DENSITY TOPOLOGIES ON THE PLANE BETWEEN ORDINARY AND STRONG. II

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ABSTRACT. Let C_0 denote a set of all non-decreasing continuous functions $f : (0, 1] \rightarrow (0, 1]$ such that $\lim_{x \rightarrow 0+} f(x) = 0$ and $f(x) \leq x$ for every $x \in (0, 1]$, and let A be a measurable subset of the plane. The notions of a density point of A with respect to f and the mapping D_f defined on the family of all measurable subsets of the plane were introduced in Wagner-Bojakowska, E.—Wilczyński, W.: *Density topologies on the plane between ordinary and strong*, Tatra Mt. Math. Publ. 44 (2009), 139–151. This mapping is a lower density, so it allowed us to introduce the topology $\mathcal{T}_f$, analogously to the density topology. In this note, properties of the topology $\mathcal{T}_f$ and functions approximately continuous with respect to f are considered. We prove that $(\mathbb{R}^2, \mathcal{T}_f)$ is a completely regular topological space and we study conditions under which topologies generated by two functions f and g are equal.

Let $\mathcal{S}$ denote a family of all Lebesgue measurable subsets of the plane and m_2 the Lebesgue plane measure. For $A \in \mathcal{S}$, let $\Phi_0(A)$ and $\Phi_s(A)$ denote a set of all ordinary density points of A and a set of all strong density points of A , respectively (see [2]).

If $A, B \in \mathcal{S}$, then we shall write $A \sim B$ if and only if $m_2(A \Delta B) = 0$ (where $A \Delta B = (A \setminus B) \cup (B \setminus A)$).

Let C_0 denote a set of all non-decreasing continuous functions $f : (0, 1] \rightarrow (0, 1]$ such that $\lim_{x \rightarrow 0+} f(x) = 0$ and $f(x) \leq x$ for every $x \in (0, 1]$.

Let $f \in C_0$ and $A \in \mathcal{S}$.

DEFINITION 1 ([3]). We shall say that $(0, 0)$ is a *density point of A with respect to a function f* if for each $\epsilon > 0$ there exists $\delta > 0$ such that for each $h, k \in (0, \delta)$ if $f(h) \leq k \leq h$ or $f(k) \leq h \leq k$, then

$$\frac{m_2(A \cap \{[-h, h] \times [-k, k]\})}{4hk} > 1 - \epsilon,$$

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2010 Mathematics Subject Classification: Primary 28A05; Secondary 54A05, 54A10, 54D15.

Keywords: density point, density topology, density point with respect to f .

The research was supported by the grant N N201 547238.

or, equivalently,

$$\frac{m_2(A' \cap \{[-h, h] \times [-k, k]\})}{4hk} < \epsilon,$$

where $A' = \mathbb{R}^2 \setminus A$.

We say that (x_0, y_0) is a density point of A with respect to f if $(0, 0)$ is a density point of $A - (x_0, y_0) = \{(x - x_0, y - y_0) : (x, y) \in A\}$ with respect to f .

For $A \in \mathcal{S}$ and $f \in C_0$, by $D_f(A)$ we shall denote a set of all points $(x, y) \in \mathbb{R}^2$ which are density points of A with respect to f . Clearly, $\Phi_s(A) \subset D_f(A) \subset \Phi_0(A)$ for every $A \in \mathcal{S}$. The mapping $D_f : \mathcal{S} \rightarrow 2^{\mathbb{R}^2}$ is a lower density (see [3, Theorem 5]), so the family

$$\mathcal{T}_f = \{A \in \mathcal{S} : A \subset D_f(A)\}$$

is a topology on the plane, essentially stronger than the strong density topology $\mathcal{T}_s$ on the plane and weaker than the ordinary density topology $\mathcal{T}_0$ on the plane. The properties of the topology $\mathcal{T}_f$ were studied in [3], (see Theorems 7–15). Among others, there was proved that $(\mathbb{R}^2, \mathcal{T}_f)$ is a regular (T_3) Baire space. In this note we shall show that $(\mathbb{R}^2, \mathcal{T}_f)$ is a completely regular $(T_{3\frac{1}{2}})$ topological space. For this purpose, observe that the theorem analogous to Lusin-Menchoff theorem holds. The proof of the next theorem is analogous to the proof of Lusin-Menchoff theorem in [2] with necessary changes.

