ON DIFFERENTIABILITY OF MAPPINGS WITH FINITE DILATION

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ABSTRACT. We study mappings $f: (a, b) \rightarrow Y$ with finite dilation having Lebesgue integrable majorant, where Y is a real normed vector space. We construct Lipschitz mapping $f: (a, b) \rightarrow Y$, $\dim Y = \infty$, which is nowhere differentiable but its graph has everywhere trivial contingent. We show that if the contingent of the graph of a mapping with finite dilation is a nontrivial space, then f is almost everywhere differentiable.

1. Definitions and preliminaries

Let Y be a real normed space. Let $f: (a, b) \rightarrow Y$ be a mapping with finite dilation, i.e.,

$$\forall x \in (a, b): \limsup_{h \rightarrow 0} \left\| \frac{f(x+h) - f(x)}{h} \right\| < \infty.$$

If $\dim Y < \infty$, then it is well-known that f is almost everywhere differentiable. In the case of infinite dimension of the space Y this is no longer true. First, we will give auxiliary definitions and theorems required to prove the main theorem.

DEFINITION 1.1 ([9]). Let $\emptyset \neq M \subset Z$, where Z is a normed space, and $z \in \text{cl}M$. The set

$$\left\{ v \in Z: \exists (z_n)_{n \in \mathbb{N}}, z_n \in M, \lim_{n \rightarrow \infty} z_n = z, \exists (\lambda_n)_{n \in \mathbb{N}}, \lambda_n > 0: \lim_{n \rightarrow \infty} \lambda_n(z_n - z) = v \right\}$$

is called a tangent cone to M at the point z and is denoted $\text{Tan}(M, z)$. Elements of $\text{Tan}(M, z)$ are called vectors tangent to M at z . The set $\text{Tan}(M, z)$ is also called a contingent of M at z (for other definitions of $\text{Tan}(M, z)$, see, e.g., [2], [4], [5], [8]).

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Let X, Y be real normed spaces. $\text{Gr}(\varphi)$ will denote the graph of a mapping $\varphi: E \rightarrow Y$, $E \subset X$, and 0_Y will stand for a zero vector of the space Y . We will need the following lemmas and definitions.

LEMMA 1.2 ([7]). *Let X and Y be real normed spaces and let $U \subset X$ be an open set. Let $F: U \rightarrow Y$ and $f: U \rightarrow Y$ be mappings continuous at a point $x_0 \in U$. Assume that F is differentiable at x_0 and $\text{Tan}(\text{Gr}(f), (x_0, f(x_0)))$ is a linear subspace of $X \times Y$. Then, $\text{Tan}(\text{Gr}(F + f), (x_0, (F + f)(x_0)))$ is a linear subspace of $X \times Y$. Moreover,*

$$\begin{aligned} \text{Tan}\left(\text{Gr}(F + f), (x_0, (F + f)(x_0))\right) = \\ \left\{ (u, F'(x_0)u + v) : (u, v) \in \text{Tan}\left(\text{Gr}(f), (x_0, f(x_0))\right) \right\}. \end{aligned}$$

Let $X \neq \emptyset$, Σ be a σ -field of subsets of the space X and let $\mu: \Sigma \rightarrow [0, \infty)$ be a measure. From now, let Y be a Banach space and X be a normed space with a finite measure μ .

DEFINITION 1.3 ([3]). A function $\varphi: X \rightarrow Y$ is called a step function if there exist points $y_1, y_2, \dots, y_n$ in Y and measurable sets $E_1, E_2, \dots, E_n$ such that

$$\varphi = \sum_{i=1}^n y_i \chi_{E_i}, \quad \text{where} \quad \chi_{E_i}(x) = \begin{cases} 1, & x \in E_i, \\ 0, & x \notin E_i. \end{cases}$$

DEFINITION 1.4 ([3]). A function $\varphi: X \rightarrow Y$ is called μ -measurable if there exists a sequence of step functions $(\varphi_n)_{n \in \mathbb{N}}$ such that $\lim_{n \rightarrow \infty} \|\varphi_n - \varphi\| = 0$ μ -almost everywhere ($\|\cdot\|$ is sup norm in the space of bounded functions).

Let $y^*: Y \rightarrow \mathbb{R}$ be a linear continuous functional on Y , i.e., $y^* \in Y^*$.

DEFINITION 1.5 ([3]). A function $\varphi: X \rightarrow Y$ is called weakly μ -measurable if $y^* \circ \varphi: X \rightarrow \mathbb{R}$ is μ -measurable for each $y^* \in Y^*$.

