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LINEAR COMBINATIONS OF THE CLASSIC CANTOR SET

Marta Pawłowicz

ABSTRACT. The sets of the form $C + mC$, where $m \in (0, 1)$ and C is the classic Cantor ternary set, are described.

1. Introduction

Georg Cantor showed the construction of a perfect nowhere dense set [1]. We will denote this set by C and call it the classic Cantor set. Each linear combination of the classic Cantor set C can be basically described as a suitable iteration of the set modified by enlarging each interval component (details of construction of C can be found on page 49).

The problem is not taken out of nowhere although the beginnings of algebraic summation of perfect sets in the real line were rather incidental. It started in 1917 when a Polish mathematican Hugo Steinhaus published his discovery of $C+C = [0, 2]$ in [14]. Unfortunately he published his result not only during the First World War, not only in an obscure mathematical journal, but also he wrote his paper in Polish, altogether making it practically inaccessible to most of the matematical community. Even the Central Library the Polish Mathematical Society in Warsaw does not posses a copy of the original Steinhaus's paper today due to losses suffered during the Second World War. No wonder that the equality $C + C = [0, 2]$ was rediscovered by Randolph in 1940 and this time published in a well-established journal [11], [12]. The next development came in 1951 when W. R. Utz proved in [15] that $C + mC$ is an interval if and only if $|m| \in [\frac{1}{3}, 3]$.

The revival of investigation on algebraic sums of perfect sets started in 1987 with the famous Palis conjecture [10]. Motivated by his research on dynamical

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systems he asked whether the sum of two regular Cantor sets must be either a set containing an interval or a set of Lebesgue measure zero. Here by a Cantor set we mean any bounded nowhere dense perfect set in the real line. Thus the classic Cantor ternary set is a Cantor set, but there are plenty of other Cantor sets.

Important examples of Cantor sets are the Cantor sets E_a for $a \in (0, \frac{1}{2})$, which are the sets of all subsums of the geometric series $\sum_{n=1}^{\infty} a^n$,

$$E_a := \left\{ x \in \mathbb{R} \ : \ \exists A \subset \mathbb{N} \quad x = \sum_{n \in A} a^n \right\}.$$

In particular the classic Cantor set C is the set $2E_{\frac{1}{3}}$ exactly. In general the set of subsums of any absolutely convergent series is a bounded perfect set [3], [4], [5], [6]. However, it does not have to be nowhere dense. A fundamental sufficient condition for the set of subsums of a monotone decreasing series of positive numbers to be a Cantor set is

$$a_n > \sum_{k=n+1}^{\infty} a_k$$

for all sufficiently large indices n. It was formulated by S. Kakeya about one hundred years ago [5] (like Steinhaus, he published his papier during the First World War, he published it in a freshly founded Japanese mathematical journal, but he wrote his notes in English, not in Japanese, fortunately). His conjecture, that the condition is also necessary was demonstrated false by an example stated without details in 1980 by Shapiro and Weinstein in [16]. Finding a characterization of absolutely convergent series that have a Cantor set as their sets of subsums remains an open and very challenging problem.

The Palis conjecture was given a negative answer by R. Sannami in [13]. However, pairs of Cantor sets C_1, C_2 such that $C_1 + C_2$ has a positive Lebesgue measure and contains an interval are very rare and Moreira and Yoccoz proved in [7] that in a generic sense, if the sum of the Hausdorff dimensions of C_1 and C_2 is larger than 1, then $C_1 + C_2$ contains intervals. On the other hand, it is known that if the sum of the Hausdorff dimensions of C_1 and C_2 is less than 1, then $C_1 + C_2$ has Lebesgue measure zero.

Topological classification of sets of subsums of absolutely convergent series was found by J. Nymman and R. Guthrie in [2]. A gap in the first proof of the result was closed by J. Nymman and R. Saenz in [9].

J. Nyman used the classification theorem to investigate linear combinations of Cantor sets of the form $E_a + xE_a$, where $x \in [0, 1]$ (see [8]). He obtained many nice but incomplete results. His findings wait for continuation still, but this research will not be easy by any means. Nymann's results on topological classification of linear combinations of the set E_a were a part of my master thesis at Szczecin University and it was in the process of writing the thesis when I noticed that Utz's method of proof from [15] can be extended to give a complete description of all linear combinations of the classical Cantor sets.

