THE BI-DIMENSIONAL SPACE OF KORENBLUM AND COMPOSITION OPERATOR

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ABSTRACT. In this paper we present the concept of total κ -variation in the sense of Hardy-Vitali-Korenblum for a real function defined in the rectangle $I_a^b \subset \mathbb{R}^2$. We show that the space $\kappa BV(I_a^b, \mathbb{R})$ of real functions of two variables with finite total κ -variation is a Banach space endowed with the norm $\|f\|_\kappa = |f(a)| + \kappa TV(f, I_a^b)$. Also, we characterize the Nemytskij composition operator H that maps the space of functions of two real variables of bounded κ -variation $\kappa BV(I_a^b, \mathbb{R})$ into another space of a similar type and is uniformly bounded (or Lipschitzian or uniformly continuous).

Introduction

The function of bounded variation was discovered by J o r d a n in 1881 while working out the proof of Dirichlet concerning the convergence of Fourier Series. This concept has been generalized in many ways since then. The desire to extend this theorem to larger classes of functions has provided much of the impetus for the study of generalizations of bounded variation. Following this idea, in 1975, B. K o r e m b l u n distorted the measurement of intervals in the domain instead of the range space, as there had been the common practice, to arrive at the concept of κ -variation.

In 1982 J. M a t k o w s k i (see [6]) showed that we cannot apply the Banach-Caccioppoli contraction mapping principle in the space $\text{Lip}[a, b]$ to find the solutions to nonlinear equations. More specifically, M a t k o w s k i showed that the composition operator H generated by a function $h: [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ acts and

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is globally Lipschitzian in the space $\text{Lip}[a, b]$ if and only if

$$h(t, x) = g_1(t)x + g_2(t), \quad t \in [a, b], \quad x \in \mathbb{R}, \quad (1)$$

where $g_1, g_2 \in \text{Lip}[a, b]$. The representation (1) is known as *Matkowski condition*.

There exists a vast literature on the problem treated in this article, mainly in spaces of bounded variation (for instance [3], [7], [10], [11]). In 1984, J. Matkowski and J. Miś (see [8]) showed that representation (1) holds in the space of functions of bounded variation $BV[a, b]$ substituting the function h by its left regularization h^- , that is

$$h^-(t, x) = g_1(t)x + g_2(t), \quad t \in [a, b], \quad x \in \mathbb{R}, \quad (2)$$

where $g_1, g_2 \in BV[a, b]$. The representation (2) is a weak version of (1) and is called the *weak Matkowski condition*.

In the present paper we introduce the notion of a function of bounded κ -variation in the sense of Hardy-Vitali-Korenblum, and we show that the space $\kappa BV(I_a^b, \mathbb{R})$ of the functions of two variables of bounded total variation is a Banach space endowed with the norm $\|f\|_\kappa = |f(a)| + \kappa TV(f, I_a^b)$. We also show that, if the Nemytskij composition operator $H: \mathbb{R}^{I_a^b} \rightarrow \mathbb{R}^{I_a^b}$ generated by h maps the space $\kappa_1 BV(I_a^b, \mathbb{R})$ into $\kappa_2 BV(I_a^b, \mathbb{R})$ and is uniformly bounded, then the one-sided regularization of the generator is affine in the third variable, that is, the composition operator H satisfies the weak Matkowski condition.

1. Mappings of bounded κ -variation in $\mathbb{R}^2$

Let: $a = (a_1, a_2), b = (b_1, b_2) \in \mathbb{R}^2$; $I_a^b = [a_1, b_1] \times [a_2, b_2] \subset \mathbb{R}^2$; $\xi = \{t_i\}_{i=0}^m$ and $\eta = \{s_j\}_{j=0}^n$ be partitions of the intervals $[a_1, b_1]$ and $[a_2, b_2]$, respectively, denoted by $\xi: a_1 = t_0 < t_1 < \dots < t_m = b_1$; and $\eta: a_2 = s_0 < s_1 < \dots < s_n = b_2$. In the sequel, the symbol I_a^b denotes the basic rectangle $[a_1, b_1] \times [a_2, b_2]$ and $\mathbb{R}^{I_a^b}$ denotes the family of all functions $f: I_a^b \rightarrow \mathbb{R}$.

DEFINITION 1.1. A κ -function is a function $\kappa: [0, 1] \rightarrow [0, 1]$ with the following properties

- (a) κ is continuous with $\kappa(0) = 0, \kappa(1) = 1$,
- (b) κ is concave and strictly increasing, and
- (c) $\lim_{t \rightarrow 0^+} \frac{\kappa(t)}{t} = \infty$.