THEOREM 1. *Let $E \in \mathcal{S}$ and let X be a closed subset of the plane such that $X \subset E \cap D_f(E)$. Then, there exists a perfect set $P \subset E$ such that*

$$X \subset P \cap D_f(P).$$

Proof. As $D_f(E) \subset \Phi_0(E)$, using in [1, Lemma 6.3, Chapter II], we obtain that there exists a perfect set P_0 such that $X \subset P_0 \subset E$. Additionally, we may suppose that $E \subset [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$. Put

$$Q_n = \left\{ (x, y) \in \mathbb{R}^2 : \frac{1}{n+1} < \text{dist}((x, y), P_0) \leq \frac{1}{n} \right\}$$

and

$$S_n = Q_n \cap E$$

for every $n \in \mathbb{N}$, where the distance $\text{dist}((x, y), P_0)$ of a point (x, y) from a set P_0 is considered in the metric ρ defined as follows

$$\rho((x_1, y_1), (x_2, y_2)) = \max(|x_1 - x_2|, |y_1 - y_2|)$$

for all $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$. Obviously, $E = P_0 \cup \bigcup_{n=1}^{\infty} S_n$. For each $n \in \mathbb{N}$, let P_n be a perfect subset of S_n such that

$$m_2(S_n \setminus P_n) < \frac{1}{2^n} \cdot f\left(\frac{1}{n+1}\right). \quad (1)$$

Put

$$P = P_0 \cup \bigcup_{n=1}^{\infty} P_n.$$

It is easy to see that P is perfect and $X \subset P \subset E$. Now, we shall prove that each point of X is a density point of P with respect to f . Let $(x, y) \in X$ and let $\{I_j\}_{j \in \mathbb{N}}$ be a sequence of intervals centered at (x, y) such that

$$\text{diam}(I_j) \xrightarrow{j \rightarrow \infty} 0, \quad \text{diam}(I_j) \leq 1 \quad \text{for } j \in \mathbb{N},$$

and if

$$I_j = (x - h_j, x + h_j) \times (y - k_j, y + k_j), \quad \text{then} \quad f(h_j) \leq k_j \leq h_j$$

or

$$f(k_j) \leq h_j \leq k_j \quad \text{for } j \in \mathbb{N}.$$

Let j be a fixed positive integer. There are two cases:

1° $I_j \cap S_n = \emptyset$ for each $n \in \mathbb{N}$. Then, $I_j \cap E = I_j \cap P_0$, and consequently,

$$m_2(E \cap I_j) = m_2(P_0 \cap I_j) \leq m_2(P \cap I_j).$$

2° $I_j \cap S_n \neq \emptyset$ for some $n \in \mathbb{N}$. Put $n_j = \min\{n \in \mathbb{N} : I_j \cap S_n \neq \emptyset\}$.

Then, $I_j \cap S_n = \emptyset$ for $n < n_j$ and $I_j \cap S_{n_j} \neq \emptyset$, so

$$(E \setminus P) \cap I_j \subset \bigcup_{n=n_j}^{\infty} (S_n \setminus P_n)$$

and, consequently, using (1), we obtain

$$\begin{aligned} m_2(E \cap I_j) &= m_2(P \cap I_j) + m_2((E \setminus P) \cap I_j) \\ &\leq m_2(P \cap I_j) + \sum_{n=n_j}^{\infty} m_2(S_n \setminus P_n) \\ &< m_2(P \cap I_j) + f\left(\frac{1}{n_j + 1}\right) \cdot \frac{1}{2^{n_j - 1}}. \end{aligned}$$

So,

$$m_2(P \cap I_j) > m_2(E \cap I_j) - f\left(\frac{1}{n_j + 1}\right) \cdot \frac{1}{2^{n_j - 1}}.$$

If $f(h_{n_j}) \leq k_{n_j} \leq h_{n_j}$, then $\frac{1}{n_j + 1} \leq h_{n_j}$, so $f\left(\frac{1}{n_j + 1}\right) \leq f(h_{n_j}) \leq k_{n_j}$, and consequently,

$$m_2(I_j) = 4h_j k_j \geq 4f\left(\frac{1}{n_j + 1}\right) \cdot \frac{1}{n_j + 1}.$$

Analogously, if $f(k_{n_j}) \leq h_{n_j} \leq k_{n_j}$. Hence,

$$\frac{m_2(P \cap I_j)}{m_2(I_j)} > \frac{m_2(E \cap I_j)}{m_2(I_j)} - \frac{f\left(\frac{1}{n_j + 1}\right) \cdot \frac{1}{2^{n_j - 1}}}{4f\left(\frac{1}{n_j + 1}\right) \cdot \frac{1}{n_j + 1}} = \frac{m_2(E \cap I_j)}{m_2(I_j)} - \frac{n_j + 1}{2^{n_j + 1}}.$$

Clearly, $n_j \xrightarrow{j \rightarrow \infty} \infty$, so

$$\liminf_{j \rightarrow \infty} \frac{m_2(P \cap I_j)}{m_2(I_j)} \geq \liminf_{j \rightarrow \infty} \frac{m_2(E \cap I_j)}{m_2(I_j)} = 1$$

because $(x, y) \in X$ and $X \subset D_f(E)$. Consequently, $X \subset D_f(P)$. $\square$

For $X, E \in \mathcal{S}$ we shall write $X \subset \cdot E$ if $X \subset E \cap D_f(E)$, i.e., X is contained in the interior of E in the topology $\mathcal{T}_f$.

