sigma(\mathcal{X})})$. Observe that the transition from fields of sets to the generated σ -fields enables us to exploit many powerful tools of mathematical analysis in probability theory.

Let $\mathcal{X} \subseteq [0, 1]^X$ be a bold algebra. Put $A(\mathcal{X} \times \mathcal{X}) = \{(u, v) \in \mathcal{X} \times \mathcal{X}; u \leq v\}$. Clearly, $A(\mathcal{X} \times \mathcal{X})$ is closed with respect to the Łukasiewicz operations $\oplus, \odot$.

Now, let us turn to the transition from $A(\mathcal{X} \times \mathcal{X})$ to $\mathcal{X} \times \mathcal{X}$, resp. from $A(\mathcal{M}(\mathbf{A}_{\sigma(\mathcal{X})}) \times \mathcal{M}(\mathbf{A}_{\sigma(\mathcal{X})}))$ to $\mathcal{M}(\mathbf{A}_{\sigma(\mathcal{X})}) \times \mathcal{M}(\mathbf{A}_{\sigma(\mathcal{X})})$. Note that if $\mathbf{A}$ is a field of sets considered as a bold algebra $\mathcal{X}$, then $\mathcal{M}(\sigma(\mathbf{A}))$ is equal to $\mathcal{M}(\mathbf{A}_{\sigma(\mathcal{X})})$.

The next lemma can be proved analogously as in [33, Lemma 4.1].

LEMMA 4.1. *Let B be a bold algebra such that $A(\mathcal{X} \times \mathcal{X}) \subseteq B \subseteq \mathcal{X} \times \mathcal{X}$. Then, $B = \mathcal{X} \times \mathcal{X}$.*

Having in mind the relationships between IF probability and fuzzy probability, consider the following question. Are there two isomorphic categories such that the objects of the first are of the form $A(\mathcal{X} \times \mathcal{X})$ and the objects of the second are of the form $\mathcal{X} \times \mathcal{X}$, where $\mathcal{X}$ is a bold algebra?

Let $\mathcal{X} \subseteq I^X$, $\mathcal{Y} \subseteq I^Y$ be bold algebras and let h be a sequentially continuous D-homomorphism of $\mathcal{X}$ into $\mathcal{Y}$. For $(u, v) \in \mathcal{X} \times \mathcal{X}$, define $\bar{h}(u, v) = (h(u), h(v))$. Denote $\bar{h}$ the resulting map of $\mathcal{X} \times \mathcal{X}$ into $\mathcal{Y} \times \mathcal{Y}$. Denote $\bar{h}_A$ the restriction of $\bar{h}$ to $A(\mathcal{X} \times \mathcal{X})$.

LEMMA 4.2.

- (i) $\bar{h}$ a sequentially continuous D-homomorphism of $\mathcal{X} \times \mathcal{X}$ into $\mathcal{Y} \times \mathcal{Y}$,
- (ii) $\bar{h}(u, v) \in A(\mathcal{Y} \times \mathcal{Y})$ whenever $(u, v) \in A(\mathcal{X} \times \mathcal{X})$.

Proof. Both assertions follow from straightforward calculations.

The preceding arguments lead to the following category **AIF**:

- (i) the objects are systems $A(\mathcal{X} \times \mathcal{X})$, where $\mathcal{X}$ is a bold algebra, equipped with the natural partial order inherited from $\mathcal{X} \times \mathcal{X}$ and two Łukasiewicz operations $\oplus, \odot$, inherited from $\mathcal{X} \times \mathcal{X}$;

- (ii) the morphisms are of the form $\bar{h}_A$, where h is a sequentially continuous D-homomorphism of the bold algebra $\mathcal{X}$ into the bold algebra $\mathcal{Y}$.

Denote $\mathbf{BID}^2$ the following subcategory of $\mathbf{BID}_2$:

- (i) the objects are bold algebras of the form $\mathcal{X} \times \mathcal{X}$;
- (ii) the morphisms are of the form $\bar{h}$, where h is a sequentially continuous D-homomorphism of a bold algebra $\mathcal{X}$ into a bold algebra $\mathcal{Y}$.

THEOREM 4.3. *The categories $\mathbf{AIF}$ and $\mathbf{BID}^2$ are isomorphic.*

Proof. It follows from the previous two lemmas that there is a one-to-one correspondence between the objects of $\mathbf{AIF}$ and $\mathbf{BID}^2$ and the morphisms of $\mathbf{AIF}$ and $\mathbf{BID}^2$, respectively, assigning $\mathcal{X} \times \mathcal{X}$ to $A(\mathcal{X} \times \mathcal{X})$ and $\bar{h}$ to $\bar{h}_A$, respectively. Clearly, this yields an isomorphism functor. $\square$

CONCLUSION. Let $\mathcal{X} \subseteq I^X$ be a bold algebra and let $\mathcal{X} \times \mathcal{X}$ be its power. Then (cf. Theorem 3.4), states on $\mathcal{X} \times \mathcal{X}$ are exactly convex combinations of states on $\mathcal{X}$. In view of Theorem 4.3, IF-probability can be interpreted within fuzzy probability theory.

4.2. Fuzzy probability

Fuzzy probability theory [34], [18] generalizes the classical one. More information can be found in [8], [17] and references therein.

The results of Section 3 and Section 4.1 indicate that it might be interesting to develop generalized probability theories based on the categories $\mathbf{BID}^2$ and $\mathbf{BID}_2$. In the first case, the resulting generalization could provide a technical support to IF-probability. In the second case, the resulting generalization could lead to a ramification of fuzzy probability.

