ON M_0 -SETS FOR SERIES WITH RESPECT TO CHARACTERS OF COMPACT ZERO-DIMENSIONAL GROUPS

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ABSTRACT. Generalizing our previous result on Walsh system, we consider the system of characters of zero-dimensional compact abelian group. For this system, we give a construction of a perfect M_0 -set whose Hausdorff h -measure equals zero, where h is any non-decreasing right-continuous function taking the value zero at the origin.

1. Introduction

A set E is said to be U -set for a system of functions (especially, for orthogonal system) if the convergence of a series with respect to this system to zero outside the set E implies that all coefficients of the series are zero.

Otherwise, the set is called M -set, i.e., a set E is said to be M -set for a system of functions if there exists a non-trivial series with respect to this system which converges to zero outside the set E . This series is called *null-series* if $\mu E = 0$.

A set is called M_0 -set, or a set of *restricted multiplicity* if it is an M -set and the corresponding non-trivial series is a Fourier–Stieltjes series of a probability measure concentrated on E .

D. E. Menshov constructed an M_0 -set of measure zero for trigonometric series. Later, rather thin M -sets were constructed for other systems.

A good survey of the theory of sets of uniqueness and sets of multiplicity for the trigonometric system is given in [4].

Generalizing our previous result on Walsh system [7], we consider the system of characters of zero-dimensional compact abelian group and give, for this system, a construction of a perfect M_0 -set whose Hausdorff h -measure equals zero,

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where h is any non-decreasing function which is right-continuous and taking the value zero at the origin. We apply our method from [7] which was also used in [2], where M -set of Haar measure zero for the system of characters of above mentioned group was constructed.

In Section 2, we recall some definitions and facts on the structure of a zero-dimensional compact abelian group and on the properties of the group of characters of this group. To justify the application of the method of [7] in the case of the group we are considering here, we need to extend to our case the theory of formal product of series, which was developed by Schneider in [6] in the case of Walsh system as an analogue of the Rajchman theory for the trigonometric case. We give this extension in Section 3. By this, we also justify results of [2], where such a justification was not presented. A construction of a perfect M_0 -set is presented in Section 4.

2. Preliminaries

Consider a zero-dimensional compact abelian group G . It is known (see [1]) that a topology in such a group can be given by a chain of subgroups

$$G = G_0 \supset G_1 \supset \cdots \supset G_n \supset \cdots$$

with $\{0\} = \bigcap_{n=0}^{+\infty} G_n$. The subgroups G_n are clopen sets with respect to this topology. As G is compact, the quotient groups G/G_n are finite for each n . Let p_n be the order of the quotient group G_n/G_{n+1} . Then, the order of G_0/G_1 is p_0 , and the order of G_0/G_n , $n = 1, 2, \dots$, is

$$m_n = p_0 \cdot p_1 \cdots p_{n-1} \quad (2.1)$$

with $p_i \geq 2$ for all $i = 0, 1, 2, \dots$ (we put $m_0 = 1$).

Typical examples of zero-dimensional compact abelian group are:

- Dyadic Cantor group G , i.e., the group of sequences of integers of the form $\tilde{x} = \{x_j\}_{j=0}^{\infty}$, where $0 \leq x_j \leq 1$ with group operation defined as component-wise addition modulo 2 for each component. Topology is defined by subgroups

$$G_n = \{\tilde{x} \in G : x_j = 0 \text{ for } j = 0, \dots, n-1\}.$$

- More general Vilenkin group G is defined by a sequence of prime numbers $\{p_j\}$, $p_j \geq 2$. It is the group of sequences of integers of the form $\tilde{x} = \{x_j\}_{j=0}^{\infty}$, where $0 \leq x_j \leq p_j - 1$ with group operation defined as component-wise addition modulo p_j for j th component. Topology is defined by the same type of subgroups as in the previous example.

- The additive group of p -adic integers (with p being a fixed prime number) is the set of formal sums

$$\sum_{j=0}^{\infty} x_j p^j,$$

where $0 \leq x_i \leq p-1$ and the addition is done with *carries*, i.e., to add two sequences $\sum_{j=0}^{\infty} x_j p^j$ and $\sum_{j=0}^{\infty} y_j p^j$, we add coordinate-wise, and if any of the sums is p or more, we take a carry of 1 to the sum in the next coordinate. Topology is given by subgroups

$$G_n = \left\{ \sum_{j=0}^{\infty} x_j p^j : x_j = 0 \text{ for } j = 1, \dots, n-1 \right\}.$$

We denote any coset of the subgroup G_n in G/G_n by K^n .

