$(\mathcal{B} \Delta \mathcal{I}, \mathcal{I})$ -SATURATED SETS AND HAMEL BASIS

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ABSTRACT. Let $\mathcal{I}$ be a proper σ -ideal of subsets of the real line. In a σ -field of Borel sets modulo sets from the σ -ideal $\mathcal{I}$ we introduce an analogue of the saturated non-measurability considered by I. Halperin. Properties of $(\mathcal{B} \Delta \mathcal{I}, \mathcal{I})$ -saturated sets are investigated.

M. Kuczma considered a problem how small or large a Hamel basis can be. We try to study this problem in the context of sets from $\mathcal{I}$.

The investigations concerning small sets have a long tradition and are connected with such mathematicians as Borel, Lebesgue, Baire, Erdős, Oxtoby and others. There is also a large group of Polish mathematicians which were interested in such problems: W. Sierpiński, K. Kuratowski, S. Ulam, E. Szpilrajn-Marczewski, M. Kuczma. In these considerations, the Lebesgue nullsets and the sets of the first category play an important role. The concept of “smallness” of sets can also be related to the microscopic sets introduced by J. Appell, E. D’Aniello and M. Văth in [1]. A set $A \subset \mathbb{R}$ is microscopic if for each $\varepsilon > 0$ there exists a sequence of intervals $\{I_n\}_{n \in \mathbb{N}}$ such that $A \subset \bigcup_{n \in \mathbb{N}} I_n$, and the length of I_n is not greater than ε^n for each $n \in \mathbb{N}$. The σ -ideal of microscopic sets will be denoted by $\mathcal{M}$. The properties of microscopic sets were considered in [2], [7]–[10]. It is worth adding that the generalization of the notion of microscopic set obtained by replacing the geometric sequence with another in the definition of microscopic set is considered by G. Horbaczewska in [5].

Let $\mathcal{B}$ denote the family of Borel sets on the real line, $\mathcal{L}$ —the family of Lebesgue measurable sets, $\mathcal{N}$ —the σ -ideal of sets of Lebesgue measure zero, $\mathcal{K}$ —the σ -ideal of sets of the first category. In the sequel, we use the following symbols: A' —the complement of the set A , $m(A)$ —the linear Lebesgue measure of the measurable set $A \subset \mathbb{R}$, $m_i(A)$ —the inner measure of A .

From now on, we will assume that $\mathcal{I}$ is a proper σ -ideal having a Borel basis (i.e., for each set $A \in \mathcal{I}$ there exists a set $B \in \mathcal{I} \cap \mathcal{B}$ such that $A \subset B$) and containing all singletons. Consider a σ -field generated by Borel sets and the σ -ideal $\mathcal{I}$:

$$\mathcal{B} \Delta \mathcal{I} := \{B \Delta I : B \in \mathcal{B} \text{ and } I \in \mathcal{I}\}.$$

Clearly, if $\mathcal{I} = \mathcal{N}$, then $\mathcal{B} \Delta \mathcal{I} = \mathcal{L}$ and if $\mathcal{I} = \mathcal{K}$, then $\mathcal{B} \Delta \mathcal{I}$ is a σ -field of sets having the Baire property.

In 1951, I. Halperin introduced the notion of saturated non-measurable set.

DEFINITION 1 ([4]). A set $A \subset \mathbb{R}$ is saturated non-measurable if and only if

$$m_i(A) = m_i(\mathbb{R} \setminus A) = 0.$$

Clearly, if A is saturated non-measurable, then so is A' . Each saturated non-measurable set is not Lebesgue measurable. Moreover, if $A \subset \mathbb{R}$ is a saturated non-measurable set, then, for every measurable set E of positive measure, the set $A \cap E$ is non-measurable.