COROLLARY 1. *Let $E \in \mathcal{T}_f$ and let X be a closed subset of E . Then, there exists a perfect set P such that*

$$X \subset \cdot P \subset \cdot E.$$

Proof. Since $E \in \mathcal{T}_f$, so E is measurable and X is a closed subset such that $X \subset E \cap D_f(E)$. According to Theorem 1, there exists a perfect set P such that $X \subset P \cap D_f(P)$ and $P \subset E$. Consequently, $X \subset \cdot P$ and $P \subset \cdot E$, because $E \subset \cdot E$. $\square$

DEFINITION 2. We shall say that a function $g: \mathbb{R}^2 \rightarrow \mathbb{R}$ is *f-approximately continuous at $p_0 \in \mathbb{R}^2$* if there exists a set $E \in \mathcal{S}$ such that $p_0 \in E \cap D_f(E)$ and $g|_E$ is continuous at p_0 . We shall say that a function $g: \mathbb{R}^2 \rightarrow \mathbb{R}$ is *f-approximately continuous* if it is *f-approximately continuous at each point $p \in \mathbb{R}^2$* .

Obviously, $D_f(E) \subset \Phi_0(E)$ for each $E \in \mathcal{S}$, so each *f-approximately continuous* function is approximately continuous and, consequently, a Baire one function.

THEOREM 2. *If g_1 and g_2 are f-approximately continuous at p_0 , then $g_1 + g_2$, $g_1 - g_2$, $g_1 \cdot g_2$ are also f-approximately continuous at p_0 . If $g_2(p) \neq 0$ on some neighbourhood of p_0 , then g_1/g_2 is f-approximately continuous at p_0 .*

THEOREM 3. *If $g_1: \mathbb{R}^2 \rightarrow \mathbb{R}$ is f-approximately continuous at p_0 and $g_2: \mathbb{R} \rightarrow \mathbb{R}$ is continuous at $g_1(p_0)$, then $g_2 \circ g_1$ is f-approximately continuous at p_0 .*

Observe that for our kind of continuity we obtain a result analogous to classical Denjoy theorem (see in [1, Theorem 5.2, Chapter II]).

Let $I = (a, b)$ be an arbitrary interval on the real line.

THEOREM 4. *A function $g: I \times I \rightarrow \mathbb{R}$ is measurable if and only if it is f-approximately continuous a.e.*

Proof. If g is *f-approximately continuous a.e.*, then it is approximately continuous a.e., so, by Denjoy theorem, it is measurable.

Suppose now that g is measurable. Then, by Lusin theorem, for each $n \in \mathbb{N}$ there exists a closed set $F_n \subset (I \times I)$ such that $m_2(I \times I \setminus F_n) < \frac{1}{n}$ and $g|_{F_n}$ is continuous at each point $p \in F_n$.

Put

$$F = \bigcup_{n=1}^{\infty} (F_n \cap D_f(F_n)).$$

We shall prove that $m_2((I \times I) \setminus F) = 0$ and g is f -approximately continuous at each point $p \in F$. Obviously,

$$\begin{aligned} m_2((I \times I) \setminus F) &\leq m_2((I \times I) \setminus (F_n \cap D_f(F_n))) \\ &\leq m_2((I \times I) \setminus F_n) + m_2(F_n \setminus D_f(F_n)) < \frac{1}{n} \end{aligned}$$

for each $n \in \mathbb{N}$, since $m_2(F_n \setminus D_f(F_n)) = 0$. Consequently, $m_2((I \times I) \setminus F) = 0$. Now, let $p \in F$. Then, there exists $n_0 \in \mathbb{N}$ such that $p \in F_{n_0} \cap D_f(F_{n_0})$ and $g|_{F_{n_0}}$ is continuous at p . Hence, g is f -approximately continuous at p . $\square$

It is clear that the last theorem holds also in the case when the domain of g is an open rectangle or the whole plane $\mathbb{R}^2$.

We can also characterize the f -approximately continuous functions in terms of associated sets (compare with [1, Theorem 5.6, Chapter II]). For this purpose, we need a following lemma.

LEMMA 1. *If $\{E_n\}_{n \in \mathbb{N}}$ is a descending sequence of measurable sets such that $(0, 0) \in D_f(E_n)$ for each $n \in \mathbb{N}$, then there exists a strictly decreasing sequence $\{l_n\}_{n \in \mathbb{N}}$ of positive numbers tending to zero such that*

$$(0, 0) \in D_f \left(\bigcup_{n=1}^{\infty} (E_n \setminus \{[-l_n, l_n] \times [-l_n, l_n]\}) \right).$$

