DENSITY TOPOLOGY INVOLVING MICROSCOPIC SETS AND CATEGORY

ELŻBIETA WAGNER-BOJAKOWSKA — WŁADYSŁAW WILCZYŃSKI —
— WOJCIECH WOJDOWSKI

*Dedicated to the memory
of Polish mathematicians: Marian Rejewski, Jerzy Różycki and Henryk Zygalski.*

ABSTRACT. Analogously to the method used for c -density point, we introduce the notion of c_μ -density point for arbitrary set A having the Baire property, replacing the coverage in measure with the convergence with respect to the σ -ideal of microscopic sets. Then, we consider the density type topology $\mathcal{T}_{c_\mu}$ generated by c_μ -density operator and study basic properties of $\mathcal{T}_{c_\mu}$ in comparison with the classical density, $\mathcal{I}$ -density and c -density topologies.

The notion of a density point has been studied and developed extensively since the notion of a density topology $\mathcal{T}_d$ was introduced by Haupt and P a u c in 1952 [8]. It is interesting that the related notion of approximate continuity, as defined by Denjoy in 1915 [3], had been known far earlier and utilized in the study of the theory of integration. The properties of the density topology were discovered gradually by Goffman, Neugebauer and Nishiura [6], Goffman and Waterman [7] and Tall [16]. The theory seemed to be mostly complete in the late seventies (see [11]). However, in 1981, the definition was reformulated without the use of the notion of measure (see [18]).

Let $\mathcal{B}$ denote a family of all sets having the Baire property and $\mathcal{I}$ the σ -ideal of sets of the first category. It is well-known (see [11, Theorem 4.6]) that any set A having the Baire property can be represented in the form $A = G \triangle P$, where G is regular open and P is of the first category. Additionally, the representation is unique in any space in which every non-empty open set is of the second category, so it is unique on the real line. The regular open set is the biggest one which can be used in this representation. Further, $n \cdot A = \{nx : x \in A\}$, $A - x_0 = \{x - x_0 : x \in A\}$, $A \sim B$ means that $A \triangle B \in \mathcal{I}$ and $m(A)$ stands for the Lebesgue measure of a measurable set A . For $A \subset \mathbb{R}$, let $\bar{A}$ ($\text{Int}(A)$) denote the closure (the interior) of A in the natural topology $\mathcal{T}$ on the real line.

Recall that x is a density point of a measurable set A if and only if

$$\lim_{h \rightarrow 0^+} \frac{m(A \cap [x-h, x+h])}{2h} = 1.$$

The set of all density points of a measurable set A will be denoted by $\Phi_d(A)$.

Let us observe that the condition given above is equivalent to any of the following:

$$\lim_{n \rightarrow \infty} \frac{m(A \cap [x - \frac{1}{n}, x + \frac{1}{n}])}{\frac{2}{n}} = 1$$

or

$$\lim_{n \rightarrow \infty} m(n \cdot (A - x) \cap [-1, 1]) = 2$$

or

$$\{\chi_{(n \cdot (A-x)) \cap [-1, 1]}\}_{n \in \mathbb{N}} \text{ converges in measure to } \chi_{[-1, 1]}.$$

Knowing this condition and using the Riesz theorem, the definition of the density point of a measurable set A can be given in terms of convergence almost everywhere of characteristic functions of dilations of the set A . Namely, x is a density point of a measurable set A if and only if for each increasing sequence $\{n_m\}_{m \in \mathbb{N}}$ of positive integers there exists a subsequence $\{n_{m_p}\}_{p \in \mathbb{N}}$ such that the sequence of characteristic functions $\{\chi_{(n_{m_p} \cdot (A-x)) \cap [-1, 1]}\}_{p \in \mathbb{N}}$ converges almost everywhere to $\chi_{[-1, 1]}$. The new definition allowed for the study of a subtler properties of the notion of the density point and the density topology, their various modifications and most of all category analogues (see [2], [13], [14] and [15]). For other generalizations of the density point concept such as ψ -density, f -density and $\mathcal{A}_d$ -density see [5], [19] and [24].

In our paper we present the generalization of the approach given in [23] where in the definition of so called c -density point, both measure and category are involved. The ideas from [23] were generalized and developed further in a series of papers [21], [22] and others.

However in the present definition of a density, defining the convergence of dilations of a set, instead of the σ -ideal of sets of measure zero, we use a smaller σ -ideal of microscopic sets. For the survey article on the microscopic sets see [1] and [10].

We shall say that some property holds $\mathcal{I}$ —almost everywhere (in abbr. $\mathcal{I}$ -a.e.) if the set of all points which do not have this property is of the first category.

Let f , f_n , $n \in \mathbb{N}$ be real functions defined on $\mathbb{R}$.

DEFINITION 1 ([17]). We say that the sequence $\{f_n\}_{n \in \mathbb{N}}$ converges with respect to the σ -ideal $\mathcal{I}$ to a function f (in abbr. $f_n \xrightarrow[n \rightarrow \infty]{\mathcal{I}} f$) if and only if every subsequence $\{f_{n_m}\}_{m \in \mathbb{N}}$ of $\{f_n\}_{n \in \mathbb{N}}$ has a subsequence $\{f_{n_{m_p}}\}_{p \in \mathbb{N}}$ converging to f $\mathcal{I}$ -a.e.

The category analogue of the density point, given by Wilczyński, is defined as follows:

DEFINITION 2 ([18]). We say that 0 is an $\mathcal{I}$ -density point of a set A having the Baire property if and only if

$$\chi_{n \cdot A \cap [-1,1]} \xrightarrow[n \rightarrow \infty]{\mathcal{I}} \chi_{[-1,1]},$$

meaning

$$\forall \{n_m\}_{m \in \mathbb{N}} \exists \{n_{m_p}\}_{p \in \mathbb{N}} \chi_{n_{m_p} \cdot A \cap [-1,1]} \xrightarrow[p \rightarrow \infty]{\mathcal{I}-a.e.} \chi_{[-1,1]}.$$

This is equivalent to the fact that for each subsequence $\{n_m\}_{m \in \mathbb{N}}$ of natural numbers there exists a subsequence $\{n_{m_p}\}_{p \in \mathbb{N}}$ such that

$$[-1, 1] \setminus \liminf_{p \rightarrow \infty} (n_{m_p} \cdot A) \in \mathcal{I}.$$

We say that x is an $\mathcal{I}$ -density point of A having the Baire property if 0 is an $\mathcal{I}$ -density point of $A - x$.

The set of all $\mathcal{I}$ -density points of A , for $A \in \mathcal{B}$, will be denoted by $\Phi_{\mathcal{I}}(A)$.

It is clear that in the definition mentioned above we can replace the set A with any open set G such that $A = G \triangle P$, where $P \in \mathcal{I}$. Particularly, we can require G to be the regular open set.

In [23], the author introduced the notion of c -density in the following way:

DEFINITION 3 ([23]). We say that 0 is a c -density point of $A \in \mathcal{B}$ if and only if

$$\lim_{h \rightarrow \infty} \frac{m(G \cap [-h, h])}{2h} = 1,$$

where G is a regular open set such that $A \sim G$.