It is easily seen that each Bernstein set (i.e., a subset B of the real line such that both B and B' meet every uncountable closed subset) is saturated non-measurable. Directly from the definition, it follows that a set $A \subset \mathbb{R}$ is saturated non-measurable if and only if

$$\forall_{E \in \mathcal{B} \Delta \mathcal{N}} [(E \subset A \vee E \subset A') \Rightarrow E \in \mathcal{N}]. \quad (1)$$

The condition (1) was a starting point to consider a similar notion in category case.

In [12], M. Ku c z m a introduced a topological analogue of the saturated non-measurability.

DEFINITION 2 ([12]). A set $A \subset \mathbb{R}$ has the $(*)$ property if and only if

$$\forall_{E \in \mathcal{B} \Delta \mathcal{K}} [(E \subset A \vee E \subset A') \Rightarrow E \in \mathcal{K}]. \quad (2)$$

It means A has the $(*)$ property if and only if each set E having the Baire property contained in A or A' is of the first category.

Clearly, if A has the $(*)$ property, then A' also has it, and each set having the $(*)$ property does not have the Baire property.

Moreover, if E is a set having the Baire property and of the second category, then intersection of E with arbitrary set having the $(*)$ property is a set which does not have the Baire property. Analogously as in measure case, each Bernstein set has the $(*)$ property.

Note that an important role in the above definitions is played by the pair $(\sigma\text{-field}, \sigma\text{-ideal})$. Therefore, we can introduce an analogous notion in a more general setting.

For the σ -field $\mathcal{B}\Delta\mathcal{I}$ and the σ -ideal $\mathcal{I}$, we can introduce the notion analogous to saturated non-measurability and the $(*)$ property.

DEFINITION 3. A set $A \subset \mathbb{R}$ is $(\mathcal{B}\Delta\mathcal{I}, \mathcal{I})$ -saturated if and only if

$$\forall_{E \in \mathcal{B}\Delta\mathcal{I}} [(E \subset A \vee E \subset A') \Rightarrow E \in \mathcal{I}]. \quad (3)$$

It means, A is $(\mathcal{B}\Delta\mathcal{I}, \mathcal{I})$ -saturated iff each set measurable with respect to $\mathcal{B}\Delta\mathcal{I}$ contained in A or A' belongs to the σ -ideal $\mathcal{I}$. Clearly, if A is $(\mathcal{B}\Delta\mathcal{I}, \mathcal{I})$ -saturated, then so is A' .

Obviously, $\mathcal{I}$ can be the σ -ideal of sets of Lebesgue measure zero or of the first category or microscopic sets. If we put $\mathcal{I} = \mathcal{N}$ in the last definition, we obtain saturated non-measurable sets. If $\mathcal{I} = \mathcal{K}$, then we obtain the sets having the $(*)$ property.

Of course, we know several examples of σ -ideals which can be considered here (σ -ideal of meager nullsets, generalized microscopic sets considered in [5], σ -ideals connected with Hausdorff measures (see [6]), the family of σ -porous sets, and others).

LEMMA 4. *If A is $(\mathcal{B}\Delta\mathcal{I}, \mathcal{I})$ -saturated, then A is not $\mathcal{B}\Delta\mathcal{I}$ measurable.*

Proof. Suppose A is $(\mathcal{B}\Delta\mathcal{I}, \mathcal{I})$ -saturated and $A \in \mathcal{B}\Delta\mathcal{I}$. Then, $A \in \mathcal{I}$. Analogously, $A' \in \mathcal{I}$. Consequently, $\mathbb{R} = A \cup A' \in \mathcal{I}$, a contradiction. $\square$

THEOREM 5. *A set $A \subset \mathbb{R}$ is $(\mathcal{B}\Delta\mathcal{I}, \mathcal{I})$ -saturated if and only if for each set $E \in (\mathcal{B}\Delta\mathcal{I}) \setminus \mathcal{I}$ both conditions $E \cap A \notin \mathcal{B}\Delta\mathcal{I}$ and $E \cap A' \notin \mathcal{B}\Delta\mathcal{I}$ hold.*