Proof. Let $\{\epsilon_n\}_{n \in \mathbb{N}}$ be a sequence of positive numbers less than 1 tending decreasingly to zero. For each $n \in \mathbb{N}$, there exists $\delta_n > 0$ such that for each $h, k \in (0, \delta_n)$ if $f(h) \leq k \leq h$ or $f(k) \leq h \leq k$, then

$$\frac{m_2(E_n \cap \{[-h, h] \times [-k, k]\})}{4hk} > 1 - \epsilon_n.$$

Let $\{h_n\}_{n \in \mathbb{N}}$ be a sequence of positive numbers tending decreasingly to zero such that $h_n \in (0, \delta_n)$ for all $n \in \mathbb{N}$. Put

$$l_n = \epsilon_n \cdot h_{n+1} \quad \text{for all } n \in \mathbb{N}$$

and

$$A = \bigcup_{n=1}^{\infty} (E_n \setminus \{[-l_n, l_n] \times [-l_n, l_n]\}).$$

We shall prove that $(0, 0) \in D_f(A)$. Let $\epsilon > 0$. Since $\epsilon_n \searrow_{n \rightarrow \infty} 0$, there exists $n_0 \in \mathbb{N}$ such that $2\epsilon_{n_0} < \epsilon$. Put $\delta = h_{n_0}$. Let $h, k \in (0, \delta)$ and assume that $f(h) \leq k \leq h$. There exists $n_1 \in \mathbb{N}$, $n_1 \geq n_0$ such that $h \in [h_{n_1+1}, h_{n_1})$.

Then,

$$\begin{aligned}
 & m_2\left(A \cap \{[-h, h] \times [-k, k]\}\right) \\
 & \geq m_2\left(E_{n_1} \cap \{[-h, h] \times [-k, k]\}\right) - m_2([-l_{n_1}, l_{n_1}] \times [-k, k]) \\
 & = m_2\left(E_{n_1} \cap \{[-h, h] \times [-k, k]\}\right) - 4k\epsilon_{n_1}h_{n_1+1}.
 \end{aligned}$$

Simultaneously, $4hk \geq 4h_{n_1+1}k$ and $h, k \in (0, h_{n_1}) \subset (0, \delta_{n_1})$, so

$$\begin{aligned}
 & \frac{m_2(A \cap \{[-h, h] \times [-k, k]\})}{4hk} \\
 & \geq \frac{m_2(E_{n_1} \cap \{[-h, h] \times [-k, k]\})}{4hk} - \frac{4k\epsilon_{n_1}h_{n_1+1}}{4hk} \\
 & \geq 1 - 2\epsilon_{n_1} \geq 1 - 2\epsilon_{n_0} > 1 - \epsilon.
 \end{aligned}$$

If $f(k) \leq h \leq k$, then the proof is analogous. Consequently, $(0, 0) \in D_f(A)$. $\square$

THEOREM 5. *A function $g : I \times I \rightarrow \mathbb{R}$ is f -approximately continuous if and only if for each $\alpha \in \mathbb{R}$ the sets*

$$E^\alpha = \{p \in \mathbb{R}^2 : g(p) < \alpha\} \quad \text{and} \quad E_\alpha = \{p \in \mathbb{R}^2 : g(p) > \alpha\}$$

are of the type F_σ and open in the topology $\mathcal{T}_f$.

Proof. Suppose that g is f -approximately continuous. Fix $\alpha \in \mathbb{R}$. As g is approximately continuous, the sets E^α and E_α are of the type F_σ . Observe that $E^\alpha \in \mathcal{T}_f$. Indeed, let $p_0 = (x_0, y_0) \in E^\alpha$. Then, $g(p_0) < \alpha$ and there exists a measurable set E such that $p_0 \in E \cap D_f(E)$ and $g|_E$ is continuous at p_0 . So, there exists $\delta > 0$ such that

$$g(p) < \alpha \quad \text{for} \quad p \in \{(x_0 - \delta, x_0 + \delta) \times (y_0 - \delta, y_0 + \delta)\} \cap E.$$

Consequently,

$$\{(x_0 - \delta, x_0 + \delta) \times (y_0 - \delta, y_0 + \delta)\} \cap E \subset E^\alpha$$

and

$$p_0 \in D_f(\{(x_0 - \delta, x_0 + \delta) \times (y_0 - \delta, y_0 + \delta)\} \cap E),$$

so $p_0 \in D_f(E^\alpha)$. For the set E_α , the proof is analogous.

Suppose now that for each $\alpha \in \mathbb{R}$ the sets E^α and E_α are of the type F_σ and each point $p \in E^\alpha$ (resp. E_α) is a density point of E^α (resp. E_α) with respect to f . We shall prove that g is f -approximately continuous on $I \times I$. Let $p_0 \in I \times I$ and let ϵ be an arbitrary positive real number. To simplify the denotation, we may assume that $I = (-1, 1)$ and $p_0 = (0, 0)$. Put

$$E_n = \left\{ p \in I \times I : |g(p) - g(0, 0)| < \frac{1}{n} \right\} \quad \text{for each } n \in \mathbb{N}.$$