2. Preparations

We want to find a the topological classification of simple linear combinations of the Cantor set C, that is, we want to describe (up to homeomorphism) sets $aC + bC$, where $a, b \in \mathbb{R}$.

The problem of topological classification $aC + bC$, where $a, b \in \mathbb{R}$, can be reduced to the study $C+mC$, where $m \in [0, 1]$ (bearing in mind that $-C = C-1$) because:

- if $ab > 0, |a| > |b|$,
  then $aC + bC = a(C + \frac{b}{a}C)$, where $\frac{b}{a} \in [0, 1]$,
- if $ab < 0, |a| > |b|, a < 0, b > 0$,
  then $aC + bC = a(C - \frac{b}{|a|}C) = a(C + \frac{b}{|a|}C - \frac{b}{|a|}) = a(C + \frac{b}{|a|}C) + b$, where $\frac{b}{|a|} \in [0, 1]$,
- if $ab < 0, |a| > |b|, a > 0, b < 0$,
  then $aC + bC = a(C + \frac{b}{a}C) = a(C - \frac{|b|}{a}C) = a(C + \frac{|b|}{a}C - \frac{|b|}{a}) = a(C + \frac{|b|}{a}C) + |b|$, where $\frac{|b|}{a} \in [0, 1]$.

Thus knowledge of the structure of the set $C + mC$, where $m \in [0, 1]$ is sufficient for understanding the topological structure of the set $\alpha(C + mC) + \beta$, where $\alpha, \beta \in \mathbb{R}$, because both dilation and translation are homeomorphisms.

Most likely, the reader knows the standard construction of the classic Cantor set but I am going to sketch it briefly in the order to introduce some specialized notation necessary to describe the proof of the main result of this note.

Let C_0 be the interval $[0, 1]$, C_1 the set obtained from C_0 by throwing away the open middle third, that is the interval $(\frac{1}{3}, \frac{2}{3})$. In general, let C_{k+1} denote the set obtained from C_k by removing the middle third (open set) of each interval component of C_k. We will call a set C_k the kth iteration of the Cantor set. Then the kth iteration C_k consists of 2^k pairwise disjoint closed subintervals. We can denote them by the letter P with an index s being a k-letter word from the alphabet $\{0,2\}$. We will use the symbol J_k to denote the family of all component intervals of C_k. Thus

$$J_k = \{P_s\}_{s \in S_k},$$

where

$$S_k = \big\{a_1 a_2 \ldots a_k : \; a_i \in \{0,2\}\big\}.$$

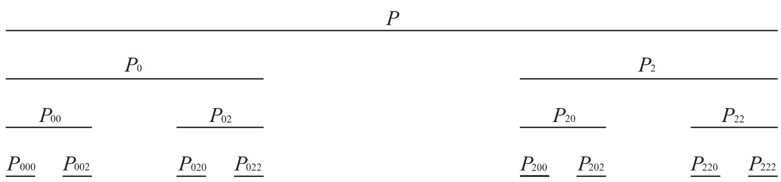

FIGURE 1.

Then we have

$$C_k = \bigsqcup_{P \in J_k} [\min P, \max P], \tag{1}$$

where $\bigsqcup$ is the symbol for a disjoint union and the classic Cantor set C is defined by

$$C = \bigcap_{k=0}^{\infty} C_k.$$

$$P_{i_1 i_2 \ldots i_k} = \left[l_k^{(n)}, r_k^{(n)}\right],$$

where $n = n_1 n_2 \ldots n_k$, $\frac{i_j}{2} = n_j$ is entered in dyadic form. Here

$$l_k^{(n)} = \frac{a_k^{(n)}}{3^k}, \qquad r_k^{(n)} = \frac{a_k^{(n)} + 1}{3^k} \quad \text{and} \quad a_k^{(n)} \tag{2}$$

are given by the recursive formula

$$a_0^{(1)} = 0, \quad a_{k+1}^{(n)} = a_k^{(n)}, \quad a_{k+1}^{(2^k+n)} = a_k^{(n)} + 2 \cdot 3^k, \quad n = 1, 2, \ldots, 2^k$$

that can be proved by induction. Thus

$$C_k = \bigsqcup_{n=1}^{2^k} \left[l_k^{(n)}, r_k^{(n)}\right]. \tag{3}$$

The classic Cantor set C can be also described as the set of all subsums of the series $\sum_{n=1}^{\infty} a_n$, where $a_n = \frac{2}{3^n}$, so as the set $2E_{\frac{1}{3}}$, where

$$E_{\frac{1}{3}} = \left\{ \sum_{n=1}^{\infty} \varepsilon_n \cdot \frac{1}{3^n} : \varepsilon_n \in \{0,1\} \right\}.$$

Intervals removed in kth iteration, which were not removed in any sth iteration with $s < k$, will be called *kth order gaps*. We enumerate these gaps from left to right and denote by $G_k^{(n)}$, where $k \in \mathbb{N}$ while $n = 1, \ldots, 2^{k-1}$. The length of each gap of the kth order is $\frac{1}{3^k}$.

