ON SEPARATION BY h -CONVEX FUNCTIONS

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ABSTRACT. In the present paper, we establish the necessary and sufficient conditions under which two functions can be separated by h -convex function, in the case, when the function h is multiplicative. This result is related to the theorem on separation by convex functions presented in Baron, K.—Matkowski, J.—Nikodem, K. [*A sandwich with convexity*, Math. Pannon. 5 (1994), 139–144].

1. Introduction

The concept of h -convexity was introduced by S. Varošanec [13] and modified by A. Hájzy [5] as follows:

DEFINITION 1. Let D be a convex subset of a real linear space, and let $h : [0, 1] \rightarrow \mathbb{R}$ be a given function. We say that $f : D \rightarrow \mathbb{R}$ is an h -convex function if, for all $x, y \in D$ and $\alpha \in [0, 1]$, we have

$$f(\alpha x + (1 - \alpha)y) \leq h(\alpha)f(x) + h(1 - \alpha)f(y).$$

This type of h -convexity is a common generalization of the convex functions, the Godunowa-Levin functions, the Breckner functions and the P -functions. In [13], S. Varošanec considered only the non-negative h -convex functions defined on real intervals. However, in that case, the h -convexity is not a full generalization of the usual convexity.

The following property (see [5]) shows that in fact we do not assume the non-negativity of the function f in the definition.

PROPOSITION 1. Assume $f : D \rightarrow \mathbb{R}$ is h -convex.

- (i) If $h(\alpha) + h(1 - \alpha) > 1$, for some $\alpha \in [0, 1]$, then f is non-negative.
- (ii) If $h(\alpha) + h(1 - \alpha) < 1$, for some $\alpha \in [0, 1]$, then f is non-positive.

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Proof. For an arbitrary point $x \in D$, and for all $\alpha \in [0, 1]$, we have

$$\begin{aligned} f(x) &= f(\alpha x + (1 - \alpha)x) \leq h(\alpha)f(x) + h(1 - \alpha)f(x) \\ &= (h(\alpha) + h(1 - \alpha))f(x), \end{aligned}$$

which implies that

$$(h(\alpha) + h(1 - \alpha) - 1)f(x) \geq 0.$$

It finishes the proof. $\square$

Fix a convex subset D of a real linear space. Recall the following definitions.

A function $f: D \rightarrow \mathbb{R}$ is a Godunowa-Levin function if, for all $x, y \in D$ and $\alpha \in (0, 1)$,

$$f(\alpha x + (1 - \alpha)y) \leq \frac{f(x)}{\alpha} + \frac{f(y)}{1 - \alpha}.$$

Godunowa-Levin functions were firstly investigated in [4] by Godunova and Levin, see also [3], [8] and [9]. Godunowa-Levin functions are h -convex, where $h: [0, 1] \rightarrow \mathbb{R}$ is given by the formula

$$h(\alpha) = \begin{cases} 0, & \alpha = 0, \\ \frac{1}{\alpha}, & \alpha \in (0, 1]. \end{cases} \quad (1)$$

The concept of s -convexity was introduced by Breckner in [2]. A function $f: D \rightarrow \mathbb{R}$ is called Breckner s -convex (or s -convex in the second sense) if

$$f(\alpha x + (1 - \alpha)y) \leq \alpha^s f(x) + (1 - \alpha)^s f(y),$$

for all $x, y \in D$ and $\alpha \in [0, 1]$, where $s \in (0, 1]$ is a fixed number. Of course, s -convexity means the usual convexity when $s = 1$. The s -convex functions are h -convex with

$$h(\alpha) = \alpha^s, \quad \alpha \in [0, 1]. \quad (2)$$

The definition of P -functions was introduced in [3]. We say that a function $f: D \rightarrow \mathbb{R}$ is P -convex if, for every $x, y \in D$ and $\alpha \in [0, 1]$, we have

$$f(\alpha x + (1 - \alpha)y) \leq f(x) + f(y).$$

P -functions are h -convex, with

$$h(\alpha) = 1, \quad \alpha \in [0, 1]. \quad (3)$$

Separation, sandwich and extension theorems have many important and interesting applications in several fields of mathematics. The aim of the present paper is to study sufficient and necessary conditions for separating given two functions by h -convex functions, for multiplicative functions h . Analogous considerations for convex functions are presented in [1]. We generalize the following result containing in [1].

THEOREM 1 (K. Baron, J. Matkowski, K. Nikodem). *Real functions f and g , defined on a convex subset D of a real vector space, satisfy*

$$f\left(\sum_{j=1}^n t_j x_j\right) \leq \sum_{j=1}^n t_j g(x_j),$$

for each positive integer n , vectors $x_1, \dots, x_n \in D$ and real numbers $t_1, \dots, t_n \in [0, 1]$ such that $t_1 + \dots + t_n = 1$ if and only if there exists a convex function $F: D \rightarrow \mathbb{R}$ such that

$$f(x) \leq F(x) \leq g(x), \quad x \in D.$$

2. Results

DEFINITION 2. A function $h: [0, 1] \rightarrow \mathbb{R}$ is said to be multiplicative if

$$h(s \cdot t) = h(s) \cdot h(t), \quad s, t \in [0, 1].$$

For multiplicative functions it is known the following obvious result.