It will be convenient to use the terminology of derivation basis theory (see, for example, [8]) and to refer to any coset K^n , $n = 0, 1, 2, \dots$, as a $\mathcal{B}$ -interval of rank n .

Let Γ denote the dual group of G , i.e., the group of characters of the group G . It is known that under the assumption imposed on G , the group Γ is a discrete abelian group with respect to the point-wise multiplication of characters (see [1]). Moreover, we can represent it as a union of increasing sequence of subgroups of finite order

$$\Gamma_0 \subset \Gamma_1 \subset \Gamma_2 \subset \dots \subset \Gamma_n \subset \dots$$

Then, $\Gamma = \bigcup_{i=0}^{+\infty} \Gamma_i$ and Γ_0 consists of one character γ_0 with $\gamma_0(g) = 1$ for all $g \in G$. For each $n \in \mathbb{N}$, the group Γ_n is the annihilator of G_n , i.e.,

$$\Gamma_n = G_n^\perp = \{\gamma \in \Gamma : \gamma(g) = 1 \text{ for all } g \in G_n\}.$$

Note that in the case of the dyadic Cantor group, Γ is the *Walsh system* and for the Vilenkin group it is the *Price-Vilenkin system* (see [3]).

The following lemma can be easily proved.

LEMMA 2.1. *If $\gamma \in \Gamma_n$, then γ is constant on each coset K^n of G_n .*

Let μ be the Haar measure on the group G normalized so that $\mu(G) = 1$ which, in our case, can be constructed by the usual Lebesgue method starting with measure on the semi-ring composed by the family of all cosets K^n . The characters γ constitute a countable orthonormal system on G with respect to measure μ . Note that $\mu(K^n) = 1/m_n$.

We can introduce a numeration of the characters in the following way.

The quotient groups $\Gamma_{n+1}/\Gamma_n = G_{n+1}^\perp/G_n^\perp$ and G_n/G_{n+1} are isomorphic (see [1]) and so, they are of the same order p_n for each $n \in \mathbb{N}$. This implies that the group Γ_n has $m_n = p_0 \cdot p_1 \cdot \dots \cdot p_{n-1}$ elements.

Having put $\gamma_0 = 1$, we pick up a character $\gamma \in \Gamma_{n+1} \setminus \Gamma_n$ for each $n = 0, 1, 2, \dots$ and denote it by γ_{m_n} . For

$$n = \sum_{k=0}^{k_n} t_k m_k, \quad 0 \leq t_k \leq p_k - 1$$

define

$$\gamma_n := \prod_{k=0}^{k_n} (\gamma_{m_k})^{t_k}. \quad (2.2)$$

This is the so-called Vilenkin–Paley numeration.

The functions γ belonging to $\Gamma_k \setminus \Gamma_{k-1}$ are called the characters of rank k . In fact, γ_n is of rank k if $m_{k-1} \leq n \leq m_k - 1$. According to Lemma 2.1, all the characters of rank k are constant on $\mathcal{B}$ -intervals of rank k . If P is a polynomial with respect to Γ , then its rank is defined as the maximal rank of its terms.

Now, we can consider a series with respect to the system Γ in the above numeration: $\sum_{i=0}^{\infty} a_i \gamma_i$ (coefficients a_n are complex numbers).

An important subsequence of partial sums of this series is

$$\sum_{i: \gamma_i \in \Gamma_n} a_i \gamma_i = S_{m_n} = \sum_{i=0}^{m_n-1} a_i \gamma_i.$$

3. Formal product and its application

Let $k(l, n)$ be an index defined by the equality $\gamma_l \gamma_n^{-1} = \gamma_{k(l, n)}$. In this notation, the *product* (formal) of a series $\sum_{n=0}^{\infty} a_n \gamma_n$ and a polynomial $\lambda = \sum_{n=0}^p b_n \gamma_n$ will mean the series $\sum_{l=0}^{\infty} c_l \gamma_l$, where $c_l =: \sum_{n=0}^p b_n a_{k(l, n)}$. This definition is justified by the formal equality

$$\left(\sum_{n=0}^p b_n \gamma_n \right) \left(\sum_{k=0}^{\infty} a_k \gamma_k \right) = \sum_{l=0}^{\infty} \sum_{n=0}^p b_n a_{k(l, n)} \gamma_n \gamma_{k(l, n)} = \sum_{l=0}^{\infty} \sum_{n=0}^p b_n a_{k(l, n)} \gamma_l.$$

LEMMA 3.1. *Let $n < m_s$, where m_s is defined by (2.1), be fixed and let $k(l, n)$ be defined as above. Then, under the assumption that $qm_s \leq l < (q+1)m_s$ for some $q \in \mathbb{N}$, we have*

$$qm_s \leq k(l, n) < (q+1)m_s \quad (3.1)$$

with the same q . Moreover, if $l \neq l'$, then $k(l, n) \neq k(l', n)$.