If $n = n_1 n_2 \ldots n_k$ is dyadic expansion of n and $i_j = 2n_j$, $j = 1, 2, \ldots, k$, then

$$G_k^{(n)} = \left(\frac{i_1}{3} + \frac{i_2}{3^2} + \cdots + \frac{i_{k-1}}{3^{k-1}} + \frac{1}{3^k}, \frac{i_1}{3} + \frac{i_2}{3^2} + \cdots + \frac{i_{k-1}}{3^{k-1}} + \frac{2}{3^k} \right).$$

Now observe that

$$\bigsqcup_{n=1}^{2^{k-1}} G_k^{(n)} = C_{k-1} \setminus C_k.$$

The number of gaps of the kth order is the same as a number of intervals in $(k-1)$th iteration.

Additionally, using previous notation, we can write

$$G_k^{(n)} = \left(r_k^{(2n-1)}, l_k^{(2n)} \right), \quad \text{where} \quad n = 1, \ldots, 2^{k-1}.$$

By $\mathcal{G}_k$ we denote the union of all gaps created in kth iteration, so

$$\mathcal{G}_k = \bigsqcup_{n=1}^{2^{k-1}} G_k^{(n)},$$

and we arrive at yet another description of the Cantor set

$$C = [0,1] \setminus \bigcup_{k=1}^{\infty} \mathcal{G}_k.$$

Now we turn our attention to the Cartesian product $C \times C$.

Sets in the form $G_k^{(n)} \times [0,1]$, where $G_k^{(n)}$ is a gap of the kth order, we will call *vertical crevices of the kth order*, while sets in the form $[0,1] \times G_k^{(n)}$, where $G_k^{(n)}$ is gap of the kth order, we will call *horizontal crevices of the kth order*, where $k \in \mathbb{N}$ and $n = 1, \ldots, 2^{k-1}$.

From elementary properties of the Cartesian product, we get that

$$C \times C = \bigcap_{k=0}^{\infty} (C_k \times C_k).$$

3. Results

The equality in the following theorem is worth comparing with (3).

Theorem 1.

$$C + mC = \bigsqcup_{n=1}^{2^k} \left[l_k^{(n)}, r_k^{(n)} + m \right], \quad \text{for all } m \in (0,1),$$

where k is such that $m \in [\frac{1}{3^{k+1}}, \frac{1}{3^k})$, $k \in \mathbb{N}_0$, where $l_k^{(n)}$ i $r_k^{(n)}$ are the left and right endpoints of nth component of the kth iteration C_k defined in (2).

The theorem has a very nice geometric interpretation. In particular, if $m \in (0, \frac{1}{3})$, we first find the smallest positive integer k such that m is smaller than the length of the shortest gap of the kth iteration C_k. Next, we enlarge each component interval P_s, $s \in S_k$, of C_k by moving its right endpoint by m to the right. The disjoint union of the resulting 2^k closed intervals is the set $C + mC$.

Using notation of the equation (1) the above theorem can be formulated as follows

$$C + mC = \bigsqcup_{P \in J_k} [\min P, \max P + m] \quad \text{for all } m \in (0,1),$$

where k is such that $m \in [\frac{1}{3^{k+1}}, \frac{1}{3^k})$, $k \in \mathbb{N}_0$.

We do not consider the case $m = 1$ in the theorem, but the case was already described by Steinhaus in his article [14]. That is why we will take care only of $m \in (0,1)$ in further considerations.

The proof of Theorem 1 is based on the following observations

Observation 1. Given $z \in \mathbb{R}$, we define a straight line

$$P_{m,z} := \{(x,y) \in \mathbb{R}^2 : y = -mx + z\},$$

Then $z \in C + mC$ if and only if $P_{m,z} \cap (C \times C) \neq \emptyset$. Thus we can reduce the problem of describing the set $C + mC$ to finding the z's for which the line $P_{m,z}$ intersects the Cartesian square $C \times C$. Because $P_{m,z} \cap (C \times C) \neq \emptyset$ implies $z \in [0, 1+m]$, we will focus only on $z \in [0, 1+m]$ in further considerations.