Remark 1. The following conditions are obviously equivalent:

- (i) 0 is a c -density point of,
- (ii) 0 is a density point of G ,
- (iii) $\lim_{n \rightarrow \infty} \frac{m(G \cap [-\frac{1}{n}, \frac{1}{n}])}{\frac{2}{n}} = 1$,
- (iv) $\chi_{n \cdot G \cap [-1,1]} \xrightarrow[n \rightarrow \infty]{in\ measure} \chi_{[-1,1]}$,
- (v) $\forall \{n_m\}_{m \in \mathbb{N}} \exists \{n_{m_p}\}_{p \in \mathbb{N}} \chi_{n_{m_p} \cdot G \cap [-1,1]} \xrightarrow[p \rightarrow \infty]{a.e.} \chi_{[-1,1]}$,
- (vi) $\forall \{n_m\}_{m \in \mathbb{N}} \exists \{n_{m_p}\}_{p \in \mathbb{N}} m([-1, 1] \setminus \liminf_{p \rightarrow \infty} (n_{m_p} \cdot G)) = 0$.

It is known ([23]) that the notion of c -density point leads us to the operator $\Phi_c : \mathcal{B} \rightarrow 2^{\mathbb{R}}$ defined as follows

$$\Phi_c(A) = \{x \in \mathbb{R} : x \text{ is a } c\text{-density point of } A\} \quad \text{for } A \in \mathcal{B},$$

Φ_c is so-called lower density and the family

$$\mathcal{T}_c = \{A \in \mathcal{B} : A \subset \Phi_c(A)\} = \{\Phi_c(A) \setminus P : A \in \mathcal{B}, P \in \mathcal{I}\}$$

is a topology on the real line, which is incomparable with the classical density topology and with the $\mathcal{I}$ -density topology.

In this paper, we introduce another notion of density, replacing the convergence *a.e.* with the convergence except for a microscopic set, i.e., in (v) the σ -ideal of sets of Lebesgue measure zero is replaced with the σ -ideal of microscopic sets.

DEFINITION 4 ([1]). The set $A \subset \mathbb{R}$ is microscopic if for each $\epsilon > 0$ there exists a sequence $\{I_n\}_{n \in \mathbb{N}}$ of intervals such that

$$A \subset \bigcup_{n=1}^{\infty} I_n \quad \text{and} \quad m(I_n) \leq \epsilon^n \quad \text{for} \quad n \in \mathbb{N}.$$

In [1], it is proved that the family $\mathcal{M}$ of microscopic sets is a σ -ideal on the real line, which is situated between countable sets and sets of Lebesgue measure zero, and the classical Cantor set is not microscopic.

We shall say that some property holds μ -a.e. if the set of all points, which do not have this property, is microscopic.

Let G be a regular open set.

DEFINITION 5. We say that 0 is a c_μ -density point of G if and only if for each increasing sequence $\{n_m\}_{m \in \mathbb{N}}$ of natural numbers there exists a subsequence $\{n_{m_p}\}_{p \in \mathbb{N}}$ such that

$$\chi_{n_{m_p} \cdot G \cap [-1, 1]} \xrightarrow[p \rightarrow \infty]{\mu\text{-a.e.}} \chi_{[-1, 1]},$$

i.e.,

$$\forall \{n_m\}_{m \in \mathbb{N}} \exists \{n_{m_p}\}_{p \in \mathbb{N}} [-1, 1] \setminus \liminf_{p \rightarrow \infty} (n_{m_p} \cdot G) \in \mathcal{M}.$$

We shall say that zero is a right c_μ -density point of a regular open set G if

$$\forall \{n_m\}_{m \in \mathbb{N}} \exists \{n_{m_p}\}_{p \in \mathbb{N}} [0, 1] \setminus \liminf_{p \rightarrow \infty} (n_{m_p} \cdot G) \in \mathcal{M}.$$

In the analogous way, we can define that zero is a left c_μ -density point of a regular open set G . Obviously, zero is a c_μ -density point of G if and only if it is a right and left c_μ -density point of G .

We shall say that x is a c_μ -density point of G if 0 is a c_μ -density point of the set $G - x$.

DEFINITION 6. We say that x is a c_μ -density point of $A \in \mathcal{B}$ if and only if x is a c_μ -density point of G , where G is a regular open set such that $A \sim G$.

We shall say that x is a dispersion ($\mathcal{I}$ -dispersion, c -dispersion, c_μ -dispersion) point of A if x is a density ($\mathcal{I}$ -density, c -density, c_μ -density) point of $\mathbb{R} \setminus A$.

We shall prove that c_μ -density defined as above is really a density, i.e., leads us to the abstract density topology introduced by the operator called lower density.

Put

$$\Phi_{c_\mu}(A) = \{x \in \mathbb{R} : x \text{ is a } c_\mu\text{-density point of } A\} \quad \text{for } A \in \mathcal{B}.$$

Clearly, $\Phi_{c_\mu}(A) = \Phi_{c_\mu}(G)$ if $A \in \mathcal{B}$ and G is a regular open set such that $A \sim G$. It is easily seen that the operator Φ_{c_μ} is monotonic, i.e., if $A, B \in \mathcal{B}$ and $A \subset B$, then $\Phi_{c_\mu}(A) \subset \Phi_{c_\mu}(B)$.

THEOREM 1. *For every $A, B \in \mathcal{B}$*

- 1) $\Phi_{c_\mu}(A) \sim A$,
- 2) if $A \sim B$, then $\Phi_{c_\mu}(A) = \Phi_{c_\mu}(B)$,
- 3) $\Phi_{c_\mu}(\emptyset) = \emptyset$, $\Phi_{c_\mu}(\mathbb{R}) = \mathbb{R}$,
- 4) $\Phi_{c_\mu}(A \cap B) = \Phi_{c_\mu}(A) \cap \Phi_{c_\mu}(B)$.

Proof. 1) Let $A = G \triangle P$, where G is a regular open set and $P \in \mathcal{I}$. It is easily seen that

$$G \subset \Phi_{c_\mu}(G) \subset \bar{G},$$

so

$$G \triangle \Phi_{c_\mu}(G) \subset \bar{G} \setminus G = FrG.$$

As FrG is nowhere dense set, we obtain $G \sim \Phi_{c_\mu}(G)$. Obviously, $A \sim G \sim \Phi_{c_\mu}(G) = \Phi_{c_\mu}(A)$, hence $A \sim \Phi_{c_\mu}(A)$.

2) Let $A = G_1 \triangle P_1$, $B = G_2 \triangle P_2$, where G_1, G_2 are regular open sets and $P_1, P_2 \in \mathcal{I}$. If $A \sim B$, then we have $G_1 = G_2$. Consequently, $B = G_1 \triangle P_2$ and

$$\Phi_{c_\mu}(A) = \Phi_{c_\mu}(G_1) = \Phi_{c_\mu}(B).$$

3) It is a consequence of the definition of Φ_{c_μ} .

4) Obviously, $\Phi_{c_\mu}(A \cap B) \subset \Phi_{c_\mu}(A) \cap \Phi_{c_\mu}(B)$.

Now, let $x_0 \in \Phi_{c_\mu}(A) \cap \Phi_{c_\mu}(B)$. Suppose that $A = G_1 \triangle P_1$, $B = G_2 \triangle P_2$, where G_1, G_2 are regular open sets and $P_1, P_2 \in \mathcal{I}$. As $x_0 \in \Phi_{c_\mu}(A)$, x_0 is a c_μ -density point of G_1 . Let $\{n_m\}_{m \in \mathbb{N}}$ be an arbitrary subsequence of natural numbers. There exists a subsequence $\{n_{m_p}\}_{p \in \mathbb{N}}$ such that

$$\chi_{n_{m_p} \cdot (G_1 - x_0) \cap [-1, 1]}(x) \xrightarrow{p \rightarrow \infty} \chi_{[-1, 1]}(x)$$

for all x except for some microscopic set Q_1 . As $x_0 \in \Phi_{c_\mu}(G_2)$, there exists a subsequence $\{n_{m_{p_k}}\}_{k \in \mathbb{N}}$ of the sequence $\{n_{m_p}\}_{p \in \mathbb{N}}$ such that

$$\chi_{n_{m_{p_k}} \cdot (G_2 - x_0) \cap [-1, 1]}(x) \xrightarrow{k \rightarrow \infty} \chi_{[-1, 1]}(x)$$

for all x except for some microscopic set Q_2 .