Now, we give some examples that illustrate Definition 1.1. Important special cases for choices of κ are

$$\kappa_0(t) = \begin{cases} t(1 - \lg(t)) & \text{if } t \neq 0, \\ 0 & \text{if } t = 0, \end{cases}$$

used by B. Korenblum in [4] and

$$\kappa_\alpha(t) = t^\alpha, \quad 0 < \alpha < 1,$$

in the literature now referred to as the Gevrey case.

The set of all κ -functions will be denoted by $\mathcal{K}$.

Remark 1.2. Every κ -function κ is subadditive. Then, for all partitions $\xi = \{t_i\}_{i=0}^m$ of $[a, b]$ ($a = t_0 < t_1 < \dots < t_m = b$), one has that

$$1 = \kappa(1) = \kappa\left(\sum_{i=1}^m \frac{|t_i - t_{i-1}|}{b - a}\right) \leq \sum_{i=1}^m \kappa\left(\frac{|t_i - t_{i-1}|}{b - a}\right). \quad (3)$$

For each function $f \in \mathbb{R}^{I_a^b}$, the following notations are usually used:

$$\begin{aligned} \Delta_{10}f(t_i, s_j) &= f(t_i, s_j) - f(t_{i-1}, s_j), \\ \Delta_{01}f(t_i, s_j) &= f(t_i, s_j) - f(t_i, s_{j-1}), \\ \Delta_{11}f(t_i, s_j) &= f(t_{i-1}, s_{j-1}) - f(t_{i-1}, s_j) - f(t_i, s_{j-1}) + f(t_i, s_j). \end{aligned}$$

2. Total bounded κ -variation in the sense of Korenblum in $\mathbb{R}^2$

In this section, we generalize the concept of κ -variation in $[a, b]$, presented by B. Korenblum in [4], to the two-dimensional Korenblum total κ -variation in I_a^b , and we prove that the Hardy-Vitali-Korenblum space $\kappa BV(I_a^b, \mathbb{R})$ of the function $f \in \mathbb{R}^{I_a^b}$ with bounded two-dimensional Korenblum total κ -variation is a Banach space.

DEFINITION 2.1. The (two-dimensional) total bounded κ -variation in the sense of Hardy-Vitali-Korenblum is defined as follows: Let $f \in \mathbb{R}^{I_a^b}$ and $\kappa \in \mathcal{K}$.

- (i) Given $x_2 \in [a_2, b_2]$, we defined the Jordan κ -variation of the function $f(\cdot, x_2)$ in $[x_1, y_1] \subset [a_1, b_1]$, denoted by $\kappa V_{[x_1, y_1]}(f) = \kappa V_{[x_1, y_1]}(f(\cdot, x_2))$, as

$$\kappa V_{[x_1, y_1]}(f(\cdot, x_2)) := \sup_{\xi} \frac{\sum_{i=1}^m |\Delta_{10}f(t_i, x_2)|}{\sum_{i=1}^m \kappa\left(\frac{|t_i - t_{i-1}|}{b_1 - a_1}\right)},$$

where the supremum is taken over all the partitions $\xi = \{t_i\}_{i=0}^m$ of $[x_1, y_1]$.

- (ii) Given $x_1 \in [a_1, b_1]$, we defined the Jordan κ -variation of the function $f(x_1, \cdot)$ in $[x_2, y_2] \subset [a_2, b_2]$, $\kappa V_{[x_2, y_2]}(f) = \kappa V_{[x_2, y_2]}(f(x_1, \cdot))$, as

$$\kappa V_{[x_2, y_2]}(f(x_1, \cdot)) := \sup_{\eta} \frac{\sum_{j=1}^n |\Delta_{01}f(x_1, s_j)|}{\sum_{j=1}^n \kappa\left(\frac{|s_j - s_{j-1}|}{b_2 - a_2}\right)},$$

where the supremum is taken over all the partitions $\eta = (s_j)_{j=0}^n$ of $[x_2, y_2]$.

- (iii) The two-dimensional Hardy-Vitali-Wiener κ -variation of the function $f \in \mathbb{R}^{I_a^b}$, denoted by $\kappa V_{I_a^b}(f, I_a^b)$, is defined as

$$\kappa V_{I_a^b}(f) := \sup_{(\xi, \eta)} \left(\frac{\sum_{i=1}^m \sum_{j=1}^n |\Delta_{11} f(t_i, s_j)|}{\sum_{i=1}^m \sum_{j=1}^n \kappa \left(\frac{|t_i - t_{i-1}| |s_j - s_{j-1}|}{(b_1 - a_1)(b_2 - a_2)} \right)} \right),$$

where the supremum is taken over all the partitions (ξ, η) of I_a^b .