Observation 2.

$$P_{m,z} \cap (C \times C) \neq \emptyset \iff \forall k \in \mathbb{N}_0 \quad P_{m,z} \cap (C_k \times C_k) \neq \emptyset.$$

Geometrically it says that if the line $P_{m,z}$ has a common point with $C \times C$, then it has a common point with product of iterations $C_k \times C_k$ for every k.

P r o o f. Necessity follows from the obvious inclusions $C \times C \subset C_k \times C_k$ for all indices k. In order to proof sufficiency, take a point

$$c_k = (x_k, y_k) \in P_{m,z} \cap (C_k \times C_k) \quad \text{for each } k.$$

Then (c_k) is a sequence of points of $[0,1] \times [0,1]$ and it has a subsequence converging to a point $c \in [0,1] \times [0,1]$ by the Bolzano-Weierstrass theorem. Since $P_{m,z} \cap ([0,1] \times [0,1])$ is a closed set, we have $c \in P_{m,z}$. On the other hand, the inclusions $C_i \times C_i \subset C_k \times C_k$ for $i \geq k$, imply that for any k almost all terms of the subsequence belong to $C_k \times C_k$. The set $C_k \times C_k$ is closed as well and hence $c \in C_k \times C_k$ for every k. Thus

$$c \in \bigcap_k (C_k \times C_k) = C \times C$$

which completes the proof. □

Observation 3.

$$P_{m,z} \cap (C \times C) = \emptyset \quad \text{if and only if} \quad \exists k(P_{m,z}) \in \mathbb{N}_0 \quad \forall i \leq k(P_{m,z}),$$

$$P_{m,z} \cap (C_i \times C_i) \neq \emptyset \quad \text{and} \quad P_{m,z} \cap \left(C_{k(P_{m,z})+1} \times C_{k(P_{m,z})+1}\right) = \emptyset.$$

The index $k(P_{m,z})$ is uniquely determined for every line $P_{m,z}$ disjoint with $C \times C$ by Observation 3. This is the number of the last iteration $C_{k(P_{m,z})} \times C_{k(P_{m,z})}$ of the set $C \times C$ having a common point with $P_{m,z}$. Then iterations $C_i \times C_i$ for $i > k(P_{m,z})$ are disjoint from line the $P_{m,z}$.

Observation 4. If $P_{m,z} \cap (C \times C) = \emptyset$, then $P_{m,z}$ lies in a horizontal crevice of the order $k+1$.

P r o o f. For simplicity we write k instead of $k(P_{m,z})$.
The phrase *a line L goes through (or: lies in) a vertical crevice* means that there are $j \in \mathbb{N}$ and $n \in \{1, \ldots, 2^{j-1}\}$, such that $P_{m,z} \cap ([0,1] \times [0,1]) \subset (G_j^{(n)} \times [0,1])$. It is easy to observe that no line $P_{m,z}$ with $m \in (0,1)$ and $z \in [0, 1+m]$ can go through any vertical crevice of any order because m is too small.

Similary, by the phrase *a line L goes through (or: lies in) a horizontal crevice* we understand that there are $j \in \mathbb{N}$ and $n \in \{1, \ldots, 2^{j-1}\}$, such that

$$P_{m,z} \cap \big([0,1] \times [0,1]\big) \subset \left([0,1] \times G_j^{(n)}\right).$$

For the purposes of the proof we will use the notation described on page 50.

With this notation, each component interval of the jth iteration defines uniquely all intervals from previous iterations which contain this interval and

$$P_{i_1 i_2 \ldots i_j} \subset P_{i_1 i_2 \ldots i_{j-1}} \subset \cdots \subset P_{i_1} \subset P, \quad \text{where} \quad i_t \in \{0, 2\}, \ \ t \in \mathbb{N}.$$

If the line $P_{m,z}$ does not have a common point with $C \times C$, then there exists $k \in \mathbb{N}_0$ by Observation 3, such that

$$P_{m,z} \cap (C_k \times C_k) \neq \emptyset \quad \text{and} \quad P_{m,z} \cap (C_{k+1} \times C_{k+1}) = \emptyset,$$

that is, there is a component square S of $C_k \times C_k$ containing a portion of the line and such that all four squares of $C_{k+1} \times C_{k+1}$ contained in the square of $C_k \times C_k$ are disjoint from $P_{m,z}$.