Obviously,

$$\chi_{n_{m_{p_k}} \cdot ((G_1 \cap G_2) - x_0) \cap [-1, 1]}(x) \xrightarrow[k \rightarrow \infty]{} \chi_{[-1, 1]}(x)$$

for all $x \in \mathbb{R} \setminus (Q_1 \cup Q_2)$ and $Q_1 \cup Q_2$ is a microscopic set. From the arbitrariness of the sequence $\{n_m\}_{m \in \mathbb{N}}$ we have $x_0 \in \Phi_{c_\mu}(G_1 \cap G_2)$. Obviously, $G_1 \cap G_2$ is a regular open set and $(A \cap B) \triangle (G_1 \cap G_2) \in \mathcal{I}$, so $x_0 \in \Phi_{c_\mu}(A \cap B)$. $\square$

From the last theorem it follows that the operator $\Phi_{c_\mu}: \mathcal{B} \rightarrow \mathcal{B}$ is a lower density ([19]). Obviously, the first property is analogous to the Lebesgue Density Theorem, which holds for metric density.

Observe that if B is an arbitrary residual set, even microscopic (see [10, Lemma 2.2]), then $B \sim \mathbb{R}$, so $\Phi_{c_\mu}(B) = \Phi_{c_\mu}(\mathbb{R}) = \mathbb{R}$.

The sets $\bigcup_{n=1}^{\infty} [a_n, b_n]$ or $\bigcup_{n=1}^{\infty} (a_n, b_n)$ are called right interval sets at zero if $b_{n+1} < a_n < b_n$ for $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} a_n = 0$. We shall now prove that there exists a right interval set G such that zero is a right c_μ -density point of G .

LEMMA 1. *If $G = \bigcup_{n=1}^{\infty} (d_{n+1}, c_n)$ is a right interval set such that*

$$(i) \quad \frac{d_n - c_n}{d_n} \searrow_{n \rightarrow \infty} 0,$$

$$(ii) \quad \frac{d_{n+1}}{d_n} \searrow_{n \rightarrow \infty} 0,$$

then zero is a right c_μ -density point of G .

Proof. It follows easily from the proof in [2, Lemma 2.1.6], since countable sets are microscopic. $\square$

As it was mentioned earlier, each microscopic set has Lebesgue measure zero, so for arbitrary set $A \in \mathcal{B}$ and $x \in \mathbb{R}$, if x is a c_μ -density point of A then x is also a c -density point of A , i.e.,

$$\Phi_{c_\mu}(A) \subset \Phi_c(A)$$

for $A \in \mathcal{B}$. Now, we shall prove that there exists a set $A \in \mathcal{B}$ for which this inclusion is proper. For this purpose, we need an auxiliary lemma.

Let $A' = \mathbb{R} \setminus A$ for $A \subset \mathbb{R}$.

LEMMA 2. *Let $\eta \in (0, 1)$ and let $\{c_n\}_{n \in \mathbb{N}}$ be a decreasing sequence tending to zero such that $\frac{c_{n+1}}{c_n} > \eta$ for $n \in \mathbb{N}$. If $\{A_n\}_{n \in \mathbb{N}}$ is a sequence of sets fulfilling conditions*

$$(i) \quad A_n \subset [c_{n+1}, c_n] \text{ for } n \in \mathbb{N},$$

$$(ii) \quad \frac{m(A_n)}{c_n - c_{n+1}} \nearrow_{n \rightarrow \infty} 1,$$

then the right-hand density at zero of the set $A = \bigcup_{n=1}^{\infty} A_n$ is equal to 1.

Proof. Put

$$\alpha_n = \frac{m([c_{n+1}, c_n] \setminus A_n)}{c_n - c_{n+1}} \quad \text{for } n \in \mathbb{N}.$$

From (ii) we obtain $\alpha_n \searrow_{n \rightarrow \infty} 0$.

Let $h \in (0, c_1]$. Then, there exists $n_0 \in \mathbb{N}$ such that

$$c_{n_0+1} < h \leq c_{n_0},$$

so

$$\begin{aligned} \frac{m(A' \cap [0, h])}{h} &\leq \frac{m(A' \cap [0, c_{n_0}])}{c_{n_0+1}} = \frac{\sum_{k=n_0}^{\infty} m([c_{k+1}, c_k] \setminus A_k)}{c_{n_0+1}} = \\ &= \frac{\sum_{k=n_0}^{\infty} \alpha_k (c_k - c_{k+1})}{c_{n_0+1}} \leq \frac{\alpha_{n_0} \sum_{k=n_0}^{\infty} (c_k - c_{k+1})}{c_{n_0+1}} = \frac{\alpha_{n_0} \cdot c_{n_0}}{c_{n_0+1}} < \alpha_{n_0} \cdot \frac{1}{\eta}. \end{aligned}$$

If h tends to zero, then n_0 tends to infinity and so α_{n_0} tends to zero. Consequently, the right-hand density at zero of the set A' is equal to zero, hence zero is the right-hand density point of A . $\square$

THEOREM 2. *There exists a regular open set G such that zero is a density, c -density and $\mathcal{I}$ -density point of G but not a c_μ -density point of G , i.e.,*

$$0 \in \Phi_d(G) \cap \Phi_c(G) \cap \Phi_{\mathcal{I}}(G) \setminus \Phi_{c_\mu}(G).$$

Proof. Let C denote the classical Cantor set. Let G_1 be a component interval of $[0, 1] \setminus C$, concentric with $[0, 1]$, which is removed in the first step of the construction of C . Let G_2 denote the union of the component intervals of $[0, 1] \setminus C$, which are removed in the first and second steps. And so on. Let G_n be the union of the component intervals of $[0, 1] \setminus C$, which are removed in the steps $1, \dots, n$, for $n \in \mathbb{N}$.

Put
$$G = (-\infty, 0) \cup \bigcup_{n=1}^{\infty} \left(\frac{1}{2^n} G_n + \frac{1}{2^n} \right). \quad (1)$$

Obviously, G is a regular open set.

As $m(G_n) \xrightarrow{n \rightarrow \infty} 1$, we have

$$\frac{m\left(\frac{1}{2^n} G_n + \frac{1}{2^n}\right)}{\frac{1}{2^n}} \xrightarrow{n \rightarrow \infty} 1 \quad \text{and} \quad \frac{1}{2^n} G_n + \frac{1}{2^n} \subset \left[\frac{1}{2^n}, \frac{1}{2^{n-1}} \right] \quad \text{for } n \in \mathbb{N},$$

so from Lemma 2, (used for $c_n = \frac{1}{2^{n-1}}$ and $A_n = \frac{1}{2^n} G_n + \frac{1}{2^n}$), zero is a density point of G and consequently, $0 \in \Phi_d(G) \cap \Phi_c(G)$.

