ON THE EXISTENCE OF RADIAL LIMITS

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ABSTRACT. We discuss conditions that ensure the existence of radial limits a.e. for harmonic functions defined on the unit disc D . We give an example of a Banach-valued harmonic function without radial limits at almost every point on the boundary of D .

1. Introduction

A holomorphic function in the unit disc D can behave very badly near the boundary ∂D , in general,. A simple illustration of this fact can be seen by considering the following example (see [6]).

Let $f: D \rightarrow \mathbb{C}$, be defined by $f(z) = \sum_{n=0}^{\infty} z^{n!}$. If we consider points $\omega_{\alpha} = e^{2\pi i \alpha}$, where $\alpha = p/q$ with $p, q \in \mathbb{N}$ and $(p, q) = 1$, we see that for $z = r\omega_{\alpha}$, $0 < r < 1$, we can write

$$f(z) = \sum_{n=1}^{q-1} z^{n!} + \sum_{n=q}^{\infty} r^{n!}.$$

Taking $M = 2q + N$, where N is an arbitrary positive integer, we obtain that

$$|f(z)| > \sum_{n=q}^M r^{n!} - \sum_{n=1}^{q-1} |z|^{n!} > (M - q) r^{M!} - (q - 1) \rightarrow M - 2q + 1 = N + 1$$

as $r \rightarrow 1^-$. This implies that f does not have radial limits in the set W of points ω_{α} , which is a denumerable and dense subset of ∂D . With much more effort, it can be proved that f does not have radial limit in any point of ∂D (a beautiful proof of this fact can be seen in [1, Th. 11]).

In this context, a natural question emerges: if f is a holomorphic function in D , what are the conditions on f that guarantee that f will converge, in an appropriate sense, to boundary values $f(e^{i\theta})$ on ∂D ? An answer to this question is given by Fatou's theorem, a remarkable result in the theory of functions of one-complex variable originally proved in [5]. This theorem states that a bounded

holomorphic function f defined on D has radial limits almost everywhere on ∂D . There is a generalization of this result for harmonic functions: every bounded harmonic function in D has radial limits a.e. on the boundary of D – it is also known as Fatou’s theorem. Even more, the study of the behaviour on the boundary of D can be extended not only in radial direction but also in the non-tangential sense, which means, to approach the boundary by means of the so-called Stoltz regions that, roughly speaking, are regions inside the disc with peaks in the boundary point. Fatou’s theorem can also be proved in this context, and it is a classical result in the theory of harmonic and analytic functions. A. Zygmund in [8] made a non-trivial construction of an analytic and bounded function F on the unit disc such that for almost every $\theta \in [0, 2\pi)$, $F(z)$ does not have a limit when z moves along a curve γ_θ which is tangent to the unit circle at $e^{i\theta}$.

We can also consider harmonic functions defined on the unit disc with values in a Banach space X . The question now is if we might assure the existence of radial limits for this kind of functions. In the next section, we present an example that answers, in a negative sense, our question.

2. Main result

Let X be a Banach space. We say that a function $\varphi: D \rightarrow X$ is harmonic if φ has continuous partial derivatives of order 2 (in the sense of the norm of X) on D and satisfies the Laplace’s equation, that is, $\Delta\varphi = 0$.

For $r > 0$ and $t \in [0, 1]$, we will denote by $P_r(t)$ the Poisson kernel in the unit disc, namely

$$P_r(t) = \frac{1 - r^2}{1 + r^2 - 2r \cos 2\pi t}.$$

Since harmonicity in the sense defined above is equivalent to weak harmonicity, i.e., the scalar functions $\langle \varphi, x^* \rangle$ are harmonic in the classical sense for each $x^* \in X^*$, it is very simple to show that if a function $f: [0, 1] \rightarrow X$ is Bochner integrable with respect to the Lebesgue measure, then

$$P(f)(re^{2\pi it}) = \int_0^1 P_r(t - s) f(s) ds$$

is an X -valued harmonic function on D . In fact, if $T: L^1[0, 1] \rightarrow X$ is a linear and continuous operator on the scalar space of Lebesgue integrable functions on $[0, 1]$, then the function

$$\Psi(re^{2\pi it}) := T(P_r(t - \cdot))$$

is also an X -valued harmonic function on D .

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When $g: [0, 1] \rightarrow \mathbb{C}$ is a Lebesgue integrable function, it is well-known that $\lim_{r \rightarrow 1} P(g)(re^{2\pi it}) = g(e^{2\pi it})$ a.e. for $t \in [0, 1]$ (see for example, [3]). This and the previous observations will allow us to show the following result.