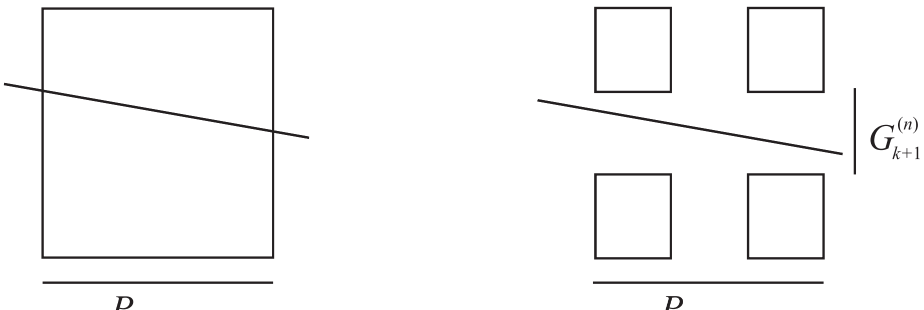

FIGURE 2.

Let $P_{i_1 i_2 \ldots i_k}$ be the projection of the square S on the x-axis and let n be the index of the gap of order $k+1$ contained in the projection of S onto the y-axis.

We want to observe that the only possibility that the line with the slope $-m \in (-1, 0)$ passes through a component square of $C_k \times C_k$ and does not intersect any of the squares belonging to $C_{k+1} \times C_{k+1}$ is that which lies in

$$P_{i_1 i_2 \ldots i_k} \times G_{k+1}^{(n)}.$$

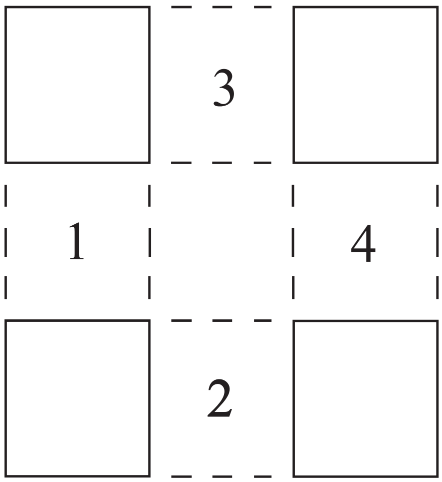

Figure 3. Areas 1, 2, 3, and 4 are open sets.

Indeed, if the line passes through the area number 1, it must have a slope of absolute value less than 1, however, if the line goes through the square 2, the slope must be of absolute value greater than 1, so there is no straight line passing through areas 1 and 2 simultaneously. Similarly, we exclude the case of a line passing through the squares 3 and 4. Since we consider lines with negative slope and of absolute value less than one only, line passing through 2 and 3 are not taken into consideration. Hence the only option is a straight line is passing through squares 1 and 4.

Let $P_{m,z}(P_{i_1 i_2 \dots i_k})$ denote the projection onto y-axis of the segment of line $P_{m,z}$ lying above the interval $P_{i_1 i_2 \dots i_k}$, that is

$$P_{m,z}(P_{i_1 i_2 \dots i_k}) = \{-mx + z : \; x \in P_{i_1 i_2 \dots i_k}\}.$$

If

$$P_{m,z} \cap (C_{k+1} \times C_{k+1}) = \emptyset \quad \text{and} \quad P_{m,z} \cap (C_k \times C_k) \neq \emptyset,$$

then

$$P_{m,z}(P_{i_1 i_2 \dots i_k}) \subset G_{k+1}^{(n)}, \quad \text{which implies that } m < \frac{1}{3}.$$

We now turn our attention to the segment of $P_{m,z}$ lying above the larger interval $P_{i_1 i_2 \dots i_{k-1}}$.

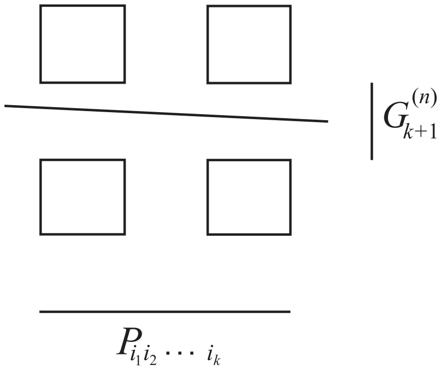

FIGURE 4.