Now, we shall prove that zero is an $\mathcal{I}$ -density point of G . According to [2, Theorem 2.2.2, condition (ii)], it is sufficient to prove that for each increasing sequence $\{t_k\}_{k \in \mathbb{N}}$ of positive numbers tending to infinity there exists a subsequence $\{t_{k_p}\}_{p \in \mathbb{N}}$ such that

$$\liminf_{p \rightarrow \infty} (t_{k_p} \cdot G) \cap [-1, 1] = \bigcup_{m=1}^{\infty} \bigcap_{p=m}^{\infty} (t_{k_p} \cdot G) \cap [-1, 1]$$

is a residual set in $[-1, 1]$.

Let $\{t_k\}_{k \in \mathbb{N}}$ be an increasing sequence of positive numbers tending to infinity, $t_1 > 1$. Put

$$h_k = \frac{1}{t_k} \quad (2)$$

for $k \in \mathbb{N}$. Then, for each $k \in \mathbb{N}$ there exists exactly one $n_k \in \mathbb{N}$ such that

$$h_k \in \left[\frac{1}{2^{n_k+1}}, \frac{1}{2^{n_k}} \right].$$

Hence,

$$1 \leq h_k \cdot 2^{n_k+1} \leq 2,$$

so the sequence $\{h_k \cdot 2^{n_k+1}\}_{k \in \mathbb{N}}$ is bounded and contains a subsequence $\{h_{k_p} \cdot 2^{n_{k_p}+1}\}_{p \in \mathbb{N}}$ convergent to some number $g \in [1, 2]$.

Clearly,

$$h_{k_p} \cdot 2^{n_{k_p}} \xrightarrow{p \rightarrow \infty} \frac{g}{2},$$

so

$$\frac{1}{h_{k_p} \cdot 2^{n_{k_p}}} \xrightarrow{p \rightarrow \infty} \frac{2}{g} \quad (3)$$

and $\frac{2}{g} \in [1, 2]$.

We shall prove that $\liminf(t_{k_p} \cdot G) \cap (0, 1)$ is a set residual in $(0, 1)$.

Put

$$G_0 = \bigcup_{n=1}^{\infty} G_n$$

and

$$B = \bigcup_{n=1}^{\infty} \left(\frac{1}{2^n} G_0 + \frac{1}{2^n} \right).$$

Clearly, G_0 is open and dense in $[0, 1]$, so B is also.

Observe that

$$\begin{aligned} \lim_{p \rightarrow \infty} \inf (t_{k_p} \cdot G) \cap (0, 1) &= \bigcup_{m=1}^{\infty} \bigcap_{p=m}^{\infty} (t_{k_p} \cdot G) \cap (0, 1) \\ &= \bigcup_{m=1}^{\infty} \bigcap_{p=m}^{\infty} \left(\frac{1}{h_{k_p}} \cdot G \right) \cap (0, 1) \supset \left(\frac{2}{g} \cdot B \right) \cap (0, 1). \end{aligned} \quad (4)$$

Let $x \in \left(\frac{2}{g} \cdot B \right) \cap (0, 1)$. Since B is open, x belongs to some component interval of $\left(\frac{2}{g} \cdot B \right) \cap (0, 1)$. Hence, there exist $n_0, m_0 \in \mathbb{N}$ such that

$$x \in \frac{2}{g} \left(\frac{1}{2^{n_0}} G_{m_0} + \frac{1}{2^{n_0}} \right). \quad (5)$$

Let $n_{k_p} \geq m_0$. Then,

$$\begin{aligned}
 \frac{1}{h_{k_p}} \cdot G \cap (0, 1) &= \frac{2^{n_{k_p}}}{h_{k_p} \cdot 2^{n_{k_p}}} \cdot G \cap (0, 1) \\
 &= \frac{1}{h_{k_p} \cdot 2^{n_{k_p}}} \cdot \bigcup_{m=1}^{\infty} \left(\frac{2^{n_{k_p}}}{2^m} G_m + \frac{2^{n_{k_p}}}{2^m} \right) \cap (0, 1) \\
 &= \frac{1}{h_{k_p} \cdot 2^{n_{k_p}}} \cdot \bigcup_{m=n_{k_p}+1}^{\infty} \left(\frac{2^{n_{k_p}}}{2^m} G_m + \frac{2^{n_{k_p}}}{2^m} \right) \\
 &= \frac{1}{h_{k_p} \cdot 2^{n_{k_p}}} \cdot \bigcup_{m=n_{k_p}+1}^{\infty} \left(\frac{1}{2^{m-n_{k_p}}} G_m + \frac{1}{2^{m-n_{k_p}}} \right) \\
 &\supset \frac{1}{h_{k_p} \cdot 2^{n_{k_p}}} \cdot \left(\frac{1}{2^{n_0}} G_{m_0} + \frac{1}{2^{n_0}} \right) \xrightarrow{p \rightarrow \infty} \frac{2}{g} \cdot \left(\frac{1}{2^{n_0}} G_{m_0} + \frac{1}{2^{n_0}} \right)
 \end{aligned}$$

according to (3).

Hence, using (2) and (5), we obtain $x \in (t_{k_p} \cdot G) \cap (0, 1)$ for each p sufficiently large and such that $n_{k_p} \geq m_0$, so (4) is fulfilled.

Since B is open and dense in $[0, 1]$ and $\frac{2}{g} \geq 1$, from (4) it follows that the set $\liminf_{p \rightarrow \infty} (t_{k_p} \cdot G) \cap (0, 1)$ is residual in $[0, 1]$. Consequently, using (1) we obtain that zero is an $\mathcal{I}$ -density point of G .

Now, we shall prove that zero is not a c_μ -density point of G . Put $n_m = 2^m$ for $m \in \mathbb{N}$.

Let us observe that $[0, 1] \setminus \liminf_{m \rightarrow \infty} (2^m \cdot G) = \limsup_{m \rightarrow \infty} ([0, 1] \setminus 2^m \cdot G)$ is not a microscopic set. Indeed, we have

$$\begin{aligned}
 \limsup_{m \rightarrow \infty} ([0, 1] \setminus 2^m \cdot G) &= \bigcap_{k=1}^{\infty} \bigcup_{m=k}^{\infty} \left([0, 1] \setminus 2^m \cdot \bigcup_{n=1}^{\infty} \left(\frac{1}{2^n} G_n + \frac{1}{2^n} \right) \right) \\
 &= \bigcap_{k=1}^{\infty} \bigcup_{m=k}^{\infty} \left([0, 1] \setminus \bigcup_{n=1}^{\infty} \left(\frac{2^m}{2^n} G_n + \frac{2^m}{2^n} \right) \right) \\
 &= \bigcap_{k=1}^{\infty} \bigcup_{m=k}^{\infty} \left([0, 1] \setminus \bigcup_{n=m+1}^{\infty} \left(\frac{1}{2^{n-m}} G_n + \frac{1}{2^{n-m}} \right) \right) \\
 &\supset \bigcap_{k=1}^{\infty} \bigcup_{m=k}^{\infty} \left([0, 1] \setminus \bigcup_{n=m+1}^{\infty} \left(\frac{1}{2^{n-m}} G_0 + \frac{1}{2^{n-m}} \right) \right)
 \end{aligned}$$

because $G_n \subset G_0$ for each $n \in \mathbb{N}$.