Proof. Let A be $(\mathcal{B}\Delta\mathcal{I}, \mathcal{I})$ -saturated and $E \in (\mathcal{B}\Delta\mathcal{I}) \setminus \mathcal{I}$. Suppose $E \cap A \in \mathcal{B}\Delta\mathcal{I}$. As $E \cap A \subset A$, we obtain $E \cap A \in \mathcal{I}$. Hence, $E \setminus (E \cap A) \in (\mathcal{B}\Delta\mathcal{I}) \setminus \mathcal{I}$ and $E \setminus (E \cap A) \subset A'$, a contradiction. Analogously, $E \cap A' \notin \mathcal{B}\Delta\mathcal{I}$.

Now, let $A \subset \mathbb{R}$ be a set such that for each set $E \in (\mathcal{B}\Delta\mathcal{I}) \setminus \mathcal{I}$ the conditions $E \cap A \notin \mathcal{B}\Delta\mathcal{I}$ and $E \cap A' \notin \mathcal{B}\Delta\mathcal{I}$ are fulfilled. Let $D \in \mathcal{B}\Delta\mathcal{I}$ and $D \subset A$. Suppose $D \notin \mathcal{I}$. Then, $D \in (\mathcal{B}\Delta\mathcal{I}) \setminus \mathcal{I}$, so $D \cap A \notin \mathcal{B}\Delta\mathcal{I}$. Simultaneously, $D \cap A = D$, a contradiction.

If $D \subset A'$, the proof is analogous. Consequently, A is $(\mathcal{B}\Delta\mathcal{I}, \mathcal{I})$ -saturated. $\square$

LEMMA 6. *Each Bernstein set is $(\mathcal{B}\Delta\mathcal{I}, \mathcal{I})$ -saturated.*

Proof. Let B be a Bernstein set, i.e., neither B nor B' contains any uncountable closed set. We shall prove that B is $(\mathcal{B}\Delta\mathcal{I}, \mathcal{I})$ -saturated. Let $E \in (\mathcal{B}\Delta\mathcal{I}) \setminus \mathcal{I}$. Suppose $E \subset B$. Then, $E = E_1 \Delta I$, where $E_1 \in \mathcal{B} \setminus \mathcal{I}$ and $I \in \mathcal{I}$. The family $\mathcal{I}$ has a Borel basis so there exists a set $I_1 \in \mathcal{I} \cap \mathcal{B}$ such that $I \subset I_1$. Hence, $E_1 \setminus I_1 \subset E_1 \setminus I \subset E \subset B$ and $E_1 \setminus I_1 \in \mathcal{B} \setminus \mathcal{I}$ which means that the Bernstein set B contains a Borel set which is uncountable. From Alexandrov-Hausdorff

theorem (see [13, Chapter III, § 37, I, Theorem 3]) it follows that every uncountable Borel set contains topologically the Cantor set, i.e., the uncountable closed subset. Hence, the Bernstein set B contains the uncountable closed subset, a contradiction.

If $E \subset B'$, the proof is analogous. Consequently, B is a $(\mathcal{B}\Delta\mathcal{I}, \mathcal{I})$ -saturated set. $\square$

LEMMA 7. *If B is $(\mathcal{B}\Delta\mathcal{I}, \mathcal{I})$ -saturated and $C \in \mathcal{I}$, then $B \cup C$ is $(\mathcal{B}\Delta\mathcal{I}, \mathcal{I})$ -saturated.*

Proof. Let $D \in \mathcal{B}\Delta\mathcal{I}$. Suppose $D \subset (B \cup C)$. Then $D = (D \cap C) \cup (D \setminus C)$. Obviously, $D \cap C \in \mathcal{I}$. On the other hand, $D \setminus C \subset B$, $D \setminus C \in \mathcal{B}\Delta\mathcal{I}$ and B is $(\mathcal{B}\Delta\mathcal{I}, \mathcal{I})$ -saturated, so $D \setminus C \in \mathcal{I}$. Therefore, $D \in \mathcal{I}$. If $D \subset (B \cup C)'$, then $D \subset B'$, hence $D \in \mathcal{I}$ as B' is $(\mathcal{B}\Delta\mathcal{I}, \mathcal{I})$ -saturated. Consequently, $B \cup C$ is $(\mathcal{B}\Delta\mathcal{I}, \mathcal{I})$ -saturated. $\square$