Because the points A, B, C are collinear (see the Figure 5), the line $P_{m,z}$ satisfies the condition $P_{m,z}(P_{i_1,i_2,\dots,i_k}) \subset G_{k+1}^{(n)}$ and by assumption

$$P_{m,z} \cap (C_{k+1} \times C_{k+1}) = \emptyset,$$

this line cannot go beyond the area $P_{i_1 i_2 \dots i_{k-1}} \times G_k^{(n)}$ due to the slope it must have - we know that the line passes through the areas 1 or 4 (marked in the Figure 5) and that $m < \frac{1}{3}$. If the line goes through the areas 2 or 3 it should have slope $m > \frac{1}{3}$ (bearing in mind that the points A, B, C are collinear), so the only possibility is that the line $P_{m,z}$ goes simultaneously through the areas 1 and 4, which implies the condition: $m < \frac{1}{9}$. So we have the situation depicted in Figure 5.

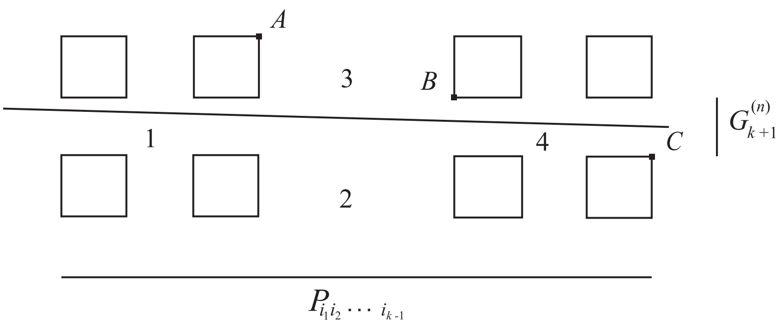

FIGURE 5.

Continuing the reasoning, if

$$P_{m,z}(P_{i_1,i_2,\ldots,i_{k-1}}) \subset G^{(n)}_{k+1}, \quad \text{then} \quad P_{m,z}(P_{i_1,i_2,\ldots,i_{k-2}}) \subset G^{(n)}_{k+1}$$

due to the slope the line must have - we know already that the line passes through the area 1' or 4' (denoted in the Figure 6) and that $m < \frac{1}{9}$, if the line goes through areas 2 'or 3' it should have $m > \frac{1}{9}$ (note that the points A', B', C' are also collinear), so the only possibility is that the line $P_{m,z}$ simultaneously passes through areas 1 'and 4', which implies the condition $m < \frac{1}{27}$. So we have (see Fig. 6).

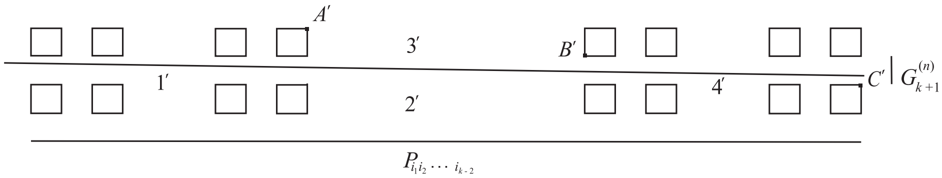

Figure 6.

and repeating the argument k-times we obtain

$$P_{m,z}(P) \subset G^{(n)}_{k+1} \quad \text{and} \quad m \leq \frac{1}{3^{k+1}},$$

i.e., the line $P_{m,z}$ lies in a horizontal slot $([0,1] \times G^{(n)}_{k+1})$, which completes the proof. □

The next observations follows from the definition of a horizontal crevice easily.

Observation 5. If the line $P_{m,z}$ lies in a horizontal crevice of the ith order, then

$$m < \frac{1}{3^i}.$$

The above observation combined with Observation 4 yields

Observation 6. If the line $P_{m,z}$ is disjoint from $C \times C$, then

$$m < \frac{1}{3^{k(P_{m,z})+1}}.$$

Remark. Consider a line $P_{m,z}$, where $m \in [\frac{1}{3}, 1]$ and $z \in [0, 1+m]$. If $P_{m,z} \cap (C \times C) = \emptyset$, then since $k(P_{m,z}) \geq 0$, we obtain by the Observation 6

$$m < \frac{1}{3^{k(P_{m,z})+1}} \leq \frac{1}{3},$$

a contradiction.