Simultaneously,

$$\bigcup_{n=m+1}^{\infty} \left(\frac{1}{2^{n-m}} G_0 + \frac{1}{2^{n-m}} \right) = \bigcup_{n=1}^{\infty} \left(\frac{1}{2^n} G_0 + \frac{1}{2^n} \right) = B$$

and

$$[0, 1] \setminus B = \bigcup_{n=1}^{\infty} \left(\frac{1}{2^n} C + \frac{1}{2^n} \right),$$

so

$$\bigcup_{n=1}^{\infty} \left(\frac{1}{2^n} C + \frac{1}{2^n} \right) \subset \limsup_{m \rightarrow \infty} ([0, 1] \setminus 2^m \cdot G).$$

Consequently, $[0, 1] \setminus \liminf_{m \rightarrow \infty} (2^m \cdot G)$ is not a microscopic set, as the Cantor set and also all sets of the form $\frac{1}{2^n} C + \frac{1}{2^n}$, $n \in \mathbb{N}$, are not microscopic. An analogous result is obtained for arbitrary subsequence of the sequence $\{2^m\}_{m \in \mathbb{N}}$. Hence, zero is not a c_μ -density point of G , i.e., $0 \notin \Phi_{c_\mu}(G)$. $\square$

THEOREM 3. *There exists a right interval set A such that zero is a c_μ -dispersion point but not an $\mathcal{I}$ -dispersion point of A .*

Proof. Let $n_1 \in \mathbb{N}$ (we can assume $n_1 = 1$). Put

$$I_1^1 = \left(\frac{1}{2n_1}, \frac{1}{n_1} \right).$$

Obviously, $m(I_1^1) = \frac{1}{2n_1} \cdot \left(\frac{1}{2}\right)^{1!}$.

Let n_2 be a natural number such that

$$\frac{1}{n_2} < \frac{1}{2n_1} \cdot \left(\frac{1}{2}\right)^{2!}.$$

In the interval $(\frac{1}{2n_2}, \frac{1}{n_2})$, we construct two open intervals I_2^1, I_2^2 with right end-points equal to

$$\frac{1}{n_2}, \quad \frac{1}{2n_2} \left(1 + \frac{1}{2}\right)$$

and with lengths equal to

$$\frac{1}{2n_2} \left(\frac{1}{3}\right)^{3!}, \quad \frac{1}{2n_2} \left(\frac{1}{4}\right)^{4!},$$

respectively.

Now, let n_3 be a natural number such that

$$\frac{1}{n_3} < \frac{1}{2n_2} \cdot \left(\frac{1}{5}\right)^{5!}.$$

In the interval $(\frac{1}{2n_3}, \frac{1}{n_3})$, we construct three open intervals I_3^1, I_3^2, I_3^3 with

right end-points equal to

$$\frac{1}{n_3} = \frac{1}{2n_3} \cdot 2, \quad \frac{1}{2n_3} \left(1 + \frac{2}{3}\right), \quad \frac{1}{2n_3} \left(1 + \frac{1}{3}\right)$$

and with lengths equal to

$$\frac{1}{2n_3} \left(\frac{1}{6}\right)^{6!}, \quad \frac{1}{2n_3} \left(\frac{1}{7}\right)^{7!}, \quad \frac{1}{2n_3} \left(\frac{1}{8}\right)^{8!},$$

respectively.

Now, we proceed by induction. Suppose that we have defined natural numbers $n_1, n_2, \dots, n_k$, and $\frac{k(k+1)}{2}$ open intervals

$$I_1^1, I_2^1, I_2^2, I_3^1, I_3^2, I_3^3, \dots, I_k^1, I_k^2, \dots, I_k^k.$$

The lengths of the intervals in k th group $(I_k^1, I_k^2, \dots, I_k^k)$ are equal to

$$\frac{1}{2n_k} \cdot \left(\frac{1}{\frac{(k-1)(k+2)}{2} + 1} \right)^{\left(\frac{(k-1)(k+2)}{2} + 1 \right)!}, \quad (6)$$

$$\frac{1}{2n_k} \cdot \left(\frac{1}{\frac{(k-1)(k+2)}{2} + 2} \right)^{\left(\frac{(k-1)(k+2)}{2} + 2 \right)!}, \dots, \frac{1}{2n_k} \cdot \left(\frac{1}{\frac{(k-1)(k+2)}{2} + k} \right)^{\left(\frac{(k-1)(k+2)}{2} + k \right)!},$$

and the right end-points are

$$\frac{1}{n_k} = \frac{1}{2n_k} \cdot 2, \frac{1}{2n_k} \left(1 + \frac{k-1}{k}\right), \dots, \frac{1}{2n_k} \left(1 + \frac{2}{k}\right), \frac{1}{2n_k} \left(1 + \frac{1}{k}\right),$$

respectively.

Let n_{k+1} be a natural number such that

$$\frac{1}{n_{k+1}} < \frac{1}{2n_k} \left(\frac{1}{\frac{k(k+3)}{2}} \right)^{\left(\frac{k(k+3)}{2} \right)!}. \quad (7)$$

In the interval $\left(\frac{1}{2n_{k+1}}, \frac{1}{n_{k+1}}\right)$, we construct the open intervals

$$I_{k+1}^1, I_{k+1}^2, \dots, I_{k+1}^{k+1}$$

with lengths equal to

$$\frac{1}{2n_{k+1}} \cdot \left(\frac{1}{\frac{k(k+3)}{2} + 1} \right)^{\left(\frac{k(k+3)}{2} + 1 \right)!}, \dots, \frac{1}{2n_{k+1}} \cdot \left(\frac{1}{\frac{k(k+3)}{2} + k + 1} \right)^{\left(\frac{k(k+3)}{2} + k + 1 \right)!}$$

and with right end-points

$$\frac{1}{n_{k+1}}, \quad \frac{1}{2n_{k+1}} \left(1 + \frac{k}{k+1}\right), \dots, \frac{1}{2n_{k+1}} \left(1 + \frac{2}{k+1}\right), \quad \frac{1}{2n_{k+1}} \left(1 + \frac{1}{k+1}\right),$$

respectively.

Put

$$A = \bigcup_{k=1}^{\infty} \bigcup_{j=1}^k I_k^j.$$

Clearly, A is a right interval set. We shall first prove that zero is a c_μ -dispersion point of A . For this purpose, we shall prove that for each increasing sequence $\{i_m\}_{m \in \mathbb{N}}$ of natural numbers there exists a subsequence $\{i_{m_p}\}_{p \in \mathbb{N}}$ such that

$$\lim_{p \rightarrow \infty} \sup (i_{m_p} \cdot A) \cap [0, 1] = [0, 1] \cap \bigcap_{r=1}^{\infty} \bigcup_{p=r}^{\infty} i_{m_p} \cdot A \in \mathcal{M}.$$

Let $\{i_m\}_{m \in \mathbb{N}}$ be an arbitrary increasing sequence of natural numbers. For each $m \in \mathbb{N}$ there exists exactly one $k \in \mathbb{N}$ such that

$$\frac{1}{2n_k} \leq \frac{1}{i_m} < \frac{1}{2n_{k-1}}$$

(for $k = 1$ we put $\frac{1}{2n_{k-1}} = 2$).