Proof. The last property is clear. Let

$$l = \sum_{r=0}^{r_l} t_r m_r = \sum_{r=0}^{s-1} t_r m_r + \sum_{r=s}^{r_l} t_r m_r = l' + l'', \quad \text{where } 0 \leq t_r < p_r.$$

Note that $l' < m_s$ and $l'' = qm_s$ here. Then, according to (2.2), $\gamma_l = \gamma_{l'}\gamma_{qm_s}$ with $\gamma_{l'} \in G_s^\perp$.

The condition $n < m_s$ implies $\gamma_n \in G_s^\perp$ and so also $\gamma_{n'} =: \gamma_n^{-1} \in G_s^\perp$, i.e., $n' < m_s$. From this we get $\gamma_{k(l,n)} = \gamma_l\gamma_{n'} = \gamma_{l'}\gamma_{n'}\gamma_{qm_s}$. With $\gamma_{l'}\gamma_{n'} \in G_s^\perp$ we have $\gamma_{l'}\gamma_{n'} =: \gamma_{k'}$, where $0 \leq k' < m_s$. Therefore by (2.2), $k(l, n) = qm_s + k'$ and this implies (3.1). $\square$

THEOREM 3.1. *Let a series*

$$\sum_{l=0}^{\infty} c_l \gamma_l \quad (3.2)$$

be obtained as a formal product of the series

$$\sum_{n=0}^{\infty} a_n \gamma_n, \quad \lim_{n \rightarrow \infty} a_n = 0, \quad (3.3)$$

and a polynomial $\lambda = \sum_{n=0}^p b_n \gamma_n$. Then the series (3.2) and

$$\lambda \sum_{n=0}^{\infty} a_n \gamma_n \quad (3.4)$$

are uniformly equi-convergent on G , i.e., the difference

$$\lambda \sum_{n=0}^N a_n \gamma_n - \sum_{n=0}^N c_n \gamma_n \quad (3.5)$$

converges to 0 uniformly on G when $N \rightarrow \infty$.

Proof. Fix s for which $p < m_s$ and note that for all $n \leq p$ and any $q \in \mathbb{N}$ we have

$$\sum_{l=0}^{qm_s-1} a_l \gamma_l = \sum_{l=0}^{qm_s-1} a_{k(l,n)} \gamma_{k(l,n)}.$$

Indeed, due to Lemma 3.1 inequality $l < qm_s$ implies $k(l, n) < qm_s$ and for different l values of $k(l, n)$ are also different. So the right hand sum in the above equality is obtained from the left hand sum by a permutation of its terms. Therefore

$$\begin{aligned} \sum_{l=0}^{qm_s-1} c_l \gamma_l &= \sum_{l=0}^{qm_s-1} \left(\sum_{n=0}^p b_n a_{k(l,n)} \right) \gamma_l = \sum_{n=0}^p b_n \gamma_n \sum_{l=0}^{qm_s-1} a_{k(l,n)} \gamma_{k(l,n)} \\ &= \sum_{n=0}^p b_n \gamma_n \sum_{l=0}^{qm_s-1} a_l \gamma_l = \lambda \sum_{l=0}^{qm_s-1} a_l \gamma_l. \end{aligned}$$

For arbitrary N we find q so that $qm_s \leq N < (q+1)m_s$. Then

$$\begin{aligned} \left| \sum_{l=0}^N c_l \gamma_l - \lambda \sum_{l=0}^N a_l \gamma_l \right| &= \left| \sum_{l=qm_s}^N c_l \gamma_l - \lambda \sum_{l=qm_s}^N a_l \gamma_l \right| \\ &\leq \max_{qm_s \leq l < (q+1)m_s} |a_l| m_s \left(\sum_{n=0}^p |b_k| + \max_{g \in G} |\lambda(g)| \right). \end{aligned}$$

The right-hand side converges to zero when N together with q tends to infinity. $\square$