- (iv) The total κ -variation of $f \in \mathbb{R}^{I_a^b}$ is defined by

$$\kappa TV(f, I_a^b) := \kappa V_{[a_1, b_1]}(f(\cdot, a_2)) + \kappa V_{[a_2, b_2]}(f(a_1, \cdot)) + \kappa V_{I_a^b}(f).$$

- (v) A function $f \in \mathbb{R}^{I_a^b}$ is of the total bounded κ -variation in I_a^b if

$$\kappa TV(f, I_a^b) < \infty.$$

- (vi) We will denote the set of all functions $f \in \mathbb{R}^{I_a^b}$ of the total bounded κ -variation on I_a^b by $\kappa BV(I_a^b, \mathbb{R})$, i.e.,

$$\kappa BV(I_a^b, \mathbb{R}) = \left\{ f \in \mathbb{R}^{I_a^b} : \kappa TV(f, I_a^b) < \infty \right\}.$$

Remark 2.2. The set $\kappa BV(I_a^b, \mathbb{R})$ is non-empty since the constant function belongs to it.

In the following result, we establish some relations between these families of functions.

Theorem 2.3. Let $\kappa, \kappa_1, \kappa_2 \in \mathcal{K}$.

- (a) $BV(I_a^b, \mathbb{R})^1 \subset \kappa BV(I_a^b, \mathbb{R})$.
 (b) If $\kappa_1 \leq \kappa_2$, then $\kappa_1 BV(I_a^b, \mathbb{R}) \subset \kappa_2 BV(I_a^b, \mathbb{R})$.

Now, we provide the class $\kappa BV(I_a^b, \mathbb{R})$ of a structure of norm space.

In $\kappa BV(I_a^b, \mathbb{R})$, we define the functional $\|\cdot\|_\kappa : \kappa BV(I_a^b, \mathbb{R}) \rightarrow \mathbb{R}$ by

$$\|f\|_\kappa := |f(a)| + \kappa TV(f, I_a^b), \quad a = (a_1, a_2).$$

Theorem 2.4. Let $\kappa \in \mathcal{K}$. Then $(\kappa BV(I_a^b, \mathbb{R}), \|\cdot\|_\kappa)$ is a normed space. Even more, $\kappa BV(I_a^b, \mathbb{R})$ is a Banach space endowed with the norm $\|\cdot\|_\kappa$.

Proof. Let $(f_\ell)_{\ell \geq 1}$ be a Cauchy sequence in $(\kappa BV(I_a^b), \|\cdot\|_\kappa)$. Then for each $\epsilon > 0$, there is a positive integer N such that

$$\|f_\ell - f_{\ell+1}\|_\kappa < \epsilon \quad \text{for } \ell \geq N.$$

¹The class of the function $f \in \mathbb{R}^{I_a^b}$ of bidimensional bounded total variation has been denoted by $BV(I_a^b, \mathbb{R})$ (see [2]).

Without loss of generality, we assume that $f_\ell(a_1, a_2) = 0$ for all $\ell \geq 1$. Therefore,

$$\kappa TV(f_\ell - f_{\ell+1}, I_a^b) < \epsilon, \quad \text{for all } \ell \geq N.$$

Hence,

$$\kappa V_{[a_1, b_1]}((f_\ell - f_{\ell+1})(\cdot, a_2)) < \epsilon,$$

$$\kappa V_{[a_2, b_2]}((f_\ell - f_{\ell+1})(a_1, \cdot)) < \epsilon,$$

and

$$\kappa V_{I_a^b}(f_\ell - f_{\ell+1}) < \epsilon, \quad \text{for all } \ell \geq N. \quad (4)$$

For the partitions $\xi: a_1 = t_0 < t = t_1 < t_2 = b_1; \eta: a_2 = s_0 < s = s_1 < s_2 = b_2$ of $[a_1, b_1]$ and $[a_2, b_2]$, respectively, by (4) and the definition of κ -variation, we get

$$|f_\ell(t, a_2) - f_{\ell+1}(t, a_2)| < 2\epsilon,$$

$$|f_\ell(a_1, s) - f_{\ell+1}(a_1, s)| < 2\epsilon$$

and

$$\begin{aligned} |(f_\ell - f_{\ell+1})(a_1, a_2) - (f_\ell - f_{\ell+1})(a_1, s) - (f_\ell - f_{\ell+1})(t, a_2) \\ + (f_\ell - f_{\ell+1})(t, s)| < 4\epsilon, \quad \text{for all } \ell \geq N. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} |f_\ell(t, s) - f_{\ell+1}(t, s)| &\leq |f_\ell(t, a_2) - f_{\ell+1}(t, a_2)| + |f_\ell(a_1, s) - f_{\ell+1}(a_1, s)| \\ &\quad + |(f_\ell - f_{\ell+1})(a_1, a_2) - (f_\ell - f_{\ell+1})(a_1, s) \\ &\quad - (f_\ell - f_{\ell+1})(t, a_2) + (f_\ell - f_{\ell+1})(t, s)| \\ &< 8\epsilon. \quad \text{for all } (t, s) \in I_a^b, \end{aligned} \quad (5)$$