If $\{m \in \mathbb{N} : \frac{1}{2n_k} \leq \frac{1}{i_m} < \frac{1}{2n_{k-1}}\} \neq \emptyset$, then we choose exactly one element from this set. In such a way, we construct a sequence $\{m_p\}_{p \in \mathbb{N}}$ having the property that for each $k \in \mathbb{N}$, if

$$\left\{m \in \mathbb{N} : \frac{1}{2n_k} \leq \frac{1}{i_m} < \frac{1}{2n_{k-1}}\right\} \neq \emptyset,$$

then there exists exactly one number m_p such that

$$\frac{1}{2n_k} \leq \frac{1}{i_{m_p}} < \frac{1}{2n_{k-1}}. \quad (8)$$

Observe that the set $(i_{m_p} \cdot A) \cap [0, 1]$ consists of at most k intervals $i_{m_p} \cdot I_k^1, \dots, i_{m_p} \cdot I_k^k$ and of the second part contained in the interval $i_{m_p} \cdot [0, \frac{1}{n_{k+1}}]$, where k fulfils (8). So,

$$(i_{m_p} \cdot A) \cap [0, 1] \subset \left[0, \frac{i_{m_p}}{n_{k+1}}\right] \cup \bigcup_{j=1}^k i_{m_p} \cdot I_k^j.$$

Obviously, from (8) and (6)

$$m(i_{m_p} \cdot I_k^j) \leq m(2n_k \cdot I_k^j) = \left(\frac{1}{\frac{(k-1)(k+2)}{2} + j}\right)^{\left(\frac{(k-1)(k+2)}{2} + j\right)!}.$$

Using (8) and (7), we obtain

$$m\left(i_{m_p} \cdot \left[0, \frac{1}{n_{k+1}}\right]\right) \leq 2n_k \cdot \frac{1}{n_{k+1}} \leq \left(\frac{1}{\frac{k(k+3)}{2}}\right)^{\left(\frac{k(k+3)}{2}\right)!},$$

hence

$$\bigcup_{p=r}^{\infty} (i_{m_p} \cdot A) \cap [0, 1] \subset \bigcup_{p=r}^{\infty} \left(\left[0, \frac{i_{m_p}}{n_{k+1}}\right] \cup \bigcup_{j=1}^k i_{m_p} \cdot I_k^j\right) \cap [0, 1]. \quad (9)$$

From (8), it follows that for each number $p \in \mathbb{N}$ there exists exactly one number $k(p)$ such that

$$\frac{1}{2n_{k(p)}} \leq \frac{1}{i_{m_p}} < \frac{1}{2n_{k(p)-1}},$$

and if $p \rightarrow \infty$, then $k(p) \rightarrow \infty$.

Let $\{I_n\}_{n \in \mathbb{N}}$ be the sequence of intervals

$$i_{m_r} \cdot I_{k(r)}^1, i_{m_r} \cdot I_{k(r)}^2, \dots, i_{m_r} \cdot I_{k(r)}^{k(r)}, \left[0, \frac{i_{m_r}}{n_{k(r)+1}}\right], i_{m_{r+1}} \cdot I_{k(r+1)}^1, i_{m_{r+1}} \cdot I_{k(r+1)}^2, \dots$$

Let $\epsilon > 0$. Then, there exists $r \in \mathbb{N}$ such that

$$\frac{1}{\frac{(r-1)(r+2)}{2} + 1} < \epsilon. \quad (10)$$

Put $s_0 = \frac{(r-1)(r+2)}{2} + 1$. From (9) the set $\bigcup_{p=r}^{\infty} (i_{m_p} \cdot A) \cap [0, 1]$ is covered by the union of intervals $\{I_n\}_{n \in \mathbb{N}}$, and lengths of these intervals are not greater than the elements of the sequence

$$\left(\frac{1}{s_0}\right)^{s_0!}, \left(\frac{1}{s_0 + 1}\right)^{(s_0+1)!}, \dots$$

so, also not greater than the elements of the sequence

$$\left(\frac{1}{s_0}\right)^{s_0!}, \left(\frac{1}{s_0}\right)^{(s_0+1)!}, \dots$$

and, finally, than the elements of the sequence

$$\left(\frac{1}{s_0}\right)^1, \left(\frac{1}{s_0}\right)^2, \dots$$

Consequently, the set $\bigcup_{p=r}^{\infty} (i_{m_p} \cdot A) \cap [0, 1]$ is covered by the intervals $\{I_n\}_{n \in \mathbb{N}}$, such that $m(I_n) < \epsilon^n$, according to (10). Hence,

$$\bigcap_{r=1}^{\infty} \bigcup_{p=r}^{\infty} i_{m_p} \cdot A \cap [0, 1]$$

is a microscopic set.

Simultaneously, for an arbitrary subsequence $\{n_{k_l}\}_{l \in \mathbb{N}}$ of the sequence $\{n_k\}_{k \in \mathbb{N}}$, the set $\bigcup_{l=r}^{\infty} (n_{k_l} \cdot A)$ is open and dense in the interval $[\frac{1}{2}, 1]$, so the set

$$\bigcap_{r=1}^{\infty} \bigcup_{l=r}^{\infty} (n_{k_l} \cdot A) = \limsup_{l \rightarrow \infty} (n_{k_l} \cdot A)$$

is residual on $[\frac{1}{2}, 1]$. Consequently, zero is not an $\mathcal{I}$ -dispersion point of A . $\square$

COROLLARY 1. *There exists a regular open set G such that zero is a c_μ -density point but not an $\mathcal{I}$ -density point of G , i.e.,*

$$0 \in \Phi_{c_\mu}(G) \setminus \Phi_{\mathcal{I}}(G).$$

Proof. Put

$$G = \text{Int}(\mathbb{R} \setminus A)$$

where A is a set from the previous theorem. Then, zero is a c_μ -density point of G but not an $\mathcal{I}$ -density point of G . $\square$

Put

$$\mathcal{T}_{c_\mu} = \{A \in \mathcal{B} : A \subset \Phi_{c_\mu}(A)\}.$$

THEOREM 4. *The family $\mathcal{T}_{c_\mu}$ is a topology on the real line such that*

$$\mathcal{T} \subsetneq \mathcal{T}_{c_\mu} \subsetneq \mathcal{T}_c.$$

Proof. The only difficulty is to prove that the family $\mathcal{T}_{c_\mu}$ is closed under arbitrary unions. For this purpose, it is sufficient to prove that if $\{A_t\}_{t \in T}$ is an arbitrary subfamily of $\mathcal{T}_{c_\mu}$, then $\bigcup_{t \in T} A_t \in \mathcal{B}$.

For each $t \in T$, there exists a regular open set G_t such that $A_t \sim G_t$. Obviously, there exists a sequence $\{t_n\}_{n \in \mathbb{N}}$ of elements of T such that

$$\bigcup_{t \in T} G_t = \bigcup_{n=1}^{\infty} G_{t_n}.$$

Then,

$$\bigcup_{n=1}^{\infty} G_{t_n} \sim \bigcup_{n=1}^{\infty} A_{t_n} \subset \bigcup_{t \in T} A_t \subset \bigcup_{t \in T} \Phi_{c_\mu}(A_t) \subset \bigcup_{t \in T} \bar{G}_t \subset \overline{\bigcup_{t \in T} G_t} = \overline{\bigcup_{n=1}^{\infty} G_{t_n}},$$

and

$$\bigcup_{n=1}^{\infty} G_{t_n} \sim \overline{\bigcup_{n=1}^{\infty} G_{t_n}}, \quad \text{so} \quad \bigcup_{t \in T} A_t \in \mathcal{B}.$$

Let $\mathbb{Q}$ denote the set of rational numbers. Obviously, $\mathbb{R} \setminus \mathbb{Q} \in \mathcal{T}_{c_\mu} \setminus \mathcal{T}$. Now, let $G_1 = G \cup \{0\}$, where G is a regular open set from Theorem 2. Then $G_1 \in \mathcal{T}_c \setminus \mathcal{T}_{c_\mu}$. $\square$

Remark 2. Analogously as for $\mathcal{I}$ -density topology (see [13, Remark 2]), it is clear that $\mathcal{T}_{c_\mu} = \{\Phi_{c_\mu}(A) \setminus P : A \in \mathcal{B}, P \in \mathcal{I}\}$.