THEOREM 3.2. *For the series (3.3) and for any $\mathcal{B}$ -interval K^n there exists a series $\sum_{l=0}^{\infty} c_l \gamma_l$ with $\lim_{l \rightarrow \infty} c_l = 0$ which is uniformly equi-convergent with (3.3) on K^n and is uniformly convergent to zero outside K^n .*

Proof. The function χ_{K^n} , the indicator of K^n , can clearly be represented as a polynomial P with respect to Γ . Then, the formal product of P and of the series (3.3) gives the series $\sum_{l=0}^{\infty} c_l \gamma_l$ we are looking for. Indeed, applying Theorem 3.1 to this polynomial, we get that series (3.3) is uniformly equi-convergent with the series $\chi_{K^n} \sum_{l=0}^{\infty} a_l \gamma_l$. However, this series is identical with series (3.3) on K^n and is vanishing outside K^n . $\square$

We need the following version of the uniqueness theorem which is a particular case of [8, Corollary 4.3].

THEOREM 3.3. *If for the series (3.3) at each point $g \in G$ some subsequence of partial sums of the form $S_{m_{n_k}}(g)$ converges to zero, then all the coefficients of the series are equal zero.*

With the above theorem we get

THEOREM 3.4. *If for the series (3.3) some subsequence of partial sums of the form $S_{m_{n_k}}(g)$ converges to zero for all g on some open set O , then this series converges to zero on O .*

Proof. As the system of all $\mathcal{B}$ -intervals forms a basis of topology in G , it is sufficient to prove the statement for any K^n . We can apply Theorem 3.2 to the series (3.3) and to the chosen K^n . The subsequence of partial sums $S'_{m_{n_k}}$ of the series

$$\sum_{l=0}^{\infty} c_l \gamma_l \tag{3.6}$$

given by Theorem 3.2 converges to zero everywhere on G . Now, by Theorem 3.3, we have $c_l = 0$ for all l . So, the series (3.6) having all coefficients zero is convergent to zero uniformly on G . Once again, applying Theorem 3.2, we get that the series (3.3) is also convergent to zero on K^n . $\square$

4. Construction of M_0 -set

DEFINITION 4.1. Let h be a nondecreasing right-continuous at zero function with $h(0) = 0$ and let μ be the Haar measure on G . The Hausdorff h -measure $\mathcal{H}^h$ is defined by

$$\mathcal{H}^h(A) = \liminf_{\delta \rightarrow 0} \sum \left\{ h(\mu(E_i)) : E_i \text{ } \mathcal{B}\text{-intervals, } \bigcup E_i \supset A, \mu(E_i) \leq \delta \right\}.$$

THEOREM 4.1. For any h as above, there exists a perfect M_0 -set E for the system Γ whose Hausdorff h -measure equals zero.

The first step in the construction is the

LEMMA 4.1. For any $A > 0$ and any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for any K^n , $n \geq N$, any $\delta > 0$, any B , $0 < B \leq A$, and any $k > n$, $k \in \mathbb{N}$, there exists a polynomial

$$P(g) = \sum_{j=m_k}^{m_l-1} a_j \gamma_j(g) \quad (4.1)$$

with the following properties:

- (1) $|a_j| < \varepsilon$, $m_k \leq j \leq m_l - 1$;
- (2) $P(g) = 0$ if $g \in G \setminus K^n$;
- (3) $P(g) \geq -B$ if $g \in K^n$;
- (4) $\frac{m_k}{m_n} h\left(\frac{1}{m_l}\right) < \delta$,

where $\frac{m_k}{m_n}$ is the number of those K^l in K^n for which $P(g) \neq -B$ with $g \in K^l$.

Proof. Take N such that $A/m_N < \varepsilon$ and having fixed arbitrary $n \geq N$ and $k > n$ choose l so that

$$\frac{m_k}{m_n} h\left(\frac{1}{m_l}\right) < \delta. \quad (4.2)$$

Let K^n be any $\mathcal{B}$ -interval of rank n . It is a union of m_k/m_n $\mathcal{B}$ -intervals of rank k . Let them be K_j^k , $j = 1, 2, \dots, m_k/m_n$. Now, we choose and fix in each K_j^k a $\mathcal{B}$ -interval of rank l . Let it be K_j^l . Consider a function

$$P(g) := \begin{cases} 0 & \text{if } g \in G \setminus K^n, \\ -B & \text{if } g \in K_j^k \setminus K_j^l, \\ (m_l/m_k - 1)B & \text{if } g \in K_j^l. \end{cases}$$

It is easy to show that this function can be represented as a polynomial of the form (4.1). Properties (2) and (3) follow directly from the definition of the function P , and property (4) follows from (4.2) and the choice of K_j^l . Property (1) was in fact checked in [2]. $\square$

Now, we sketch the proof of Theorem 4.1 once again following the scheme given in [2] and [7].