THEOREM 1.6 ([6]). *A necessary and sufficient condition that $\varphi: X \rightarrow Y$ is μ -measurable is that φ is weakly μ -measurable and separably-valued (separably-valued means that the range $\varphi(X)$ is separable).*

DEFINITION 1.7 ([3]). Let $\varphi: X \rightarrow Y$, $\varphi = \sum_{i=1}^n y_i \chi_{E_i}$, be a step function. We define μ -integral $\int_X \varphi d\mu$ of φ over X by $\sum_{i=1}^n y_i \mu(E_i)$.

DEFINITION 1.8 ([3]). A μ -measurable function $\varphi: X \rightarrow Y$ is called Bochner integrable if there exists a sequence of step functions $(\varphi_n)_{n \in \mathbb{N}}$ such that

$$\lim_{n \rightarrow \infty} \int_X \|\varphi_n - \varphi\| d\mu = 0.$$

In this case, the integral $\int_E \varphi d\mu$ is defined uniquely for each $E \in \Sigma$ in the following way

$$\int_E \varphi d\mu = \lim_{n \rightarrow \infty} \int_E \varphi_n d\mu.$$

THEOREM 1.9 ([3]). *A μ -measurable function $\varphi: X \rightarrow Y$ is Bochner integrable if and only if $\int_X \|\varphi\| d\mu < \infty$.*

Remark 1.10 ([1]). For measurable sets in $\mathbb{R}^k$, the Bochner integral of real functions with respect to Lebesgue measure coincides with Lebesgue integral.

THEOREM 1.11 ([3]). *Let $\varphi: (a, b) \rightarrow Y$ be Bochner integrable on (a, b) with respect to Lebesgue measure. Then, for almost all $s \in (a, b)$,*

$$\lim_{h \rightarrow 0} \frac{1}{h} \int_s^{s+h} \|\varphi(t) - \varphi(s)\| dt = 0.$$

Consequently, for almost all $s \in (a, b)$,

$$\lim_{h \rightarrow 0} \frac{1}{h} \int_s^{s+h} \varphi(t) dt = \varphi(s).$$

DEFINITION 1.12 ([8]). We say that a mapping $\varphi: (a, b) \rightarrow \mathbb{R}$ is absolutely continuous on (a, b) , $\varphi \in AC(a, b)$ if

$$\forall \varepsilon > 0 \exists \delta > 0 \forall (a_1, b_1), \dots, (a_n, b_n): \sum_{i=1}^n (b_i - a_i) < \delta \Rightarrow \sum_{i=1}^n |\varphi(b_i) - \varphi(a_i)| < \varepsilon.$$

DEFINITION 1.13 ([6]). We say that an additive mapping $\varphi: \Sigma \rightarrow Y$ is absolutely continuous on X , $\varphi \in AC(X)$ if

$$\forall \varepsilon > 0 \exists \delta > 0 \forall A \in \Sigma: \mu(A) < \delta \Rightarrow \|\varphi(A)\| < \varepsilon.$$

THEOREM 1.14 ([6]). *If $\varphi: X \rightarrow Y$ is integrable, then $\Phi: \Sigma \rightarrow Y$, $\Phi(E) = \int_E \varphi d\mu$ for $E \in \Sigma$, is absolutely continuous on X .*

COROLLARY 1.15. *If $X = (a, b)$ and $\varphi: \Sigma \rightarrow Y$ is absolutely continuous on (a, b) and $x_0 \in (a, b)$ is fixed, then the mapping $\Phi: (a, b) \rightarrow Y$,*

$$\Phi(x) = \begin{cases} \varphi([x_0, x]), & x \geq x_0, \\ \varphi([x, x_0]), & x < x_0 \end{cases}$$

is absolutely continuous on (a, b) .

LEMMA 1.16. *Let $g: (a, b) \rightarrow Y$ be a mapping. Assume that for each $y^* \in Y^*$, the mapping $y^* \circ g: (a, b) \rightarrow \mathbb{R}$ is constant. Then, g is constant.*

Proof. Suppose that the mapping g is not constant. There exist t_1, t_2 , $a < t_1 < t_2 < b$, such that $g(t_1) \neq g(t_2)$. By Hahn-Banach theorem, there exists a functional $y^* \in Y^*$ such that $y^*(g(t_1) - g(t_2)) \neq 0$. Therefore, $(y^* \circ g)(t_1) \neq (y^* \circ g)(t_2)$, a contradiction. $\square$

2. Main result

In this section, we will show that if the contingent is not trivial at each point of $\text{Gr}(f)$, then f is almost everywhere differentiable provided that f satisfies some simple conditions (e.g., Lipschitz, finite dilation).