THEOREM 5. *The topology $\mathcal{T}_{c_\mu}$ is incomparable with both $\mathcal{T}_{\mathcal{I}}$ and $\mathcal{T}_d$ topologies.*

Proof. Let G be a regular open set from Theorem 2. Put

$$G_1 = G \cup \{0\}.$$

Then, $G_1 \in \mathcal{T}_{\mathcal{I}} \cap \mathcal{T}_d \setminus \mathcal{T}_{c_\mu}$.

If G is a set from Corollary 1, then $G \in \mathcal{T}_{c_\mu} \setminus \mathcal{T}_{\mathcal{I}}$.

Let B be a residual set of measure zero. Then, $B \sim \mathbb{R}$, so $\Phi_{c_\mu}(B) = \mathbb{R}$ and, consequently, $B \in \mathcal{T}_{c_\mu}$. Simultaneously, $B \notin \mathcal{T}_d$, as $\Phi_d(B) = \emptyset$. $\square$

THEOREM 6. *If $A \in \mathcal{B}$ and G is regular open set such that $A \sim G$, then*

- 1) $\text{Int}(\Phi_{c_\mu}(A)) = \emptyset$ if and only if $A \in \mathcal{I}$,
- 2) $\Phi_{c_\mu}(A) \setminus G$ is a nowhere dense set of Lebesgue measure zero,
- 3) $G = \text{Int}(\Phi_{c_\mu}(A)) = \text{Int}(\overline{G})$.

The proof is analogous to the proof of [23, Theorem 3].

Remark 3. Since for each $A \in \mathcal{B}$ we have $\Phi_{c_\mu}(A) \sim A$, it is easy to see that, analogously as for the density topology, for arbitrary set $B \subset \mathbb{R}$

$$\text{Int}_{\mathcal{T}_{c_\mu}}(B) = B \cap \Phi_{c_\mu}(C),$$

where C is a measurable (it means having the Baire property) kernel of B .

In particular, if $A \in \mathcal{B}$, we have $\text{Int}_{\mathcal{T}_{c_\mu}}(A) = A \cap \Phi_{c_\mu}(A)$.

Also, a set $A \in \mathcal{B}$ is $\mathcal{T}_{c_\mu}$ -regular open if and only if $A = \Phi_{c_\mu}(D)$ for some $D \in \mathcal{B}$.

THEOREM 7. *For arbitrary set $A \subset \mathbb{R}$, the following conditions are equivalent:*

- 1) $A \in \mathcal{I}$,
- 2) A is $\mathcal{T}_{c_\mu}$ -nowhere dense set,
- 3) A is $\mathcal{T}_{c_\mu}$ -first category set,
- 4) A is $\mathcal{T}_{c_\mu}$ -closed and $\mathcal{T}_{c_\mu}$ -discrete set.

The proof is analogous to the proof in [19, Theorem 2.8], (see also [9]).

THEOREM 8.

- 1) *The family of $\mathcal{T}_{c_\mu}$ -Borel sets coincides with the family of sets having the Baire property.*
- 2) *Every $\mathcal{T}_{c_\mu}$ -Borel set is a union of $\mathcal{T}_{c_\mu}$ -open set and $\mathcal{T}_{c_\mu}$ -closed set.*
- 3) *Every $\mathcal{T}_{c_\mu}$ -Borel set is $\mathcal{T}_{c_\mu} - F_{\sigma\delta}$ set.*

Proof.

- 1) The proof is analogous to the proof of Theorem 2.9 in [19].
- 2) This is clear since for each $A \in \mathcal{B}$ we have $A = (A \cap \Phi_{c_\mu}(A)) \cup (A \setminus \Phi_{c_\mu}(A))$.
- 3) If $A \in \mathcal{B}$, then $A = E \cup P$, where E is a set of type G_δ (in natural topology) and $P \in \mathcal{I}$. Here, E is a set of type $F_{\sigma\delta}$ in natural topology and also $F_{\sigma\delta}$ in $\mathcal{T}_{c_\mu}$ -topology. Simultaneously, P is $\mathcal{T}_{c_\mu}$ -closed, so is $\mathcal{T}_{c_\mu} - F_{\sigma\delta}$ set. Consequently, A is $\mathcal{T}_{c_\mu} - F_{\sigma\delta}$ set. $\square$

Obviously, the space $(\mathbb{R}, \mathcal{T}_{c_\mu})$ is not separable, because all countable sets are closed. Consequently, it is not the second countable (see [4, Corollary 1.3.8., p. 44]). It is not the first countable, too (compare [19, Theorem 2.11]).

Additionally, each compact set in this space is finite and $(\mathbb{R}, \mathcal{T}_{c_\mu})$ is not Lindelöf space (see the proof in [18, Corollary 2]). Furthermore, $(\mathbb{R}, \mathcal{T}_{c_\mu})$ is a Baire space according to Theorem 7, and is not regular (T_3) (see the proof in [18, Theorem 6]).

Now, we shall discuss some properties of the continuity of real functions with respect to the $\mathcal{T}_{c_\mu}$ -topology.

DEFINITION 7. We say that a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is $\mathcal{T}_{c_\mu}$ -continuous at a point x_0 , if and only if for each $\varepsilon > 0$ there exists a set $A_{x_0} \in \mathcal{T}_{c_\mu}$, such that $x_0 \in A_{x_0}$ and $f(A_{x_0}) \subset (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$.

A function f is $\mathcal{T}_{c_\mu}$ -continuous if it is $\mathcal{T}_{c_\mu}$ -continuous at each point $x \in \mathbb{R}$.

Using a standard argument, we obtain

THEOREM 9. A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is $\mathcal{T}_{c_\mu}$ -continuous if and only if $f^{-1}(G) \in \mathcal{T}_{c_\mu}$ for each open set $G \subset \mathbb{R}$ (i.e., f is continuous as a function from $(\mathbb{R}, \mathcal{T}_{c_\mu})$ to $(\mathbb{R}, \mathcal{T})$).

THEOREM 10. For a function $f : \mathbb{R} \rightarrow \mathbb{R}$, the following conditions are equivalent:

- (i) f has the Baire property,
- (ii) f is $\mathcal{T}_{c_\mu}$ -continuous $\mathcal{I}$ -almost everywhere on $\mathbb{R}$.

Proof. (Compare [13], see also [9].)

(ii) $\Rightarrow$ (i) Suppose f has the Baire property. From [11, Theorem 8.1], there exists a residual set $E \subset \mathbb{R}$ such that the restriction $f|_E$ is continuous. Then, f is $\mathcal{T}_{c_\mu}$ -continuous at every point of E , i.e., it is $\mathcal{T}_{c_\mu}$ -continuous $\mathcal{I}$ -almost everywhere on $\mathbb{R}$.

(ii) $\Rightarrow$ (i) Suppose now, that f is $\mathcal{T}_{c_\mu}$ -continuous $\mathcal{I}$ -almost everywhere. Let $a, b \in \mathbb{R}$, $a < b$ and $B = f^{-1}((a, b))$. We shall show that B has the Baire property. Let C be the set of all $\mathcal{T}_{c_\mu}$ -continuity points of f .