We will prove that for a σ -ideal $\mathcal{I}$ being a proper subset of $\mathcal{N}$, the family of $(\mathcal{B}\Delta\mathcal{I}, \mathcal{I})$ -saturated sets is contained in the family of saturated non-measurable sets (i.e., $(\mathcal{B}\Delta\mathcal{N}, \mathcal{N})$ -saturated sets), and this inclusion is proper.

THEOREM 8. *If $\mathcal{I} \subsetneq \mathcal{N}$, then the family of $(\mathcal{B}\Delta\mathcal{I}, \mathcal{I})$ -saturated sets is a proper subset of the family of $(\mathcal{B}\Delta\mathcal{N}, \mathcal{N})$ -saturated sets.*

Proof. Let A be $(\mathcal{B}\Delta\mathcal{I}, \mathcal{I})$ -saturated and let E be a measurable set in the Lebesgue sense. Suppose $E \subset A$. Then, there exists a Borel (even of type F_σ) set $D \subset E$ such that $E \setminus D$ is a nullset. Hence, $D \in \mathcal{B}\Delta\mathcal{I}$ and $D \subset A$. As A is $(\mathcal{B}\Delta\mathcal{I}, \mathcal{I})$ -saturated, we obtain $D \in \mathcal{I}$. Consequently, E is a nullset. Analogously, if $E \subset A'$, so A is a $(\mathcal{B}\Delta\mathcal{N}, \mathcal{N})$ -saturated set.

Observe now that if $\mathcal{I}$ is a proper subset of $\mathcal{N}$, there exists a $(\mathcal{B}\Delta\mathcal{N}, \mathcal{N})$ -saturated set which is not $(\mathcal{B}\Delta\mathcal{I}, \mathcal{I})$ -saturated. Put $A = B \cup C$, where B is a Bernstein set and $C \in \mathcal{B} \cap (\mathcal{N} \setminus \mathcal{I})$. From Lemmas 6 and 7, A is $(\mathcal{B}\Delta\mathcal{N}, \mathcal{N})$ -saturated. Simultaneously, A is not $(\mathcal{B}\Delta\mathcal{I}, \mathcal{I})$ -saturated, as $C \notin \mathcal{I}$. $\square$

We say that two σ -ideals $\mathcal{I}$ and $\mathcal{J}$ of subsets of $\mathbb{R}$ are Borel orthogonal if there exists a decomposition of the real line onto two sets A and B such that $A \in \mathcal{B} \cap \mathcal{I}$ and $B \in \mathcal{B} \cap \mathcal{J}$.

We will prove that for σ -ideals Borel orthogonal to $\mathcal{K}$, the families of $(\mathcal{B}\Delta\mathcal{I}, \mathcal{I})$ -saturated sets and $(\mathcal{B}\Delta\mathcal{K}, \mathcal{K})$ -saturated sets are incomparable. For this purpose, we use the Bernstein set and sets from the decomposition of the real line onto two Borel sets, where one is of the first category and the second one belongs to $\mathcal{I}$.

THEOREM 9. *Let $\mathcal{I}$ be a σ -ideal such that $\mathcal{I}$ and $\mathcal{K}$ are Borel orthogonal. There exists a set $A_1 \subset \mathbb{R}$ which is $(\mathcal{B}\Delta\mathcal{I}, \mathcal{I})$ -saturated but not $(\mathcal{B}\Delta\mathcal{K}, \mathcal{K})$ -saturated,*

and there exists a set $A_2 \subset \mathbb{R}$ which is $(\mathcal{B}\Delta\mathcal{K}, \mathcal{K})$ -saturated but not $(\mathcal{B}\Delta\mathcal{I}, \mathcal{I})$ -saturated.