Let us consider the following question. Must the mapping $f: (a, b) \rightarrow Y$, Y normed space, $\dim Y = \infty$, with finite dilation be almost everywhere differentiable?

Professor B. Kirchheim suggested to investigate the following example.

EXAMPLE 2.1. There is a nowhere differentiable mapping with finite dilation and with trivial contingent at every point. Let us consider $K: (0, 1) \rightarrow L^1[0, 1]$,

$$K(t) := \chi_{[0, t]}.$$

Finite dilation is evident from the definition of K , since

$$\|K(t_1) - K(t_2)\| = \|\chi_{(t_1, t_2]}\| = \int_{t_1}^{t_2} dt = t_2 - t_1 \quad \text{for } 0 < t_1 < t_2 < 1.$$

Next, we will show that K is nowhere differentiable indeed. Suppose to the contrary that there exists a point $t \in (0, 1)$ and there exists a mapping $y \in L^1[0, 1]$ such that $K'(t) = y$. In other words,

$$\int_0^1 \left| \frac{K(t+h)(\tau) - K(t)(\tau)}{h} - y(\tau) \right| d\tau \rightarrow 0 \quad \text{when } h \rightarrow 0. \quad (2.1)$$

First, assume that $h > 0$. We have

$$\begin{aligned} \int_0^1 \left| \frac{K(t+h)(\tau) - K(t)(\tau)}{h} - y(\tau) \right| d\tau &= \int_0^1 \left| \frac{1}{h} \chi_{(t, t+h]}(\tau) - y(\tau) \right| d\tau \\ &= \int_t^{t+h} \left| \frac{1}{h} - y(\tau) \right| d\tau + \int_{[0, 1] \setminus (t, t+h]} |y(\tau)| d\tau \\ &\geq \int_{[0, 1] \setminus (t, t+h]} |y(\tau)| d\tau. \end{aligned}$$

This yields, by (2.1), that

$$\int_{[0,1] \setminus (t, t+h)} |y(\tau)| d\tau \rightarrow 0 \quad \text{when } h \rightarrow 0.$$

It follows that $y = 0$ almost everywhere on $[0, 1]$. Therefore,

$$\int_0^1 \left| \frac{1}{h} \chi_{(t, t+h]}(\tau) - y(\tau) \right| d\tau = \int_0^1 \left| \frac{1}{h} \chi_{(t, t+h]}(\tau) \right| d\tau = \frac{1}{h} \int_t^{t+h} d\tau = 1$$

which contradicts (2.1). The case $h < 0$ is similar. Hence, $K: (0, 1) \rightarrow L^1[0, 1]$ is nowhere differentiable.

Now, we will check that the contingent of the graph of K is trivial at each point $(t, K(t))$, which means that it is zero vector $(0, 0_{L^1[0,1]})$ of $\mathbb{R} \times L^1[0, 1]$.

Clearly, $(0, 0_{L^1[0,1]}) \in \text{Tan}(\text{Gr}(K), (t, K(t)))$ for each $t \in (0, 1)$. Fix $t \in (0, 1)$. Let

$$(\alpha, w) \in \text{Tan}(\text{Gr}(K), (t, K(t))), \quad (\alpha, w) \in \mathbb{R} \times L^1[0, 1].$$

Then, by Definition 1.1, there exist sequences

$$(t_n)_{n \in \mathbb{N}}, \quad t_n \in (0, 1), \quad t_n \rightarrow t, \quad \text{and} \quad (\lambda_n)_{n \in \mathbb{N}}, \quad \lambda_n > 0,$$

such that

$$\lambda_n(t_n - t, K(t_n) - K(t)) \rightarrow (\alpha, w) \quad \text{in } \mathbb{R} \times L^1[0, 1].$$

First, assume that $t_n > t$. Then, $K(t_n) - K(t) = \chi_{(t, t_n]}$. Therefore,

$$\lambda_n(K(t_n) - K(t)) = \lambda_n \chi_{(t, t_n]} \rightarrow w \quad \text{in } L^1[0, 1],$$

i.e.,

$$\lim_{n \rightarrow \infty} \int_0^1 |\lambda_n \chi_{(t, t_n]}(\xi) - w(\xi)| d\xi = 0. \quad (2.2)$$