Obviously,

$$E_n = \left\{ p \in I \times I : g(p) < g(0,0) + \frac{1}{n} \right\} \cap \left\{ p \in I \times I : g(p) > g(0,0) - \frac{1}{n} \right\},$$

so $E_n \subset D_f(E_n)$ and $(0,0) \in E_n$ for each $n \in \mathbb{N}$. Clearly, $\{E_n\}_{n \in \mathbb{N}}$ is a descending sequence of measurable sets such that $(0,0) \in D_f(E_n)$ for each $n \in \mathbb{N}$. By Lemma 1, there exists a strictly decreasing sequence $\{l_n\}_{n \in \mathbb{N}}$ of positive numbers tending to zero such that $(0,0)$ is a density point of a set

$$A = \bigcup_{n=1}^{\infty} \left(E_n \setminus \{[-l_n, l_n] \times [-l_n, l_n]\} \right) \cup \{(0,0)\}$$

with respect to f , so $(0,0) \in A \cap D_f(A)$ and $g|_A$ is continuous at $(0,0)$. $\square$

From the last theorem, it follows that each f -approximately continuous function is continuous with respect to the topology $\mathcal{T}_f$ in the domain.

Using Corollary 1, we can prove the next lemmas analogously to the proofs in [4, Lemma 11 and Lemma 12].

LEMMA 2. *Let E be a set of the type F_σ such that $E \subset D_f(E)$. There exists an f -approximately continuous function g such that*

- (i) $0 < g(p) \leq 1$ for $p \in E$,
- (ii) $g(p) = 0$ for $p \notin E$.

LEMMA 3. *Let E_1, E_2, H be pairwise disjoint subsets of the plane such that $E_1 \cup E_2 \cup H = \mathbb{R}^2$ and $E_1 \cup H, E_2 \cup H$ are $\mathcal{T}_f$ -open sets of the type F_σ . There exists f -approximately continuous function $g: \mathbb{R}^2 \rightarrow \mathbb{R}$ such that*

- (i) $g(p) = 0$ for $p \in E_1$,
- (ii) $0 < g(p) < 1$ for $p \in H$,
- (iii) $g(p) = 1$ for $p \in E_2$.

From the last lemma we obtain the next result similarly as in [2] for ordinary density topology.

THEOREM 6. *The space $(\mathbb{R}^2, \mathcal{T}_f)$ is completely regular $(T_{3\frac{1}{3}})$.*

Clearly, the space $(\mathbb{R}^2, \mathcal{T}_f)$ is not normal, analogously as the ordinary density topology on the plane. If A and B are two disjoint dense (in the Euclidean topology) subsets of the plane of measure zero, then no f -approximately function g separates these sets, since each function separating A and B is discontinuous at each point and g is Baire 1.

Observe that each measurable subset of the plane is of the type G_δ in the topology $\mathcal{T}_f$. Let $A \in \mathcal{S}$. Then,

$$A = (A \cap D_f(A)) \cup (A \setminus D_f(A)).$$

The first summand is the interior of A in the topology $\mathcal{T}_f$ (see [3, Theorem 7]). From [3, Theorem 5], we have $m_2(A \setminus D_f(A)) = 0$. Let E be a set of the type G_δ (in the Euclidean topology) of measure zero such that $A \setminus D_f(A) \subset E$. Obviously,

$$m_2(E \setminus (A \setminus D_f(A))) = 0, \quad A \setminus D_f(A) = E \setminus (E \setminus (A \setminus D_f(A)))$$

and $E \setminus (A \setminus D_f(A))$ is closed in the topology $\mathcal{T}_f$. Consequently, $A \setminus D_f(A)$ is of the type G_δ in the topology $\mathcal{T}_f$, so A is of this type, too.

Let $f, g \in C_0$ and $f(x) \leq g(x)$ for every $x \in (0, 1]$. Then, for arbitrary measurable subset A of the plane, $D_f(A) \subset D_g(A)$ and, consequently, $\mathcal{T}_f \subset \mathcal{T}_g$. In [3] we were looking for conditions under which two topologies $\mathcal{T}_f$ and $\mathcal{T}_g$ are equal. We proved that if $f \in C_0$ and $c \in (0, 1]$, then $D_f(A) = D_{cf}(A)$ for $A \in \mathcal{S}$, so $\mathcal{T}_f = \mathcal{T}_{cf}$ (see [3, Theorem 3]). More generally, if

$$f, g \in C_0, \quad f(x) \leq g(x) \quad \text{for every } x \in (0, 1]$$

and

$$\liminf_{x \rightarrow 0^+} \frac{f(x)}{g(x)} > 0,$$

then $\mathcal{T}_f = \mathcal{T}_g$ ([3, Theorem 17]). Particularly, if

$$\liminf_{x \rightarrow 0^+} \frac{f(x)}{x} > 0,$$

then $\mathcal{T}_f = \mathcal{T}_0$.