Proof. Put $A_1 = B \cup M$, where B is a Bernstein set and $M \in \mathcal{B} \cap \mathcal{I}$ is a residual set. From Lemmas 6 and 7, the set A_1 is $(\mathcal{B}\Delta\mathcal{I}, \mathcal{I})$ -saturated. Since $M \subset A_1$ and $M \in (\mathcal{B}\Delta\mathcal{K}) \setminus \mathcal{K}$, we obtain that A_1 is not $(\mathcal{B}\Delta\mathcal{K}, \mathcal{K})$ -saturated.

Put $A_2 = B \cup N$, where B is a Bernstein set and $N \in \mathcal{B} \cap \mathcal{K}$ is such that $\mathbb{R} \setminus N \in \mathcal{I}$. Lemmas 6 and 7 imply that A_2 is $(\mathcal{B}\Delta\mathcal{K}, \mathcal{K})$ -saturated. Observe that A_2 is not $(\mathcal{B}\Delta\mathcal{I}, \mathcal{I})$ -saturated, as $N \subset A_2$ and $N \in (\mathcal{B}\Delta\mathcal{I}) \setminus \mathcal{I}$. $\square$

It is well-known that each set with positive measure contains some non-measurable subset and each set of the second category contains a subset without the Baire property. Analogous property holds for sets which are not in $\mathcal{I}$.

THEOREM 10. *For each set $A \notin \mathcal{I}$, there exists a set $E \subset A$ such that $E \notin \mathcal{B}\Delta\mathcal{I}$.*

Proof. If $A \notin \mathcal{B}\Delta\mathcal{I}$, then put $E = A$. Otherwise, we put $E = A \cap D$, where D is an arbitrary $(\mathcal{B}\Delta\mathcal{I}, \mathcal{I})$ -saturated set. From Theorem 5, we obtain that $A \cap D \notin \mathcal{B}\Delta\mathcal{I}$. $\square$

In the next part of this paper, we will focus on Hamel and Burstin bases. So, let us shortly recall this notions.

A Hamel basis is any basis of the linear space $(\mathbb{R}^N; \mathbb{Q}; +; \cdot)$. A Hamel basis $H \subset \mathbb{R}^N$ is called a Burstin basis relative to a set $X \subset \mathbb{R}$ if and only if $H \subset X$ and H intersects every uncountable Borel subset of X . A Burstin basis relative to $\mathbb{R}^N$ is simply called a Burstin basis. The existence of a Burstin basis was proved in [12, Corollary 11.4.2].

In the first edition of [12], in 1985, M. Kuczm a considered a problem how large a Hamel basis can be. Observe that there exist very small Hamel bases: of measure zero (see [16]) or of the first category, even nowhere dense (see [12, Corollary 11.3.1]). In [9] we proved that there exists a Hamel basis which is a microscopic set. So, the Hamel basis can be small.

On the other hand, in [12, Theorem 11.4.1], it is proved that if the set $X \subset \mathbb{R}^N$ is of full measure (i.e., $\mathbb{R}^N \setminus X$ is a set of N -dimensional Lebesgue measure zero), then every Burstin basis relative to X is saturated non-measurable. A topological analogue of this result [12, Theorem 11.4.2] says that if the set $X \subset \mathbb{R}^N$ is residual, then every Burstin basis relative to X has the $(*)$ property.

From these theorems, it follows (see [12, Corollary 11.4.3]) that there exists a Hamel basis of measure zero which has the $(*)$ property and there exists a Hamel basis which is of the first category and is saturated non-measurable.

Now, we will prove that the analogous theorem holds in the context of sets from each σ -ideal $\mathcal{I}$ which is closed under multiplication (i.e., if $A \in \mathcal{I}$ and $\alpha \in \mathbb{R}$, then $\alpha \cdot A = \{\alpha \cdot x : x \in A\} \in \mathcal{I}$).