On the other hand, for each $n \in \mathbb{N}$, we get

$$\begin{aligned} \int_0^1 |\lambda_n \chi_{(t, t_n]}(\xi) - w(\xi)| d\xi &= \int_{(t, t_n]} |\lambda_n \chi_{(t, t_n]}(\xi) - w(\xi)| d\xi + \int_{[0,1] \setminus (t, t_n]} |w(\xi)| d\xi \\ &\geq \int_{[0,1] \setminus (t, t_n]} |w(\xi)| d\xi, \end{aligned}$$

whence by (2.2),

$$\int_{[0,1] \setminus (t, t_n]} |w(\xi)| d\xi \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Since

$$\lim_{n \rightarrow \infty} \int_{[0,1] \setminus (t, t_n]} |w(\xi)| d\xi = \int_0^1 |w(\xi)| d\xi,$$

we obtain $w = 0_{L^1[0,1]}$ almost everywhere, and therefore

$$\int_0^1 |\lambda_n \chi_{(t, t_n]}(\xi) - w(\xi)| d\xi = \int_0^1 |\lambda_n \chi_{(t, t_n]}(\xi)| d\xi = \int_{(t, t_n]} \lambda_n d\xi = \lambda_n(t_n - t) \rightarrow \alpha.$$

It follows by (2.2) that $\alpha = 0$.

The case $t_n < t$ is treated analogously and yields the same result. We conclude that $(\alpha, w) = (0, 0_{L^1[0,1]})$. Thus, $\text{Tan}(\text{Gr}(K), (t, K(t)))$ is trivial for each $t \in (0, 1)$.

THEOREM 2.2. *Let Y be a separable Banach space and $\dim Y = \infty$. Assume that $f: (a, b) \rightarrow Y$ satisfies conditions:*

- (1) *there exists a Lebesgue integrable mapping $g: (a, b) \rightarrow [0, \infty)$ such that for each $t \in (a, b)$,*

$$\limsup_{h \rightarrow 0} \left\| \frac{f(t+h) - f(t)}{h} \right\| \leq g(t);$$

- (2) *for each $t \in (a, b)$, the contingent $\text{Tan}(\text{Gr}(f), (t, f(t)))$ is a one-dimensional vector subspace of $\mathbb{R} \times Y$.*

Under these assumptions, f is almost everywhere differentiable.

Proof. Let $t \in (a, b)$ and assume that $(\alpha(t), w(t)) \in \text{Tan}(\text{Gr}(f), (t, f(t)))$ is a non-zero vector. Such a vector exists by assumption that the contingent is not trivial. By Definition 1.1 there exists a sequence $(t_n)_{n \in \mathbb{N}}$, $t_n \in (a, b)$, $t_n \rightarrow t$, and there exists a sequence $(\lambda_n)_{n \in \mathbb{N}}$, $\lambda_n > 0$, such that $\lambda_n(t_n - t) \rightarrow \alpha(t)$ and $\lambda_n(f(t_n) - f(t)) \rightarrow w(t)$. It is impossible to have $\alpha(t) = 0$, $w(t) \neq 0_Y$. This follows at once from condition (1) because

$$\lambda_n(f(t_n) - f(t)) = \lambda_n(t_n - t) \frac{f(t_n) - f(t)}{t_n - t} \rightarrow w(t). \quad (2.3)$$

Since $(\alpha(t), w(t)) \neq (0, 0_Y)$, we have by (2.3) that there exists

$$\lim_{n \rightarrow \infty} \frac{f(t_n) - f(t)}{t_n - t} = \frac{w(t)}{\alpha(t)}. \quad (2.4)$$

Set $v(t)$ to denote this vector $\frac{w(t)}{\alpha(t)}$. Note that, since $\dim \text{Tan}(\text{Gr}(f), (t, f(t))) = 1$, it is clear that $v(t)$ is defined uniquely for each $t \in (a, b)$ and we have

$$\text{Tan}(\text{Gr}(f), (t, f(t))) = \left\{ \xi(1, v(t)) : \xi \in \mathbb{R} \right\}. \quad (2.5)$$

Next, we will show that the mapping $v: (a, b) \rightarrow Y$ is Bochner integrable. Let $y^* \in Y^*$. It follows by (1) that

$$\forall t \in (a, b): \limsup_{h \rightarrow 0} \left| \frac{(y^* \circ f)(t+h) - (y^* \circ f)(t)}{h} \right| \leq \|y^*\|g(t) < \infty. \quad (2.6)$$

Since $y^* \circ f$ is real-valued, it is differentiable almost everywhere on (a, b) . Let $y^* \circ f$ be differentiable at $t \in (a, b)$, and $(t_n)_{n \in \mathbb{N}}$ be a sequence, $t_n \in (a, b)$, $t_n \rightarrow t$, such that (cf. (2.4))

$$v(t) = \lim_{n \rightarrow \infty} \frac{f(t_n) - f(t)}{t_n - t}.$$

Then, clearly,

$$y^*(v(t)) = \lim_{n \rightarrow \infty} y^* \left(\frac{f(t_n) - f(t)}{t_n - t} \right) = (y^* \circ f)'(t).$$