Making use of (5), the sequence $(f_\ell(t, s))_{\ell \geq 1}, (t, s) \in I_a^b$ is Cauchy in $\mathbb{R}$, so, by completeness of $\mathbb{R}$, there exists a function $f: I_a^b \rightarrow \mathbb{R}$ such that

$$f(t, s) = \lim_{\ell \rightarrow \infty} f_\ell(t, s), \quad \text{for all } (t, s) \in I_a^b. \quad (6)$$

Since $(f_\ell)_{\ell \geq 1}$ is a Cauchy sequence in $(\kappa BV(I_a^b), \|\cdot\|_\kappa)$ and by (6), we get

$$\begin{aligned} \kappa TV(f_\ell - f) &= \sup_{\xi} \lim_{\ell' \rightarrow \infty} \frac{\sum_{i=1}^m |\Delta_{10} f_\ell(t_i, a_2) - \Delta_{10} f_{\ell'}(t_i, a_2)|}{\sum_{i=1}^m \kappa\left(\frac{|t_i - t_{i-1}|}{b_1 - a_1}\right)} \\ &\quad + \sup_{\eta} \lim_{\ell' \rightarrow \infty} \frac{\sum_{j=1}^n |\Delta_{01} f_\ell(a_1, s_j) - \Delta_{01} f_{\ell'}(a_1, s_j)|}{\sum_{j=1}^n \kappa\left(\frac{|s_j - s_{j-1}|}{b_2 - a_2}\right)} \\ &\quad + \sup_{(\xi, \eta)} \lim_{\ell' \rightarrow \infty} \frac{\sum_{i=1}^m \sum_{j=1}^n |\Delta_{11} f_\ell(t_i, s_j) - \Delta_{11} f_{\ell'}(t_i, s_j)|}{\sum_{i=1}^m \sum_{j=1}^n \kappa\left(\frac{|t_i - t_{i-1}| |s_j - s_{j-1}|}{(b_1 - a_1)(b_2 - a_2)}\right)} \\ &\leq 3\epsilon, \quad \text{for all } \ell \geq N. \end{aligned} \quad (7)$$

By (7), we get that $f \in \kappa BV(I_a^b)$ (note that: $f - f_\ell, f_\ell \in \kappa BV(I_a^b)$ and $f = (f - f_\ell) + f_\ell$ and $\|f_\ell - f\|_\kappa < \epsilon$). $\square$

3. The regularization functions

In this section, we present the concept of regularizations of real functions of two variables.

DEFINITION 3.1. Let $f \in \mathbb{R}^{I_a^b}$. We define the left-left regularization $f^- \in \mathbb{R}^{I_a^b}$ of f by

$$f^-(x_1, x_2) := \begin{cases} \lim_{\substack{y_1 \rightarrow x_1^- \\ y_2 \rightarrow x_2^-}} f(y_1, y_2) & \text{for } (x_1, x_2) \in (a_1, b_1] \times (a_2, b_2], \\ \lim_{\substack{y_1 \rightarrow x_1^- \\ y_2 \rightarrow a_2^+}} f(y_1, y_2) & \text{for } x_1 \in (a_1, b_1] \text{ and } x_2 = a_2, \\ \lim_{\substack{y_1 \rightarrow a_1^+ \\ y_2 \rightarrow x_2^-}} f(y_1, y_2) & \text{for } x_1 = a_1 \text{ and } x_2 \in (a_2, b_2], \\ \lim_{\substack{y_1 \rightarrow a_1^+ \\ y_2 \rightarrow a_2^+}} f(y_1, y_2) & \text{for } x_1 = a_1 \text{ and } x_2 = a_2. \end{cases}$$

DEFINITION 3.2. A function $f \in \mathbb{R}^{I_a^b}$ is said to be *left-left continuous* if

$$\lim_{\substack{x \rightarrow t^- \\ y \rightarrow s^-}} f(x, y) = f(t, s), \quad \text{for all } (t, s) \in (a_1, b_1] \times (a_2, b_2].$$

We denote the subspace of $\kappa BV(I_a^b, \mathbb{R})$ of those functions f which are left-left continuous on $(a_1, b_1] \times (a_2, b_2]$ by $\kappa BV^-(I_a^b, \mathbb{R})$.