Hence the line intersects $C \times C$ for any $z \in [0, 1+m]$. Thus $C + mC = [0, 1+m]$ for $m \geq \frac{1}{3}$ and we can now deduce the Utz's theorem [15] easily, because

$$C + \alpha C = \alpha\Big(C + \frac{1}{\alpha}C\Big) \quad \text{for } \ \alpha \in [1, 3].$$

Observation 7. If $m \geq \frac{1}{3^{i+1}}$ and $P_{m,z} \cap (C \times C) = \emptyset$, then

$$k(P_{m,z}) \leq i - 1.$$

Observation 8.

$$P_{m,z} \cap [0,1]^2 \subset \Big[[0,1] \ \times \ \Big(r_k^{(i)}, l_k^{(i+1)}\Big)\Big] \Longleftrightarrow z \in \Big(r_k^{(i)} + m, l_k^{(i+1)}\Big).$$

So the line $P_{m,z}$ goes through a crevice

$$\Big[[0,1] \ \times \ \Big(r_k^{(i)}, l_k^{(i+1)}\Big)\Big] \quad \text{if and only if} \quad z \in \Big(r_k^{(i)} + m, l_k^{(i+1)}\Big).$$

Proof of Theorem 1.
Taking $m \in [\frac{1}{3^{k+1}}, \frac{1}{3^k})$ and $k \in \mathbb{N}$. The case $m \in [\frac{1}{3}, 1)$ has already been discussed in the remark following Observation 6, so we get:

$$\begin{aligned}
&\{z \in [0, 1+m] : P_{m,z} \cap (C \times C) = \emptyset\} = \\
&\overset{Obs.\ 3}{=} \bigcup_{j \in \mathbb{N}_0} \{z \in [0, 1+m] : P_{m,z} \cap (C \times C) = \emptyset \quad \text{and} \quad k(P_{m,z}) = j\} = \\
&\overset{Obs.\ 7}{=} \bigcup_{j=0}^{k-1} \{z \in [0, 1+m] : P_{m,z} \cap (C \times C) = \emptyset \quad \text{and} \quad k(P_{m,z}) = j\} = \\
&\overset{Obs.\ 4}{=} \bigcup_{j=0}^{k-1} \{z \in [0, 1+m] : P_{m,z} \\
&\qquad\qquad \text{lies in a horizontal crevice of the} (j+1)\text{th order}\} = \\
&\overset{Obs.\ 8}{=} \bigcup_{i=1}^{2^k - 1} \Big(r_k^{(i)} + m, \ l_k^{(i+1)}\Big).
\end{aligned}$$

Thus, for the complement of the set we have the equality

$$\{z \in [0, 1+m]; P_{m,z} \cap (C \times C) \neq \emptyset\} = \bigcup_{i=1}^{2^k} \Big[l_k^{(i)}, r_k^{(i)} + m\Big].$$

We have obtained the thesis of Theorem 1 (see Observation 1). $\square$

The choice of k described in Theorem 1 is not the only possible option. Actually, any iteration C_s, where $s \geq k$ can be used. The only difference is that in result increased intervals will not be disjoint.

Corollary 1.

$$C + mC = \bigsqcup_{i=1}^{2^k} \left[l_k^{(i)}, r_k^{(i)} + m \right] = \bigcup_{i=1}^{2^s} \left[l_s^{(i)}, r_s^{(i)} + m \right], \quad \text{for all} \quad m \in (0,1),$$

where k is such that $m \in [\frac{1}{3^{k+1}}, \frac{1}{3^k})$ and $\forall s \geq k$, where $k, s \in \mathbb{N}_0$.

Corollary 2. *If $m \in [\frac{1}{3^{k+1}}, \frac{1}{3^k})$, then the set $C + mC$ contains as many intervals as kth iteration, i.e., 2^k.*

Corollary 3.

$$C + mC = \bigcup_{x \in C} [x, x + m], \quad \text{for all} \quad m \in (0,1).$$

The theorem together with the proof presented here can be easily extended to linear combinations of the form $E_a + xE_a$ for $a \in [\frac{1}{3}, \frac{1}{2})$ and $x \in (0,1)$.

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Institute of Mathematics
University of Szczecin
ul. Wielkopolska 15
70-453 Szczecin
POLAND
E-mail: marta_pawlowicz@o2.pl