THEOREM 11. *Let $\mathcal{I}$ be a σ -ideal closed under multiplication. If a set $X \subset \mathbb{R}$ is such that $\mathbb{R} \setminus X \in \mathcal{I}$, then every Burstin basis relative to X is $(\mathcal{B} \Delta \mathcal{I}, \mathcal{I})$ -saturated.*

Proof. Let H be a Burstin basis relative to X . Then, H is a Hamel basis in $\mathbb{R}$, $H \subset X$ and the intersection of H with an arbitrary uncountable Borel subset of X is a non-empty set. We will prove that for each set $A \in \mathcal{B} \Delta \mathcal{I}$, if $A \subset H$ or $A \subset \mathbb{R} \setminus H$, then $A \in \mathcal{I}$.

Let $A \in \mathcal{B} \Delta \mathcal{I}$ and assume $A \notin \mathcal{I}$. Then,

$$A = (B \setminus I) \cup (I \setminus B),$$

where $B \in \mathcal{B} \setminus \mathcal{I}$ and $I \in \mathcal{I}$. As $\mathcal{I}$ has a Borel basis, there exists a set $I_1 \in \mathcal{B} \cap \mathcal{I}$ such that $I \subset I_1$. Hence, $A_1 = B \setminus I_1 \subset A$ and $A_1 \in \mathcal{B} \setminus \mathcal{I}$.

Analogously, $\mathbb{R} \setminus X \in \mathcal{I}$, so there exists a set $Y \in \mathcal{B} \cap \mathcal{I}$ such that $\mathbb{R} \setminus X \subset Y$. Hence, $X_1 = \mathbb{R} \setminus Y \subset X$, $X_1 \in \mathcal{B}$ and $\mathbb{R} \setminus X_1 \in \mathcal{I}$.

We will now prove that if $A \subset H$ or $A \subset \mathbb{R} \setminus H$, then $A \in \mathcal{I}$. Suppose $A \subset \mathbb{R} \setminus H$. We have $A_1 \cap X_1 = A_1 \setminus Y$, so $A_1 \cap X_1 \in \mathcal{B} \setminus \mathcal{I}$ and $A_1 \cap X_1 \subset A \cap X$. Hence, $A_1 \cap X_1$ is a Borel uncountable subset of X disjoint with H , a contradiction.

Now, let $A \subset H$. Clearly, $2A_1 = \{2x : x \in A_1\} \in \mathcal{B} \setminus \mathcal{I}$, so $2A_1 \cap X_1 = 2A_1 \setminus Y$ is a Borel uncountable subset of X and, consequently, it intersects H . Let $y \in 2A_1 \cap X_1 \cap H$. Then, $y \in H$ and $\frac{1}{2}y \in A_1 \subset A \subset H$, which is impossible, as y and $\frac{1}{2}y$ are linearly dependent on $\mathbb{Q}$. $\square$

In [12], M. K u c z m a proved (see Corollary 11.4.1) that every Burstin basis is a saturated non-measurable set and has the $(*)$ property. Taking $X = \mathbb{R}$ and using Theorem 11, we obtain

COROLLARY 12. *If $\mathcal{I}$ is a σ -ideal closed under multiplication, then every Burstin basis in $\mathbb{R}$ is a $(\mathcal{B} \Delta \mathcal{I}, \mathcal{I})$ -saturated set having the $(*)$ property.*

Note that for a σ -ideal $\mathcal{I} \subset \mathcal{N}$ closed under multiplication, this corollary is the strengthening of Corollary 11.4.1 in [12], as, by Theorem 8, each $(\mathcal{B} \Delta \mathcal{I}, \mathcal{I})$ -saturated set is saturated non-measurable. At the same time, it says a Burstin basis is a “large” Hamel basis.