LEMMA 3.3. If $f \in \kappa BV(I_a^b, \mathbb{R})$, then $f^- \in \kappa BV^-(I_a^b, \mathbb{R})$.

Proof. From the first line of the definition of the left-left regularization, it follows directly that f^- is left-left continuous on $(a_1, b_1] \times (a_2, b_2]$. We prove that $\kappa TV(f^-, I_a^b) < \infty$. In fact, for every $\epsilon > 0$, let $a_1 = t_0 < t_1 < \dots < t_m = b_1$ and $a_2 = s_0 < s_1 < \dots < s_n = b_2$ be partition of $[a_1, b_1]$ and $[a_2, b_2]$, respectively. From the definition of f^- , we obtain that

$$\begin{aligned} \kappa V_{[a_1, b_1]}(f^-(\cdot, a_2)) &\leq \kappa V_{[a_1, b_1]}(f(\cdot, a_2)) + \epsilon, \\ \kappa V_{[a_2, b_2]}(f^-(a_1, \cdot)) &\leq \kappa V_{[a_2, b_2]}(f(a_1, \cdot)) + \epsilon, \\ \kappa V_{I_a^b}(f^-) &\leq \kappa V_{I_a^b}(f) + \epsilon. \end{aligned}$$

Therefore, $\kappa TV(f^-, I_a^b) < \infty$. $\square$

4. An auxiliary result and its consequences

Following the idea given by J. Matkowski in [9], we prove Theorem 4.1.

THEOREM 4.1. *Let $\kappa_1, \kappa_2 \in \mathcal{K}$ and $h : I_a^b \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function with respect to the third variable. If the composition operator $H : \mathbb{R}^{I_a^b} \rightarrow \mathbb{R}^{I_a^b}$ generated by h maps the space $\kappa_1 BV(I_a^b, \mathbb{R})$ into $\kappa_2 BV(I_a^b, \mathbb{R})$ and satisfies the inequality*

$$\|H(f_1) - H(f_2)\|_{\kappa_2} \leq \gamma(\|f_1 - f_2\|_{\kappa_1}), \quad f_1, f_2 \in \kappa_1 BV(I_a^b, \mathbb{R}), \quad (8)$$

for some function $\gamma : [0, \infty) \rightarrow [0, \infty)$, then there are functions

$$a_0, a_1 \in \kappa_2 BV^-(I_a^b, \mathbb{R}),$$

such that

$$h^-(z, y) = a_0(z) + a_1(z)y, \quad z = (t, s) \in I_a^b, \quad y \in \mathbb{R}.$$

Proof. For each $y \in \mathbb{R}$, the constant function $f(z) = y (z = (t, s) \in I_a^b)$ belongs to $\kappa_1 BV(I_a^b, \mathbb{R})$. Since H maps $\kappa_1 BV(I_a^b, \mathbb{R})$ into $\kappa_2 BV(I_a^b, \mathbb{R})$, we have that the function $(Hf)(z) = h(z, f(z)) = h(z, y) \in \kappa_2 BV(I_a^b, \mathbb{R})$. By Lemma 3.3, the function regularization left-left $h^-(\cdot, y) \in \kappa_2 BV^-(I_a^b, \mathbb{R})$ for all $y \in \mathbb{R}$.

For $s', t' \in \mathbb{R}$, $s' < t'$, let us define the function $\eta_{s', t'} : \mathbb{R} \rightarrow [0, 1]$ by

$$\eta_{s', t'}(t) := \begin{cases} 0 & \text{if } t \leq s', \\ \frac{t-s'}{t'-s'} & \text{if } s' \leq t \leq t', \\ 1 & \text{if } t' \leq t. \end{cases}$$

Let $\mathcal{H} = Hf_1 - Hf_2$. Fix $r_1 \in (a_1, b_1]$, $r_2 \in (a_2, b_2]$, $m \in \mathbb{N}$, and take an arbitrary finite sequence

$$\begin{aligned} a_1 &< \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_m < \beta_m < r_1, \\ a_2 &< \bar{\alpha}_1 < \bar{\beta}_1 < \bar{\alpha}_2 < \bar{\beta}_2 < \cdots < \bar{\alpha}_m < \bar{\beta}_m < r_2. \end{aligned}$$