Another condition sufficient for the equality $\mathcal{T}_f = \mathcal{T}_g$, which is described in terms of porosity, is given in [3, Theorem 20]. This theorem says that if $f, g \in C_0$, $g(x) \leq f(x)$ for every $x \in (0, 1]$ and for each decreasing sequence $\{c_n\}_{n \in \mathbb{N}}$ convergent to zero

$$\lim_{n \rightarrow \infty} p^+(M_{c_n}, 0) = 0,$$

where $M_c = \{x \in (0, 1] : g(x) \geq c \cdot f(x)\}$ for $c \in (0, 1)$ and $p^+(A, x)$ denotes the right-hand porosity of A at x , then $\mathcal{T}_f = \mathcal{T}_g$.

Here, we give some condition, which is necessary for the equality $\mathcal{T}_f = \mathcal{T}_g$ for arbitrary functions $f, g \in C_0$ such that $g(x) \leq f(x)$ for every $x \in (0, 1]$. We also prove that this condition is sufficient if g is a strictly increasing function.

Let $f, g \in C_0$ be such that $g(x) \leq f(x)$ for every $x \in (0, 1]$. Let $x \in (0, 1]$ and $n \in \mathbb{N}$. If $g(x) \leq f(x)/n$, then put

$$z_n(x) = \min \left\{ \sup \left\{ z \in [x, 1] : g(z) \leq \frac{1}{n} f(x) \right\}, 2x \right\}.$$

If $g(x) > f(x)/n$, then $z_n(x) = x$.

Let n be a fixed positive integer, and let $f, g \in C_0$. If there exists $\delta > 0$ such that

$$\frac{1}{n} f(x) \leq g(x) \leq f(x) \quad \text{for every } x \in (0, \delta],$$

then

$$D_{\frac{1}{n}f}(A) \subset D_g(A) \subset D_f(A) \quad \text{for every } A \in \mathcal{S}.$$

Simultaneously, $D_{\frac{1}{n}f}(A) = D_f(A)$ by [3, Theorem 3] for every $A \in \mathcal{S}$, so $D_f(A) = D_g(A)$ for every $A \in \mathcal{S}$ and $\mathcal{T}_f = \mathcal{T}_g$.

DEFINITION 3. Let $f, g \in C_0$ be such that $g(x) \leq f(x)$ for every $x \in (0, 1]$. We shall say that the pair (f, g) fulfills condition (W) if and only if, for each real number $a > 0$, there exist $n \in \mathbb{N}$ and $\delta > 0$ such that

$$\frac{z_n(x) - x}{z_n(x)} \leq a \quad \text{for every } x \in (0, \delta).$$

THEOREM 7. Let $f, g \in C_0$ be such that $g(x) \leq f(x)$ for every $x \in (0, 1]$ and for each $\delta > 0$ and each $n \in \mathbb{N}$ there exists $x \in (0, \delta)$ such that $g(x) < f(x)/n$. If $D_g(A) = D_f(A)$ for arbitrary $A \in \mathcal{S}$, then the pair (f, g) fulfills condition (W).

Proof. Suppose, on the contrary, that there exists a number $a > 0$ such that for each $n \in \mathbb{N}$ and $\delta > 0$ there exists a point $x \in (0, \delta)$ for which

$$\frac{z_n(x) - x}{z_n(x)} > a.$$

We shall prove that $D_f(A) \neq D_g(A)$ for some $A \in \mathcal{S}$. For this purpose, we construct a set $A \subset [0, 1] \times [0, 1]$ such that $(0, 0)$ is a dispersion point of A with respect to f and is not a dispersion point of A with respect to g .

First, by easy induction, we define a sequence $\{x_n\}_{n \in \mathbb{N}}$ such that

$$\frac{z_n(x_n) - x_n}{z_n(x_n)} > a \quad \text{and} \quad x_{n+1} < \frac{1}{2}x_n \quad \text{for every } n \in \mathbb{N}.$$

Obviously,

$$z_n(x_n) \neq x_n,$$

so

$$g(z_n(x_n)) \leq \frac{1}{n}f(x_n) \quad \text{and} \quad z_n(x_n) \leq 2x_n \quad \text{for every } n \in \mathbb{N}.$$

Put

$$A = \bigcup_{n=1}^{\infty} \left\{ [x_n, z_n(x_n)] \times [0, g(z_n(x_n))] \right\}. \quad (2)$$

Obviously,

$$\begin{aligned} & \frac{m_2(A \cap \{[0, z_n(x_n)] \times [0, g(z_n(x_n))]\})}{m_2([0, z_n(x_n)] \times [0, g(z_n(x_n))])} \\ & > \frac{(z_n(x_n) - x_n) \cdot g(z_n(x_n))}{z_n(x_n) \cdot g(z_n(x_n))} = \frac{z_n(x_n) - x_n}{z_n(x_n)} > a \end{aligned}$$

for every $n \in \mathbb{N}$, so $(0, 0)$ is not a dispersion point of A with respect to g .