In [11, Chapter 3, Theorem 4], A. B. K h a r a z i s h v i l i proved that there exists a Hamel basis H in $\mathbb{R}$ which simultaneously is a Bernstein subset of $\mathbb{R}$. As it was observed in Lemma 6, H is $(\mathcal{B} \Delta \mathcal{I}, \mathcal{I})$ -saturated, so also saturated non-measurable, and has the $(*)$ property. Hence, H is a large set both in topological and metric sense.

Now, we will prove that there exists a Hamel basis (exactly a Burstin basis) which is large in one sense and small in another. Put

$$\mathcal{U} = \{T \subset \mathbb{R} : \text{every convex function } f : D \rightarrow \mathbb{R}, \text{ where } T \subset D \subset \mathbb{R}$$

and D is convex and open, bounded above on T , is continuous in $D\}$

(compare [3]).

In 1929, A. Ostrowski proved in [15] (compare also [12, Theorem 9.3.1]) that if $T \subset \mathbb{R}$ is a set such that $m_i(T) > 0$, then $T \in \mathcal{U}$.

The topological analogue of the result of Ostrowski was given by M. R. Mehdi in [14], (see also [12, Theorem 9.3.2]). In 1964, Mehdi proved that if $T \subset \mathbb{R}$ contains a set having the Baire property and of the second category, then $T \in \mathcal{U}$. From [12, Corollary 9.3.2], it follows that if $T \in \mathcal{U}$, then T contains a Hamel basis. In [12, Theorem 11.4.3], it is proved that if $X \subset \mathbb{R}^N$ is a Borel set such that $\text{Lin}(X) = \mathbb{R}^N$ (where $\text{Lin}(A)$ denotes the linear subspace spanned by A , i.e., the set of all finite linear combinations of elements of A) then there exists a Burstin basis relative to X .

Using these facts, we obtain

COROLLARY 13. *Let $\mathcal{I} \subset \mathcal{N}$ be a σ -ideal closed under multiplication and Borel orthogonal to $\mathcal{K}$. There exists a Hamel basis which is of the first category and is $(\mathcal{B} \Delta \mathcal{I}, \mathcal{I})$ -saturated, and there exists a Hamel basis belonging to $\mathcal{I}$ which is $(\mathcal{B} \Delta \mathcal{K}, \mathcal{K})$ -saturated.*

Proof. Let $\mathbb{R} = A \cup B$, where $A \in \mathcal{B} \cap \mathcal{K}$ and $B \in \mathcal{B} \cap \mathcal{I}$. From Theorem of Ostrowski, $A \in \mathcal{U}$, and from Theorem of Mehdi, $B \in \mathcal{U}$. By [12, Corollary 9.3.2], both A and B contain a Hamel basis, so $\text{Lin}(A) = \text{Lin}(B) = \mathbb{R}$. Using Theorem 11.4.3 from [12], we obtain that A contains a Burstin basis H_1 relative to A and B contains a Burstin basis H_2 relative to B . As $\mathbb{R} \setminus A \in \mathcal{I}$, from Theorem 11, it follows that H_1 is a $(\mathcal{B} \Delta \mathcal{I}, \mathcal{I})$ -saturated set. Analogously, as B is residual, from [12, Theorem 11.4.2] H_2 fulfills the $(*)$ property. Since $H_1 \subset A$ and $H_2 \subset B$, H_1 is a set of the first category and H_2 belongs to $\mathcal{I}$. $\square$

Note that the last result is an improvement of [12, Corollary 11.4.3] which says that there exists a Hamel basis which is of measure zero and has the $(*)$ property, and there exists a Hamel basis which is of the first category and is saturated non-measurable. In [9], we proved that there exists a Hamel basis which is a microscopic set and has the $(*)$ property. The σ -ideal $\mathcal{M}$ of microscopic sets is closed under multiplication and Borel orthogonal to $\mathcal{K}$, so taking the above corollary into account, we can write even more: there exists a Hamel basis which is of the first category and is a $(\mathcal{B} \Delta \mathcal{M}, \mathcal{M})$ -saturated set, and there exists a Hamel basis which is a microscopic set and has the $(*)$ property.

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