THEOREM 1. *There exists a Banach space X and a harmonic and bounded function $\varphi: D \rightarrow X$ such that φ does not have radial limits almost everywhere on ∂D .*

PROOF. Let $X = c_0$, the space of complex sequences converging to 0. Consider the norm $\|\cdot\|_\infty$ in c_0 , that is, for a sequence $x = (x_n)_{n=0}^\infty$, $\|x\|_\infty = \sup_{n \geq 0} |x_n|$.

Let $T: L^1[0, 1] \rightarrow c_0$ be the linear and continuous operator defined by

$$T(f) = \left(\int_{[0,1]} f(\theta) \sin(2^n \pi \theta) d\theta \right)_{n=0}^\infty.$$

Now, let φ be the Poisson integral of T , that is $\varphi: D \rightarrow c_0$ such that

$$\varphi(re^{2\pi it}) = T(P_r(t - \cdot)).$$

Then, φ is a harmonic and bounded function on D .

We will show that there exists a Lebesgue measurable subset E of $[0, 1]$ with positive Lebesgue measure such that $\lim_{r \rightarrow 1} \varphi(re^{2\pi it})$ does not exist for every $t \in E$ with respect to the norm $\|\cdot\|_\infty$.

Notice that

$$\varphi(re^{2\pi it}) = \left(\int_{[0,1]} P_r(t - \theta) \sin(2^n \pi \theta) d\theta \right)_{n=0}^\infty.$$

Now, for every $n = 0, 1, 2, \dots$,

$$\int_{[0,1]} P_r(t - \theta) \sin(2^n \pi \theta) d\theta$$

represents a harmonic and bounded function with values on $\mathbb{R}$. By Fatou's theorem, for each $n = 0, 1, 2, \dots$, there exists a measurable set $A_n \subset [0, 1]$ with Lebesgue measure equal to 0 such that

$$\lim_{r \rightarrow 1} \int_0^1 P_r(t - \theta) \sin(2^n \pi \theta) d\theta = \sin(2^n \pi t) \quad (1)$$

for every $t \in [0, 1] \setminus A_n$.

Let $A = \bigcup_{n=0}^\infty A_n$, thus A has Lebesgue measure equal to 0, and for each $t \in [0, 1] \setminus A$, the limit (1) exists for every $n = 0, 1, 2, \dots$

Take $t \in [0, 1] \setminus A$ and let it be fixed. If there were an element $f(t) \in c_0$, let us say $f(t) = (f_n(t))_{n=0}^\infty$ such that

$$\lim_{r \rightarrow 1} \varphi(re^{2\pi it}) = f(t) \quad (2)$$

with respect to the norm $\|\cdot\|_\infty$, then using the fact that the convergence with respect to $\|\cdot\|_\infty$ implies usual convergence on each coordinate, we would have

$$f_n(t) = \sin(2^n \pi t)$$

for each $n = 0, 1, 2, \dots$. However, the sequence $(\sin(2^n \pi t))_{n=0}^\infty \notin c_0$ unless $t \in D$ where

$$D = \{k/2^m : k \in \mathbb{Z} \text{ and } m \in \mathbb{N} \cup \{0\}\}$$

which is a denumerable set and hence, $D \cap [0, 1]$ has Lebesgue measure equal to 0.

This shows that for at most a denumerable subset of $[0, 1] \setminus A$, the limit (2) might exist, or, in other words, there is a Lebesgue subset $E \subset [0, 1]$ with positive measure, where φ does not have radial limits. $\square$

Before concluding, it should be mentioned what is really happening in the previous example: the Banach space under consideration $(c_0, \|\cdot\|_\infty)$ does not satisfy the Radon-Nikodym property; roughly speaking, it states that the classical Radon-Nikodym theorem holds in the space. The Radon-Nikodym property has been extensively studied (for a detailed exposition see [4]) and has many formulations, among them that one stated in the following result proved by several authors, for example [2]

THEOREM 2 ([2]). *A Banach space X has the Radon-Nikodym property if and only if for every harmonic and bounded function $u: D \rightarrow X$ there exists an essentially bounded function $f: [0, 1] \rightarrow X$ such that*

$$\lim_{r \rightarrow 1} u(re^{2\pi it}) = f(t)$$

for almost every $t \in [0, 1]$.

In view of Theorem 2, the space $(c_0, \|\cdot\|_\infty)$ does not have the Radon-Nikodym property.

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