Proof. Having taken a sequence

$$\{\varepsilon_k\}, \varepsilon_k > 0, \lim_{k \rightarrow \infty} \varepsilon_k = 0,$$

we construct by induction a sequence of polynomials $\{Q_k\}$ with respect to Γ having the following properties:

- (1) $Q_1(g) = 1$ for all $g \in G$;
- (2) maximal rank s_k of the characters γ constituting the polynomial Q_k is less than the minimal rank of those constituting Q_{k+1} ;
- (3) coefficients a_i of the polynomial Q_k satisfy the inequality $|a_i| \leq \varepsilon_k$;
- (4) $\sum_{i=1}^k Q_i(g) \geq 0$ for any $g \in G$;
- (5) $\sum_{i=1}^k Q_i(g) = 0$ for $g \in G \setminus E_k$, where $E_k = \bigcup_{j=1}^{j_k} I_j$, $j_k \geq 2^{k-1}$ with I_j being $\mathcal{B}$ -intervals of various ranks greater than k (starting with $k = 2$) so that

$$\sum_{j=1}^{j_k} h(\mu(I_j)) < \varepsilon_k \quad \text{and} \quad E_{k+1} \subset E_k.$$

Construction of polynomial Q_{k+1} can be realized by a repeated application of Lemma 4.1 to $\mathcal{B}$ -intervals constituting E_k following the lines of the proof of [7, Theorem 1].

Property (2) implies that the sum $\sum_{k=1}^{\infty} Q_k$ defines a series

$$\sum_{i=0}^{\infty} a_i \gamma_i \tag{4.3}$$

with partial sums

$$S_{m_{s_k}} = \sum_{i=1}^k Q_i. \tag{4.4}$$

By property (3), $\lim_{i \rightarrow \infty} a_i = 0$. Property (5) implies that the Hausdorff h -measure of the set $E = \bigcap_{k=1}^{\infty} E_k$ is zero. It is clear from the construction that E is a perfect set in G . By the same property, the partial sums (4.4) tend to zero on $G \setminus E$ and this set is open. So, we can apply Theorem 3.4 to this set to obtain that the series (4.3) converges to zero outside E .

To prove that this null-series is a Fourier–Stieltjes series, we define a $\mathcal{B}$ -interval function by putting for each interval of rank s_k , $k = 1, 2, \dots$,

$$F(K^{s_k}) = \int_{K^{s_k}} S_{m_{s_k}}(g) d\mu.$$

It can be easily checked that F defines an additive function on the family of all $\mathcal{B}$ -intervals. It is usually referred to as a *quasi-measure* associated with the series (see [9]). By property (4), F is non-negative, by property (1),

$$F(G) = \int_G S_{m_0}(g) d\mu = 1$$

and by property (5), $F(I) = 0$ for any $\mathcal{B}$ -interval $I \subset G \setminus E$. So, F generates a probability measure μ_F on G supported by E , and for any $\mathcal{B}$ -interval I we have $\mu_F(I) = F(I)$. Let $i < m_n$ and let $\gamma_i(K_j^n)$, $j = 1, 2, \dots, m_n$, denote the constant values which γ_i assumes on K_j^n . Then,

$$\begin{aligned} a_i &= \int_G S_{m_n} \gamma_i d\mu = \sum_{j=1}^{m_n} \gamma_i(K_j^n) F(K_j^n) \\ &= \sum_{j=1}^{m_n} \gamma_i(K_j^n) \mu_F(K_j^n) = \int_G \gamma_i d\mu_F. \end{aligned}$$

So, the series (3.4) is the Fourier–Stieltjes series of the measure μ_F and E is an M_0 -set. $\square$

In the conclusion, we note that Piatetski-Shapiro gave an example of a closed M -set for the trigonometric system which is not an M_0 -set (see [4]). An analogous result for the Vilenkin system is obtained in [5]. It is an open problem whether such an example can be constructed in the general case of the system of characters we are considering here, and, in particular, for the system of characters of the group of p -adic integers.

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