We have $B = (B \cap C) \cup (B \setminus C)$ and $B \setminus C \in \mathcal{I}$. The proof is completed by showing that $B \cap C$ has the Baire property. Let $x \in B \cap C$. Then $f(x) \in (a, b)$. From our assumption, there exists a set $A_x \in \mathcal{T}_{c_\mu}$ such that $x \in A_x$ and $f(A_x) \subset (a, b)$. Clearly, $A_x \cap C \in \mathcal{T}_{c_\mu}$, $x \in A_x \cap C$ and $f(A_x \cap C) \subset (a, b)$. We finally obtain

$$B \cap C = \bigcup_{x \in B \cap C} (A_x \cap C),$$

so $B \cap C \in \mathcal{B}$, as $\mathcal{T}_{c_\mu} \subset \mathcal{B}$. □

THEOREM 11. *Every $\mathcal{T}_{c_\mu}$ -continuous function is approximately continuous, so is a Darboux Baire 1 function.*

Proof. It is a simple consequence of the inclusion

$$\mathcal{T}_{c_\mu} \subsetneq \mathcal{T}_c$$

and the fact that every $\mathcal{T}_c$ -continuous function is approximately continuous (see [23, Corollary 6]). $\square$

THEOREM 12. *The family of $\mathcal{T}_{c_\mu}$ -connected sets coincides with the family of sets connected in the natural topology.*

Proof. We can follow here the proof of [19, Theorem 3.7]. $\square$

THEOREM 13. *If a function $f: \mathbb{R} \rightarrow \mathbb{R}$ is $\mathcal{T}_{c_\mu}$ -continuous, then $f^{-1}((a, b))$ is a countable union of $\mathcal{T}_{c_\mu}$ -regular open sets for arbitrary interval $(a, b) \subset \mathbb{R}$.*

Proof. Suppose $f: \mathbb{R} \rightarrow \mathbb{R}$ is $\mathcal{T}_{c_\mu}$ -continuous function and $(a, b) \subset \mathbb{R}$. We have $f^{-1}((a, b)) = \Phi_{c_\mu}(A) \setminus P$ for some $A \in \mathcal{B}$ and $P \in \mathcal{I}$, and we can assume $P \subset \Phi_{c_\mu}(A)$.

Let $x \in \Phi_{c_\mu}(A)$. We shall prove that $f(x) \in [a, b]$. Let $\epsilon > 0$. Put

$$D = \Phi_{c_\mu}(A) \cap f^{-1}((f(x) - \epsilon, f(x) + \epsilon)).$$

Obviously, $x \in D$ and $D \in \mathcal{T}_{c_\mu}$, so D is a non-empty set open in $\mathcal{T}_{c_\mu}$. Consequently, $D \notin \mathcal{I}$. Hence

$$D \setminus P = (\Phi_{c_\mu}(A) \setminus P) \cap f^{-1}((f(x) - \epsilon, f(x) + \epsilon)) \neq \emptyset.$$

It means $f^{-1}((a, b)) \cap f^{-1}((f(x) - \epsilon, f(x) + \epsilon)) \neq \emptyset$, so

$$(a, b) \cap (f(x) - \epsilon, f(x) + \epsilon) \neq \emptyset.$$

From the arbitrariness of ϵ , we obtain $f(x) \in [a, b]$. Consequently,

$$f^{-1}((a, b)) \subset \Phi_{c_\mu}(A) \subset f^{-1}([a, b]).$$

We can repeat the above considerations for every set of the form $(a + \frac{1}{n}, b - \frac{1}{n})$, for $n \in \mathbb{N}$. We have

$$f^{-1}\left(\left(a + \frac{1}{n}, b - \frac{1}{n}\right)\right) = \Phi_{c_\mu}(A_n) \setminus P_n,$$

where $A_n \in \mathcal{B}$, $P_n \in \mathcal{I}$ and $P_n \subset \Phi_{c_\mu}(A_n)$ for $n \in \mathbb{N}$. Clearly,

$$\begin{aligned} f^{-1}((a, b)) &= \bigcup_{n=1}^{\infty} f^{-1}\left(\left(a + \frac{1}{n}, b - \frac{1}{n}\right)\right) \subset \bigcup_{n=1}^{\infty} \Phi_{c_\mu}(A_n), \\ &\subset \bigcup_{n=1}^{\infty} f^{-1}\left(\left[a + \frac{1}{n}, b - \frac{1}{n}\right]\right) = f^{-1}((a, b)). \end{aligned}$$

Thus,

$$f^{-1}((a, b)) = \bigcup_{n=1}^{\infty} \Phi_{c_{\mu}}(A_n),$$

and according to Remark 3, the proof is finished. $\square$

In [12], O'Malley proved that the family of sets

$$\left\{ A \in \mathcal{T}_d : m(A) = m(\text{Int}(A)) \right\}$$

forms a topology, called *a.e.*-topology. He proved that *a.e.*-topology is completely regular but not normal.

The sets open in *a.e.*-topology can be characterized as sets of the form $G \cup S$, where G is open in the natural topology $\mathcal{T}$ and $S \subset \Phi_d(G)$. Clearly, S can be nowhere dense here and of Lebesgue measure zero.

Simultaneously,

$$\begin{aligned} \{ A \in \mathcal{T}_d : A = G \cup S, G \in \mathcal{T} \text{ and } m(S) = 0 \} \\ = \{ B = G \cup S : G \in \mathcal{T} \text{ and } S \subset \Phi_d(G) \}. \end{aligned}$$

By analogy, we define $\mathcal{T}_{c_{\mu}}$ -*a.e.*-topology as a family of sets of the form $G \cup S$, where $G \in \mathcal{T}$ and $S \subset \Phi_{c_{\mu}}(G)$ (compare [20]).

THEOREM 14. *If a function $f: \mathbb{R} \rightarrow \mathbb{R}$ is $\mathcal{T}_{c_{\mu}}$ -continuous, then f is continuous with respect to the $\mathcal{T}_{c_{\mu}}$ -*a.e.*-topology.*

Proof. Let $(a, b) \subset \mathbb{R}$. From the proof of the previous theorem, we have

$$f^{-1}((a, b)) = \bigcup_{n=1}^{\infty} \Phi_{c_{\mu}}(A_n) = \bigcup_{n=1}^{\infty} \Phi_{c_{\mu}}(G_n) \supset \bigcup_{n=1}^{\infty} G_n,$$

where $A_n \in \mathcal{B}$ and G_n is regular open set such that $A_n \sim G_n$, for $n \in \mathbb{N}$.

Put

$$G = \bigcup_{n=1}^{\infty} G_n \quad \text{and} \quad S = f^{-1}((a, b)) \setminus G.$$

Obviously, $f^{-1}((a, b)) = G \cup S$ and $G \in \mathcal{T}$. On the other hand,

$$\bigcup_{n=1}^{\infty} \Phi_{c_{\mu}}(G_n) \subset \Phi_{c_{\mu}}(G),$$

so

$$S \subset f^{-1}((a, b)) = \bigcup_{n=1}^{\infty} \Phi_{c_{\mu}}(G_n) \subset \Phi_{c_{\mu}}(G).$$

$\square$

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Elżbieta Wagner-Bojakowska
Władysław Wilczyński
University of Łódź
Faculty of Mathematics and
Computer Science
Banacha 22
PL-90-238 Łódź
POLAND
E-mail: wagner@math.uni.lodz.pl
wwil@uni.lodz.pl

Wojciech Wojdowski
Institute of Mathematics
Technical University of Łódź
Wólczańska 215
PL-90-924 Łódź
POLAND
E-mail: wojwoj@gmail.com