For arbitrary numbers $y_1, y_2 \in \mathbb{R}$, $y_1 \neq y_2$, the functions $f_1, f_2 : I_a^b \rightarrow \mathbb{R}$ defined by

$$f_\ell(t, s) := \frac{1}{2} \left[\eta_{\alpha_i, \beta_i}(t) + \eta_{\bar{\alpha}_i, \bar{\beta}_i}(s) \right] y_1 + (2 - \ell) y_2, \quad (t, s) \in I_a^b, \quad \ell = 1, 2, \quad (9)$$

belong to $\kappa_1 BV(I_a^b, \mathbb{R})$. From (9), we have

$$f_1 - f_2 \equiv y_2,$$

therefore,

$$\|f_1 - f_2\|_{\kappa_1} = |y_2|.$$

Since

$$\|\mathcal{H}\|_{\kappa_2} \geq \frac{\sum_{i=1}^m |\mathcal{H}(\beta_i, \bar{\beta}_i) - \mathcal{H}(\alpha_i, \bar{\alpha}_i)|}{C_m},$$

where

$$C_m = \max \left\{ \begin{array}{l} \sum_{i=1}^{2m+1} \kappa_1 \left(\frac{t_i - t_{i-1}}{b_1 - a_1} \right), \\ \sum_{j=1}^{2m+1} \kappa_1 \left(\frac{s_j - s_{j-1}}{b_2 - a_2} \right), \\ \sum_{i=1}^{2m+1} \sum_{j=1}^{2m+1} \kappa_1 \left(\frac{t_i - t_{i-1}}{b_1 - a_1} \frac{s_j - s_{j-1}}{b_2 - a_2} \right) \end{array} \right\}$$

and

$$\begin{aligned} t_0 &= a_1, & t_{2k-1} &= \alpha_k, & t_{2k} &= \beta_k, & k &= 1, \dots, m, & t_{2m+1} &= b_1, \\ s_0 &= a_2, & s_{2k-1} &= \bar{\alpha}_k, & s_{2k} &= \bar{\beta}_k, & k &= 1, \dots, m, & s_{2m+1} &= b_2. \end{aligned}$$

Therefore,

$$\sum_{i=1}^m |\mathcal{H}(\beta_i, \bar{\beta}_i) - \mathcal{H}(\alpha_i, \bar{\alpha}_i)| \leq C_m \gamma(|y_2|).$$

As

$$f_1(\alpha_i, \bar{\alpha}_i) = y_2, \quad f_2(\alpha_i, \bar{\alpha}_i) = 0, \quad f_1(\beta_i, \bar{\beta}_i) = y_1 + y_2 \quad \text{and} \quad f_2(\beta_i, \bar{\beta}_i) = y_1,$$

the last inequality can be rewritten in the form

$$\begin{aligned} \sum_{i=1}^m & \left| h(\beta_i, \bar{\beta}_i, y_1 + y_2) - h(\beta_i, \bar{\beta}_i, y_1) - h(\alpha_i, \bar{\alpha}_i, y_2) + h(\alpha_i, \bar{\alpha}_i, 0) \right| \\ & \leq C_m \gamma(|y_2|). \end{aligned} \quad (10)$$

So, if $(\alpha_1, \bar{\alpha}_1) \longrightarrow (r_1^-, r_2^-)$, then

$$C_m \longrightarrow C = \max \left\{ \begin{array}{l} \kappa_1 \left(\frac{b_1 - r_1}{b_1 - a_1} \right) + \kappa_1 \left(\frac{r_1 - a_1}{b_1 - a_1} \right), \\ \kappa_1 \left(\frac{b_2 - r_2}{b_2 - a_2} \right) + \kappa_1 \left(\frac{r_2 - a_2}{b_2 - a_2} \right), \\ \kappa_1 \left(\frac{b_1 - r_1}{b_1 - a_1} \frac{b_2 - r_2}{b_2 - a_2} \right) + \kappa_1 \left(\frac{b_1 - r_1}{b_1 - a_1} \frac{b_2 - r_2}{b_2 - a_2} \right) + \\ \kappa_1 \left(\frac{r_1 - a_1}{b_1 - a_1} \frac{r_2 - a_2}{b_2 - a_2} \right) + \kappa_1 \left(\frac{r_1 - a_1}{b_1 - a_1} \frac{r_2 - a_2}{b_2 - a_2} \right) \end{array} \right\}$$

and

$$\begin{aligned} \sum_{i=1}^m & \left| h^-(r_1, r_2, y_1 + y_2) + h^-(r_1, r_2, y_1) - h^-(r_1, r_2, y_2) + h^-(r_1, r_2, 0) \right| \\ & \leq C \gamma(|y_2|), \end{aligned}$$

whence

$$\begin{aligned} & \left| h^-(r_1, r_2, y_1 + y_2) + h^-(r_1, r_2, y_1) - h^-(r_1, r_2, y_2) + h^-(r_1, r_2, 0) \right| \\ & \leq \frac{C \gamma(|y_2|)}{m}. \end{aligned}$$

Since $m \in \mathbb{N}$ is arbitrary, we have

$$|h^-(r_1, r_2, y_1 + y_2) + h^-(r_1, r_2, y_1) - h^-(r_1, r_2, y_2) + h^-(r_1, r_2, 0)| = 0.$$

Therefore,

$$h^-(r_1, r_2, y_1 + y_2) + h^-(r_1, r_2, y_1) - h^-(r_1, r_2, y_2) + h^-(r_1, r_2, 0) = 0 \quad (11)$$

for all $r_1 \in (a_1, b_1]$, $r_2 \in (a_2, b_2]$ and $y_1, y_2 \in \mathbb{R}$, $y_1 \neq y_2$.