Remark 1. If h is multiplicative, then it is non-negative and either $h \equiv 0$ or $h(1) = 1$.

Now, we are going to prove the Jensen-type inequality for an h -convex function. The following theorem generalizes the classical Jensen inequality for the usual convexity.

THEOREM 2. *If h is multiplicative, then a real valued function $f: D \rightarrow \mathbb{R}$ is h -convex if and only if, for all $n \in \mathbb{N}$, $x_1, \dots, x_n \in D$, $\alpha_1, \dots, \alpha_n \in [0, 1]$ with $\alpha_1 + \dots + \alpha_n = 1$, we have*

$$f\left(\sum_{j=1}^n \alpha_j x_j\right) \leq \sum_{j=1}^n h(\alpha_j) f(x_j). \quad (4)$$

Proof. Let us suppose that f is h -convex. The case $n = 1$ is trivial, while, for $n = 2$, the inequality (4) follows from the definition of h -convexity.

Suppose that (4) is true for all convex combinations with at most $n \geq 2$ points. If $x_1, \dots, x_{n+1} \in D$, whereas $\alpha_1, \dots, \alpha_{n+1} \in (0, 1)$, and

$$\sum_{j=1}^{n+1} \alpha_j = 1,$$

then

$$\begin{aligned}
 f\left(\sum_{j=1}^{n+1} \alpha_j x_j\right) &= f\left((1 - \alpha_{n+1}) \sum_{j=1}^n \frac{\alpha_j}{1 - \alpha_{n+1}} x_j + \alpha_{n+1} x_{n+1}\right) \\
 &\leq h(1 - \alpha_{n+1}) f\left(\sum_{j=1}^n \frac{\alpha_j}{1 - \alpha_{n+1}} x_j\right) + h(\alpha_{n+1}) f(x_{n+1}) \\
 &\leq h(1 - \alpha_{n+1}) \sum_{j=1}^n h\left(\frac{\alpha_j}{1 - \alpha_{n+1}}\right) f(x_j) + h(\alpha_{n+1}) f(x_{n+1}) \\
 &= \sum_{j=1}^n h(1 - \alpha_{n+1}) h\left(\frac{\alpha_j}{1 - \alpha_{n+1}}\right) f(x_j) + h(\alpha_{n+1}) f(x_{n+1}) \\
 &= \sum_{j=1}^{n+1} h(\alpha_j) f(x_j).
 \end{aligned}$$

Clearly, if f satisfies (4), for all $n \in \mathbb{N}$, and for all $x_1, \dots, x_n \in D$ and $\alpha_1, \dots, \alpha_n \in [0, 1]$ with $\sum_{j=1}^n \alpha_j = 1$, then it is h -convex. $\square$

Remark 2. A corresponding result for non-negative super-multiplicative h and non-negative h -convex functions f is given in [13, Theorem 19].

Now, we are able to prove our main result.

THEOREM 3. *Let D be a non-empty convex subset of a real linear space, $h: [0, 1] \rightarrow \mathbb{R}$ a non-zero multiplicative function, and let $f, g: D \rightarrow \mathbb{R}$. Then, the following conditions are equivalent:*

- (i) *there exists an h -convex function $p: D \rightarrow \mathbb{R}$ such that $f \leq p \leq g$;*
- (ii) *the inequality*

$$f\left(\sum_{j=1}^n \alpha_j x_j\right) \leq \sum_{j=1}^n h(\alpha_j) g(x_j), \quad (5)$$

holds true for any $n \in \mathbb{N}$, $x_1, \dots, x_n \in D$, $\alpha_1, \dots, \alpha_n \in [0, 1]$ such that $\alpha_1 + \dots + \alpha_n = 1$.

Proof. Assume (i). Using the h -convexity of p , non-negativity of h and Theorem 2, for any $n \in \mathbb{N}$, $x_1, \dots, x_n \in D$, $\alpha_1, \dots, \alpha_n \in [0, 1]$ such that $\alpha_1 + \dots + \alpha_n = 1$, we have

$$f\left(\sum_{j=1}^n \alpha_j x_j\right) \leq p\left(\sum_{j=1}^n \alpha_j x_j\right) \leq \sum_{j=1}^n h(\alpha_j) p(x_j) \leq \sum_{j=1}^n h(\alpha_j) g(x_j).$$

Now, suppose that (ii) is fulfilled. Define the function $p : D \rightarrow \mathbb{R}$ by the formula

$$p(x) := \inf \left\{ \sum_{j=1}^n h(\alpha_j)g(x_j) : x = \sum_{j=1}^n \alpha_j x_j, \ x_1, \dots, x_n \in D, \right. \\ \left. \alpha_1, \dots, \alpha_n \in [0, 1], \ \alpha_1 + \dots + \alpha_n = 1, \ n \in \mathbb{N} \right\}.$$

By (5), this definition is correct and $f \leq p$ on D . Since $h(1) = 1$ (see Remark 1), we get $p \leq g$ on D .