Fix $n \in \mathbb{N}$. For $t \in [z_n(x_n), x_{n-1}]$ and $y \in [f(t), t]$, using (2), we have

$$\frac{m_2(A \cap \{[0, t] \times [0, y]\})}{m_2([0, t] \times [0, y])} \leq \frac{z_n(x_n) \cdot g(z_n(x_n))}{z_n(x_n) \cdot f(z_n(x_n))} \leq \frac{\frac{1}{n}f(x_n)}{f(x_n)} = \frac{1}{n}$$

because

$$g(z_n(x_n)) \leq \frac{1}{n}f(x_n) \quad \text{and} \quad x_n \leq z_n(x_n).$$

For $t \in [x_n, z_n(x_n)]$ and $y \in [f(t), t]$, using (2), we obtain

$$\frac{m_2(A \cap \{[0, t] \times [0, y]\})}{m_2([0, t] \times [0, y])} \leq \frac{z_n(x_n) \cdot g(z_n(x_n))}{x_n \cdot f(x_n)} \leq \frac{2 \cdot \frac{1}{n}f(x_n)}{f(x_n)} = \frac{2}{n}$$

because

$$z_n(x_n) \leq 2x_n \quad \text{and} \quad g(z_n(x_n)) \leq \frac{1}{n}f(x_n).$$

If $[0, t] \times [0, y]$ is such that $f(y) \leq t \leq y$, then

$$A \cap \{[0, t] \times [0, y]\} = A \cap \{[0, t] \times [0, t]\},$$

so

$$\frac{m_2(A \cap \{[0, t] \times [0, y]\})}{m_2([0, t] \times [0, y])} \leq \frac{m_2(A \cap \{[0, t] \times [0, t]\})}{m_2([0, t] \times [0, t])},$$

and finally,

$$\frac{m_2(A \cap \{[0, t] \times [0, y]\})}{m_2([0, t] \times [0, y])} \leq \frac{1}{n} \quad \text{for } t \in [z_n(x_n), x_{n-1}]$$

and

$$\frac{m_2(A \cap \{[0, t] \times [0, y]\})}{m_2([0, t] \times [0, y])} \leq \frac{2}{n} \quad \text{for } t \in [x_n, z_n(x_n)].$$

Since $n \in \mathbb{N}$ was arbitrary, $(0, 0)$ is a dispersion point of A with respect to f . $\square$

THEOREM 8. *Let $f, g \in C_0$ be such that $g(x) \leq f(x)$ for every $x \in (0, 1]$ and for each $\delta > 0$ and each $n \in \mathbb{N}$ there exists $x \in (0, \delta)$ such that $g(x) < f(x)/n$. If g is a strictly increasing function and the pair (f, g) fulfills condition (W), then $D_g(A) = D_f(A)$ for every $A \in \mathcal{S}$.*

Proof. Obviously, $D_g(A) \subset D_f(A)$ for every $A \in \mathcal{S}$. Now, we shall prove that if g is strictly increasing and condition (W) is fulfilled, then $D_f(A) \subset D_g(A)$ for every $A \in \mathcal{S}$.

Let $A \in \mathcal{S}$ and let $(0, 0)$ be a dispersion point of A with respect to f . We shall prove that $(0, 0)$ is a dispersion point of A with respect to g , i.e., for each $\epsilon > 0$ there exists $\eta > 0$ such that for each $x, y \in (0, \eta)$ if $g(x) \leq y < f(x)$ or $g(y) \leq x < f(y)$, then

$$\frac{m_2(A \cap \{[-x, x] \times [-y, y]\})}{4xy} < \epsilon$$

(if $f(x) \leq y \leq x$ or $f(y) \leq x \leq y$, then the inequality follows from our assumption).

Let $\epsilon > 0$. We may assume that $\epsilon < 1$. Put $a = \epsilon/2$. From (W), there exist a positive integer n_0 and a positive real number $\delta_1 < 1/2$ such that for each $x \in (0, \delta_1)$ we have

$$\frac{z_{n_0}(x) - x}{z_{n_0}(x)} < \frac{\epsilon}{2}. \quad (3)$$

Then, $z_{n_0}(x) - x < z_{n_0}(x)/2$, so $x \leq z_{n_0}(x) < 2x < 1$, and consequently, from definition of $z_{n_0}(x)$ and the continuity of f and g , we have $g(z_{n_0}(x)) = f(x)/n_0$ for every $x \in (0, \delta_1)$.

Since $(0, 0)$ is a dispersion point of A with respect to f , there exists $\delta_2 > 0$ such that for each $x \in (0, \delta_2)$ and $y \in [f(x), x]$

$$\frac{m_2(A \cap \{[-x, x] \times [-y, y]\})}{4xy} < \frac{\epsilon}{2n_0}. \quad (4)$$

Put $\delta = \min(\delta_1, \delta_2/2)$. Let $x \in (0, \delta)$. Then, $z_{n_0}(x) < 2x < \delta_2$.