For $r_1 \in (a_1, b_1]$ and $r_2 = a_2$, let

$$a_1 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_m < \beta_m < r_1$$

and

$$a_2 < \bar{\alpha}_1 < \bar{\beta}_1 < \bar{\alpha}_2 < \bar{\beta}_2 < \cdots < \bar{\alpha}_m < \bar{\beta}_m < b_2,$$

the above arguments provide the estimate (10). Taking the limit when

$$(\alpha_1, \bar{\beta}_m) \longrightarrow (r_1^-, a_2^+),$$

we arrive at equality (11).

The cases when $r_1 = a_1$ and $r_2 \in (a_2, b_2]$ or $r_1 = a_1$ and $r_2 = a_2$ are treated in a similar way. In consequence, equality (11) holds for all $(t, s) \in I_a^b$, $y_1, y_2 \in \mathbb{R}$, $y_1 \neq y_2$.

To complete the proof of the theorem, we apply a standard argument (adapted) used in M. K u c z m a [5, pp. 314–315]. For all $z \in I_a^b$, we define the function $g_z: \mathbb{R} \rightarrow \mathbb{R}$ by

$$g_z(y) := h^-(z, y) - h^-(z, 0), \quad y \in \mathbb{R}. \quad (12)$$

For all $y_1, y_2 \in \mathbb{R}$, from (11) and (12), we have

$$\begin{aligned} g_z(y_1 + y_2) &= h^-(z, y_1 + y_2) - h^-(z, 0) \\ &= h^-(z, y_1) + h^-(z, y_2) - 2h^-(z, 0) \\ &= g_z(y_1) + g_z(y_2). \end{aligned}$$

This shows that g_z is an affine Jensen function in $\mathbb{R}$ and $g_z(0) = 0$.

Let us define the function $f(y) = g_z(y)$, $y \in \mathbb{R}$. Since g_z is an affine Jensen function, for all $y_1, y_2 \in \mathbb{R}$, we get

$$f(y_1 + y_2) = f(y_1) + f(y_2).$$

In other words, f is an additive function. Then, the continuity of $h^-(z, \cdot)$ implies the continuity of f . Consequently, since f is additive, f has to be linear; i.e., there is a function $a_1: I_a^b \longrightarrow \mathbb{R}$ such that

$$f(y) = g_z(y) = a_1(z)y$$

for all $z \in I_a^b$ and all $y \in \mathbb{R}$. Taking $a_0(z) = h^-(z, 0)$, we have

$$h^-(z, y) - h^-(z, 0) = g_z(y) = f(y) = a_1(z)y,$$

and so,

$$h^-(z, y) = a_0(z) + a_1(z)y, \quad z \in I_a^b, \quad y \in \mathbb{R}.$$

Since $a_0(\cdot) = h^-(\cdot, 0)$ and $a_1(\cdot) = h^-(\cdot, 1) - h^-(\cdot, 0)$, by Lemma 3.3, we obtain $a_0, a_1 \in \kappa_2 BV^-(I_a^b, \mathbb{R})$. $\square$

As a consequence of Theorem 4.1, we obtain the following result.

THEOREM 4.2. *Let $\kappa_1, \kappa_2 \in \mathcal{K}$ and $h : I_a^b \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function with respect to the third variable. If the composition operator $H : \mathbb{R}^{I_a^b} \rightarrow \mathbb{R}^{I_a^b}$ generated by h maps the space $\kappa_1 BV(I_a^b, \mathbb{R})$ into $\kappa_2 BV(I_a^b, \mathbb{R})$ and*

- (i) *H is Lipschitzian, or*
- (ii) *H is uniformly continuous,*

then there are functions $a_0, a_1 \in \kappa_2 BV^-(I_a^b, \mathbb{R})$, such that

$$h^-(z, y) = a_0(z) + a_1(z)y, \quad z = (t, s) \in I_a^b, \quad y \in \mathbb{R}.$$

Proof. (i) Since H is a Lipschitzian operator, there is a constant $\mu > 0$ such that

$$\|H(f_1) - H(f_2)\|_{\kappa_2} \leq \mu \|f_1 - f_2\|_{\kappa_1}$$

for all $f_1, f_2 \in \kappa_1 BV(I_a^b, \mathbb{R})$.