We will check that $p : D \rightarrow \mathbb{R}$ is an h -convex function. Take arbitrary $x, y \in D$, $\alpha \in [0, 1]$, and arbitrary $\varepsilon > 0$. By the definition of p , there exist $n \in \mathbb{N}$, $x_1, \dots, x_n \in D$, $s_1, \dots, s_n \in [0, 1]$, $s_1 + \dots + s_n = 1$ with

$$\sum_{j=1}^n s_j x_j = x,$$

and there exist $k \in \mathbb{N}$, $y_1, \dots, y_k \in D$, $t_1, \dots, t_k \in [0, 1]$, $t_1 + \dots + t_k = 1$ with

$$\sum_{i=1}^k t_i y_i = y$$

such that

$$p(x) + \varepsilon > \sum_{j=1}^n h(s_j)g(x_j),$$

and

$$p(y) + \varepsilon > \sum_{i=1}^k h(t_i)g(y_i).$$

On the other hand,

$$\alpha x + (1 - \alpha)y = \alpha \sum_{j=1}^n s_j x_j + (1 - \alpha) \sum_{i=1}^k t_i y_i = \sum_{l=1}^{n+k} c_l z_l,$$

where

$$c_l = \begin{cases} \alpha s_l, & l = 1, \dots, n \\ (1 - \alpha)t_{l-n}, & l = n + 1, \dots, n + k, \end{cases},$$

$$z_l = \begin{cases} x_l, & l = 1, \dots, n \\ y_{l-n}, & l = n + 1, \dots, n + k. \end{cases}.$$

Clearly,

$$\sum_{l=1}^{n+k} c_l = \alpha \sum_{j=1}^n s_j + (1 - \alpha) \sum_{i=1}^k t_i = 1,$$

moreover,

$$\begin{aligned} & h(\alpha)p(x) + h(1 - \alpha)p(y) + [h(\alpha) + h(1 - \alpha)]\varepsilon \\ & > h(\alpha) \sum_{j=1}^n h(s_j)g(x_j) + h(1 - \alpha) \sum_{i=1}^k h(t_i)g(y_i) \\ & = \sum_{j=1}^n h(\alpha)h(s_j)g(x_j) + \sum_{i=1}^k h(1 - \alpha)h(t_i)g(y_i) \\ & = \sum_{j=1}^n h(\alpha s_j)g(x_j) + \sum_{i=1}^k h((1 - \alpha)t_i)g(y_i) \\ & = \sum_{l=1}^{n+k} h(c_l)g(z_l) \geq p(\alpha x + (1 - \alpha)y). \end{aligned}$$

Tending $\varepsilon \rightarrow 0_+$, we obtain the h -convexity of p . □

DEFINITION 3. Let D be a non-empty convex subset of a real linear space, $h: [0, 1] \rightarrow \mathbb{R}$ a given function, and let $\varepsilon > 0$ be a fixed number. A function $f: D \rightarrow \mathbb{R}$ is called ε - h -convex if the inequality

$$f\left(\sum_{j=1}^n \alpha_j x_j\right) \leq \sum_{j=1}^n h(\alpha_j)f(x_j) + \varepsilon$$

holds true for any $n \in \mathbb{N}$, $x_1, \dots, x_n \in D$, $\alpha_1, \dots, \alpha_n \in [0, 1]$ such that

$$\alpha_1 + \dots + \alpha_n = 1.$$

As an immediate consequence of Theorem 3, we obtain the following stability result for h -convex function.

THEOREM 4. Let D be a non-empty convex subset of a real linear space, and let $h: [0, 1] \rightarrow \mathbb{R}$ be a non-zero multiplicative function. A function $g: D \rightarrow \mathbb{R}$ is ε - h -convex if and only if there exists an h -convex function $\phi: D \rightarrow \mathbb{R}$ such that

$$g(x) - \varepsilon \leq \phi(x) \leq g(x), \quad x \in D. \quad (6)$$

Remark 3. All the functions given by formulas (1)–(3) are multiplicative, thus, we obtain the separation theorem for convex functions, Godunowa-Levin functions, s -Breckner functions and P -functions.

Remark 4. The multiplicative functions may have many pathological properties, in particular, they can be discontinuous at every point. Indeed, it is easy to see that every function of the form

$$h(s) = \begin{cases} e^{a(\log(s))}, & s \in (0, 1], \\ 0, & s = 0, \end{cases}$$

where $a: \mathbb{R} \rightarrow \mathbb{R}$ is an additive function, is multiplicative.

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