First, we shall prove that the required condition holds for the point $(z_{n_0}(x), y)$, where $y \in [g(z_{n_0}(x)), f(z_{n_0}(x))]$. There are three cases:

1° $g(z_{n_0}(x)) \leq y < f(x)$. Then using (4) for $y = f(x)$ and (3), we obtain

$$\begin{aligned} & \frac{m_2(A \cap \{[-z_{n_0}(x), z_{n_0}(x)] \times [-y, y]\})}{4z_{n_0}(x)y} \\ & \leq \frac{m_2(A \cap \{[-x, x] \times [-y, y]\})}{4z_{n_0}(x)g(z_{n_0}(x))} + \frac{4(z_{n_0}(x) - x)y}{4z_{n_0}(x)y} \\ & \leq n_0 \cdot \frac{m_2(A \cap \{[-x, x] \times [-y, y]\})}{4xf(x)} + \frac{z_{n_0}(x) - x}{z_{n_0}(x)} < \epsilon \end{aligned}$$

since $g(z_{n_0}(x)) = f(x)/n_0$ and $x \leq z_{n_0}(x)$.

2° $f(x) \leq y \leq x$. Then from (4) and (3), we obtain

$$\begin{aligned} & \frac{m_2(A \cap \{[-z_{n_0}(x), z_{n_0}(x)] \times [-y, y]\})}{4z_{n_0}(x)y} \\ & \leq \frac{m_2(A \cap \{[-x, x] \times [-y, y]\})}{4z_{n_0}(x)y} + \frac{4(z_{n_0}(x) - x)y}{4z_{n_0}(x)y} \\ & \leq \frac{m_2(A \cap \{[-x, x] \times [-y, y]\})}{4xy} + \frac{z_{n_0}(x) - x}{z_{n_0}(x)} < \epsilon \end{aligned}$$

since $x \leq z_{n_0}(x)$.

3° $x \leq y < f(z_{n_0}(x))$. Then using (4) for $y = x$ and (3), we obtain

$$\begin{aligned} & \frac{m_2(A \cap \{[-z_{n_0}(x), z_{n_0}(x)] \times [-y, y]\})}{4z_{n_0}(x)y} \\ & \leq \frac{m_2(A \cap \{[-y, y] \times [-y, y]\})}{4z_{n_0}(x)y} + \frac{4(z_{n_0}(x) - y)y}{4z_{n_0}(x)y} \\ & \leq \frac{m_2(A \cap \{[-y, y] \times [-y, y]\})}{4y^2} + \frac{z_{n_0}(x) - x}{z_{n_0}(x)} < \epsilon \end{aligned}$$

since $x \leq y < f(z_{n_0}(x)) \leq z_{n_0}(x)$.

Consequently, if $x \in (0, \delta)$, then

$$\frac{m_2(A \cap [-z_{n_0}(x), z_{n_0}(x)] \times [-y, y])}{4z_{n_0}(x)y} < \epsilon \quad (5)$$

for each y fulfilling inequalities $g(z_{n_0}(x)) \leq y \leq z_{n_0}(x)$ (for $y \in [f(z_{n_0}(x)), z_{n_0}(x)]$ it follows immediately from (4)).

Let us consider the interval $(0, \delta)$. From our assumption, there exists a point $\eta \in (0, \delta)$ such that $g(\eta) < f(\eta)/n_0$.

We shall prove that for each point $t \in (0, \eta)$ there exists $x \in (0, t]$ such that $z_{n_0}(x) = t$. Let $t \in (0, \eta)$. Then, $0 < g(t) < g(\eta) < f(\eta)/n_0$. The function f/n_0 is continuous, so there exists $x \in (0, \eta)$ such that

$$\frac{1}{n_0}f(x) = g(t).$$

Simultaneously, since $x < \eta < \delta \leq \delta_1$, we have $x \leq z_{n_0}(x) < 2x < 1$ and

$$z_{n_0}(x) = \sup \left\{ z \geq x : g(z) \leq \frac{1}{n_0}f(x) \right\}.$$

Obviously, $g(t) \leq f(x)/n_0$ and $t \geq x$, so $t \in \{z \geq x : g(z) \leq f(x)/n_0\}$. Since g is strictly increasing, we have $g(t_1) > g(t) = f(x)/n_0$ for $t_1 > t$. Consequently,

$$z_{n_0}(x) = t.$$

Finally, let

$$t \in (0, \eta) \quad \text{and} \quad g(t) \leq y \leq t.$$

There exists $x \in (0, \eta)$ such that $z_{n_0}(x) = t$. From (5), we obtain

$$\frac{m_2(A \cap \{[-t, t] \times [-y, y]\})}{4ty} = \frac{m_2(A \cap \{[-z_{n_0}(x), z_{n_0}(x)] \times [-y, y]\})}{4z_{n_0}(x)y} < \epsilon.$$

We can obtain analogous inequality if $y \in (0, \eta)$ and $g(y) \leq x \leq y$. Consequently, $(0, 0)$ is a dispersion point of A with respect to g . $\square$

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Received November 28, 2013

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