REFERENCES

- [1] ADÁMEK, J.: *Theory of Mathematical Structures*. Reidel, Dordrecht, 1983.
- [2] CHOVANEC, F.—KÔPKA, F.: *D-posets*, in: Handbook of Quantum Logic and Quantum Structures: Quantum Structures. (K. Engesser et al., eds.), Elsevier, Amsterdam, 2007, pp. 367–428.
- [3] DVUREČENSKIJ, A.—PULMANNOVÁ, S.: *New Trends in Quantum Structures*. Kluwer Academic Publ., Dordrecht; Ister Science, Bratislava, 2000.
- [4] FRIČ, R.: *Remarks on sequential envelopes*, Rend. Istit. Math. Univ. Trieste **26** (1988), 604–612.
- [5] FRIČ, R.: *Stone-type duality and its applications to probability*, Topology Proc. **22** (Summer 1997), 125–137.
- [6] FRIČ, R.: *Lukasiewicz tribes are absolutely sequentially closed bold algebras*, Czechoslovak Math. J. **52** (2002), 861–874.
- [7] FRIČ, R.: *Duality for generalized events*, Math. Slovaca **54** (2004), 49–60.

- [8] FRIČ, R.: Remarks on statistical maps and fuzzy (operational) random variables. *Tatra Mt. Math. Publ.* **30** (2005), 21–34.
- [9] FRIČ, R.: *Extension of measures: a categorical approach*, *Math. Bohemica* **130** (2005), 397–407.
- [10] FRIČ, R.: *Measures: continuity, measurability, duality, extension*, *Tatra Mt. Math. Publ.* **42** (2009), 161–174.
- [11] FRIČ, R.: *Extension of domains of states*, *Soft Comput.* **13** (2009), 63–70.
- [12] FRIČ, R.: *From probability to sequences and back*, *Rend. Istit. Mat. Univ. Trieste* **44** (2012), 285–296.
- [13] FRIČ, R.: *On D-posets of fuzzy sets*, *Math. Slovaca* **64** (2014), 545–554.
- [14] FRIČ, R.—PAPČO, M.: *On probability domains*, *Internat. J. Theoret. Phys.* **49** (2010), 3092–3100.
- [15] FRIČ, R.—PAPČO, M.: *A categorical approach to probability theory*, *Studia Logica* **94** (2010), 215–230.
- [16] FRIČ, R.—PAPČO, M.: *On probability domains II*, *Internat. J. Theoret. Phys.* **50** (2011), 3778–3786.
- [17] FRIČ, R.—PAPČO, M.: *On probability domains III*, *Internat. J. Theoret. Phys.* (to appear).
- [18] GUDDER, S.: *Fuzzy probability theory*, *Demonstratio Math.* **31** (1998), 235–254.
- [19] HAVLÍČKOVÁ, J.: *Real functions and the extension of generalized probability measure*, *Tatra Mt. Math. Publ.* **55** (2013), 85–94.
- [20] KÔPKA, F.—CHOVANEK, F.: *D-posets*, *Math. Slovaca* **44** (1994), 21–34.
- [21] JUREČKOVÁ, M.: *The measure extension theorem for MV-algebras*, *Tatra Mt. Math. Publ.* **6** (1995), 56–61.
- [22] NOVÁK, J.: *Ueber die eindeutigen stetigen Erweiterungen stetiger Funktionen*, *Czechoslovak Math. J.* **8** (1958), 344–355.
- [23] NOVÁK, J.: *On the sequential envelope*, in: *General Topology and its Relations to Modern Analysis and Algebra*, Proc. of the 1st Prague Topological Symp., 1961, Publ. House of the Czechoslovak Academy of Sciences, Prague, 1962, pp. 292–294.
- [24] NOVÁK, J.: *On convergence spaces and their sequential envelopes*, *Czechoslovak Math. J.* **15** (1965), 74–100.
- [25] NOVÁK, J.: *On sequential envelopes defined by means of certain classes of functions*, *Czechoslovak Math. J.* **18** (1968), 450–456.
- [26] PAPČO, M.: *On measurable spaces and measurable maps*, *Tatra Mt. Math. Publ.* **28** (2004), 125–140.
- [27] PAPČO, M.: *On effect algebras*, *Soft Comput.* **12** (2007), 26–35.
- [28] PAPČO, M.: *Fuzzification of probabilistic objects*, in: 8th Conference of the European Society for Fuzzy Logic and Technology—EUSFLAT '13 (G. Pasi et al., eds.), Milano, Italy, 2013, Atlantis Press, Amsterdam, 2013, pp. 67–71.
- [29] RIEČAN, B.—MUNDICI, D.: *Probability on MV-algebras*, in: *Handbook of Measure Theory*, Vol. I and II (E. Pap, ed.), North-Holland, Amsterdam, 2002, pp. 869–910.
- [30] RIEČAN, B.—NEUBRUNN, T.: *Integral, Measure, and Ordering*. Kluwer Acad. Publ., Dordrecht, 1997.
- [31] RIEČAN, B.: *Analysis of fuzzy logic models*, in: *Intelligent Systems* (V. M. Koleshko, ed.), InTech, Rijeka, 2012, pp. 219–244.
- [32] SKŘIVÁNEK, V.: *Real functions in generalized probability* (submitted).

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- [33] SKŘIVÁNEK, V.—FRIČ, R.: *Generalized random events*, Internat. J. Theoret. Phys. (to appear).
- [34] ZADEH, L. A.: *Probability measures of fuzzy events*, J. Math. Anal. Appl. **23** (1968), 421–427.

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