(ii) On the other hand, taking the uniform continuity of H into account, the modulus of continuity of H , $\gamma : [0, \infty) \rightarrow [0, \infty)$, i.e.,

$$\gamma(\tau) = \sup \left\{ \|H(f_1) - H(f_2)\|_{\kappa_2} : \|f_1 - f_2\|_{\kappa_1} \leq \tau, f_1, f_2 \in \kappa_1 BV(I_a^b, \mathbb{R}) \right\}$$

for $\tau \geq 0$, is well defined. Also, from the definition of the $\|\cdot\|_{\kappa}$, we obtain

$$\|H(f_1) - H(f_2)\|_{\kappa_2} \leq \gamma(\|f_1 - f_2\|_{\kappa_1}), \quad f_1, f_2 \in \kappa_1 BV(I_a^b, \mathbb{R}).$$

Hence, by Theorem 4.1, we get the required result. $\square$

5. Uniformly bounded composition operator

DEFINITION 5.1 ([9, Def. 1]). Let X and Y be two metric (or normed) spaces. We say that a mapping $H : X \rightarrow Y$ is uniformly bounded if, for any $t > 0$, there is a real number $\gamma(t)$ such that, for any nonempty set $B \subset X$, we have

$$\text{diam}(B) \leq t \implies \text{diam}H(B) \leq \gamma(t).$$

Remark 5.2. Obviously, every uniformly continuous operator or Lipschitzian operator is uniformly bounded. Note that, under the assumptions of this definition, every bounded operator is uniformly bounded.

Our main result reads as follows

THEOREM 5.3. *Let $\kappa_1, \kappa_2 \in \mathcal{K}$ and $h: I_a^b \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous with respect to the third variable. If the composition operator $H: \mathbb{R}^{I_a^b} \rightarrow \mathbb{R}^{I_a^b}$ generated by h maps the space $\kappa_1 BV(I_a^b, \mathbb{R})$ into $\kappa_2 BV(I_a^b, \mathbb{R})$ and is uniformly bounded, then there are functions $a_0, a_1 \in \kappa_2 BV^-(I_a^b, \mathbb{R})$, such that*

$$h^-(z, y) = a_0(z) + a_1(z)y, \quad z = (t, s) \in I_a^b, \quad y \in \mathbb{R}.$$

Proof. Take any $\tau \geq 0$ and arbitrary $f_1, f_2 \in \kappa_1 BV(I_a^b, \mathbb{R})$ such that

$$\|f_1 - f_2\|_{\kappa_1} \leq \tau.$$

Since $\text{diam}\{f_1, f_2\} \leq \tau$, by the uniform boundedness of H , we have

$$\text{diam}H(\{f_1, f_2\}) \leq \gamma(\tau),$$

that is

$$\|H(f_1) - H(f_2)\|_{\kappa_2} = \text{diam}H(\{f_1, f_2\}) \leq \gamma(\|f_1 - f_2\|_{\kappa_1}),$$

and the result follows from Theorem 4.1. $\square$

Remark 5.4. If the function $\gamma: [0, \infty) \rightarrow [0, \infty)$ in Definition 5.1 is right continuous at 0 and $\gamma(0) = 0$ (or if only $\gamma(0^+) = 0$) then, clearly, the uniform boundedness of the involved operator reduces to its uniform continuity.

Consider the following:

DEFINITION 5.5 ([9, Def. 2]). Let X and Y be two metric (or normed) spaces. We say that a mapping $H: X \rightarrow Y$ is equidistantly uniformly bounded if, for any $t > 0$, there is a nonnegative real number $\gamma(t)$ such that for all $u, v \in B \subset X$,

$$\text{diam}\{u, v\} = t \implies \text{diam}\{H(u), H(v)\} \leq \gamma(t).$$

Of course, the equidistant uniform boundedness is a weaker condition than the uniform boundedness. The following result follows from Theorem 4.1.

THEOREM 5.6. *Let $\kappa_1, \kappa_2 \in \mathcal{K}$ and $h: I_a^b \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function with respect to the third variable. If the composition operator $H: \mathbb{R}^{I_a^b} \rightarrow \mathbb{R}^{I_a^b}$ generated by h maps the space $\kappa_1 BV(I_a^b, \mathbb{R})$ into $\kappa_2 BV(I_a^b, \mathbb{R})$ and is equidistantly uniformly bounded, then there are functions $a_0, a_1 \in \kappa_2 BV^-(I_a^b, \mathbb{R})$, such that*

$$h^-(z, y) = a_0(z) + a_1(z)y, \quad z = (t, s) \in I_a^b, \quad y \in \mathbb{R}.$$

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