Therefore,

$$\forall y^* \in Y^*: (y^* \circ f)'(t) = y^*(v(t)) \quad \text{almost everywhere on } (a, b). \quad (2.7)$$

Thus, $y^* \circ f$ satisfies Lusin condition (N) (by (2.6) and Theorem 4.6 from [8]) which yields together with (1) and Theorem 7.7 from [8] that

$$\forall y^* \in Y^*: y^* \circ f \in AC(a, b).$$

Due to (2.7), it follows that $y^* \circ v$ is measurable, hence v is weakly measurable. Since Y is separable and v is weakly measurable, we get by Theorem 1.6 that the mapping v is measurable, i.e., there exists a sequence $(v_n)_{n \in \mathbb{N}}$ of step functions $v_n: (a, b) \rightarrow Y$ taking a finite number of values such that

$$\lim_{n \rightarrow \infty} \|v_n - v\| = 0 \quad \text{almost everywhere on } (a, b).$$

It follows that $\|v_n\| \rightarrow \|v\|$ almost everywhere on (a, b) . However, for each $n \in \mathbb{N}$, the mapping $t \mapsto \|v_n(t)\|$ is measurable (it is a finitely-valued real function, whence $t \mapsto \|v(t)\|$ is measurable). By (1), we get $\|v(t)\| \leq g(t)$ for each $t \in (a, b)$, whence

$$\int_a^b \|v(\tau)\| d\tau < \infty. \quad (2.8)$$

Therefore, $v: (a, b) \rightarrow Y$ is Bochner integrable (Theorem 1.9).

Fix $t_0 \in (a, b)$ and consider the mapping $F: (a, b) \rightarrow Y$ defined by

$$F(t) = - \int_{t_0}^t v(\tau) d\tau.$$

From Corollary 1.15, we obtain that F is $AC(a, b)$, and from Theorem 1.11, we get, in view of (2.8), that $F'(t) = -v(t)$ for almost all $t \in (a, b)$. Put $t \in (a, b)$ such that $F'(t)$ exists and $F'(t) = -v(t)$. Fix $y^* \in Y^*$. By (2.5), we have

$$\text{Tan}\left(\text{Gr}(y^* \circ f), (t, (y^* \circ f)(t))\right) = \left\{ \xi \left(1, y^*(v(t)) \right) : \xi \in \mathbb{R} \right\}.$$

Applying Lemma 1.2 for $y^* \circ F$ and $y^* \circ f$, we get

$$\begin{aligned} & \text{Tan}\left(\text{Gr}(y^* \circ F + y^* \circ f), (t, (y^* \circ F + y^* \circ f)(t))\right) \\ &= \left\{ (\alpha, (y^* \circ F)'(t)\alpha + \beta) : (\alpha, \beta) \in \text{Tan}\left(\text{Gr}(y^* \circ f), (t, (y^* \circ f)(t))\right) \right\} \\ &= \left\{ (\xi, -y^*(v(t))\xi + y^*(v(t))\xi) : \xi \in \mathbb{R} \right\} = \{(\xi, 0) : \xi \in \mathbb{R}\}. \end{aligned}$$

In other words, for each $y^* \in Y^*$,

$$\text{Tan}\left(\text{Gr}(y^* \circ F + y^* \circ f), (t, (y^* \circ F + y^* \circ f)(t))\right) = \{(\xi, 0) : \xi \in \mathbb{R}\}$$

for almost all $t \in (a, b)$.

Therefore, for each $y^* \in Y^*$, we get that $y^* \circ (F + f)$ is $AC(a, b)$ and

$$(y^* \circ (F + f))'(t) = 0 \quad \text{for almost each } t \in (a, b).$$

It follows that the mapping $y^* \circ (F + f)$ is constant for each $y^* \in Y^*$. By Lemma 1.16, $F + f$ is constant. Since F is almost everywhere differentiable, we conclude that so is f . $\square$

Remark 2.3. In particular, if f is Lipschitz (in Theorem 2.2) and (2) is fulfilled, then condition (1) is obviously satisfied, whence f is almost everywhere differentiable and $\|v\|$ is bounded. Also observe that the mapping g (in Theorem 2.2) may be considered only locally integrable.

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ON DIFFERENTIABILITY OF MAPPINGS WITH FINITE DILATION

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