VERSITA

TATRA MOUNTAINS
Mathematical Publications
DOI: 10.2478/tmmp-2013-0003
Tatra Mt. Math. Publ. **54** (2013), 31–43

CONVERGENCE OF THE SOLUTIONS FOR A NEUTRAL DIFFERENCE EQUATION WITH NEGATIVE COEFFICIENTS

George E. Chatzarakis — George N. Miliaras

ABSTRACT. In this paper, we investigate the asymptotic behavior of the solutions of a neutral type difference equation of the form

$$\Delta\left[x(n)+\sum_{j=1}^{w}c_j x\big(\tau_j(n)\big)\right]+\big(-p(n)\big)x\big(\sigma(n)\big)=0, \qquad n\geq 0,$$

where $\tau_j(n)$, $j=1,\dots,w$ are general retarded arguments, $\sigma(n)$ is a general deviated argument, $c_j\in\mathbb{R}$, $j=1,\dots,w$, $\big(p(n)\big)_{n\geq 0}$ is a sequence of positive real numbers such that $p(n)\geq p$, $p\in\mathbb{R}_+$, and Δ denotes the forward difference operator $\Delta x(n)=x(n+1)-x(n)$.

1. Introduction

A neutral difference equation is a difference equation in which the higher order difference of the unknown sequence appears in the equation both with and without delays or advances. See, for example, [1], [4], [5], [12] and the references cited therein. We should note that, the theory of neutral difference equations presents complications, and results which are true for non-neutral difference equations may not be true for neutral equations [19].

The study of the asymptotic and oscillatory behavior of the solutions of neutral difference equations presents a strong theoretical interest. Aside from the mathematical interest, the study of those equations is motivated by their applications. Neutral difference equations arise in several areas of applied mathematics, including circuit theory, bifurcation analysis, population dynamics, stability theory, the dynamics of delayed network systems and others. Neutral difference equations are used in the analysis of computer networks containing lossless

2010 Mathematics Subject Classification: 39A13.
Keywords: neutral difference equations, retarded argument, deviated argument, oscillatory solutions, nonoscillatory solutions, bounded solutions, unbounded solutions.

transmission lines, as in high speed networks where lossless transmission lines serve to connect switching circuits in the network. Neutral difference equations also come up in the study of vibrating masses attached to an elastic bar, as for example, the Euler equation is used in some variational problems and in the theory of automatic control. As a result of the wide range of applications, neutral difference equations have attracted a great interest in the literature.

Consider the neutral difference equation in which the difference of the unknown sequence appears in the equation both with and without more than one delays

$$\Delta\left[x(n)+\sum_{j=1}^{w} c_j x\big(\tau_j(n)\big)\right]+\big(-p(n)\big)x\big(\sigma(n)\big)=0, \qquad n\ge 0, \tag{E}$$

where $\big(p(n)\big)_{n\ge 0}$ is a sequence of positive real numbers such that

$$p(n)\ge p, \quad p\in\mathbb{R}_+, \quad c_j\in\mathbb{R}, \quad j=1,\dots,w, \quad \big(\tau_j(n)\big)_{n\ge 0}, \quad j=1,\dots,w$$

are increasing sequences of integers that satisfy

$$\begin{gathered}\tau_j(n)\le n-1, \qquad j=1,\dots,w \quad \forall n\ge 0, \qquad \lim_{n\to\infty}\tau_j(n)=+\infty\\ \text{and}\\ \tau_\ell(n)<\tau_m(n+1), \qquad \forall \ell,m\in[1,w]\cap\mathbb{N}\end{gathered} \tag{1.1}$$

and $\big(\sigma(n)\big)_{n\ge 0}$ is an increasing sequence of integers such that

$$\begin{gathered}\sigma(n)\le n-1 \qquad \forall n\ge 0, \qquad \lim_{n\to\infty}\sigma(n)=+\infty,\\ \text{or}\\ \sigma(n)\ge n+1 \qquad \forall n\ge 0.\end{gathered} \tag{1.2}$$

Define

$$k_1=-\min_{\substack{n\ge 0\\ 1\le j\le w}}\tau_j(n), \quad k_2=-\min_{n\ge 0}\sigma(n) \quad \text{and} \quad k=\max\{k_1,k_2\}.$$

(Clearly, k is a positive integer.)

By a *solution* of the neutral difference equation (E) we mean a sequence of real numbers $\big(x(n)\big)_{n\ge -k}$ which satisfies (E) for all $n\ge 0$. It is clear that, for each choice of real numbers c_{-k}, c_{-k+1}, …, c_{-1}, c_0, there exists a unique solution $\big(x(n)\big)_{n\ge -k}$ of (E) which satisfies the initial conditions

$$x(-k)=c_{-k}, \quad x(-k+1)=c_{-k+1}, \ \dots, \ x(-1)=c_{-1}, \quad x(0)=c_0.$$

A solution $\big(x(n)\big)_{n\ge -k}$ of the neutral difference equation (E) is called *oscillatory* if the terms $x(n)$ of the sequence are neither eventually positive nor eventually negative. Otherwise, the solution is said to be *nonoscillatory*.

In the special case, where $\tau_j(n) = n - a_j$ and $\sigma(n) = n \pm b$, $a_j, b \in \mathbb{N}$, the equation (E) takes the form

$$\Delta\left[x(n) + \sum_{j=1}^{w} c_j x(n - a_j)\right] + \bigl(-p(n)\bigr)x(n \pm b) = 0, \qquad n \geq 0. \tag{E$_1$}$$

In the last few decades the asymptotic behavior of neutral difference equations has been extensively researched and developed. Hence, a large number of related papers have been published. See [2], [3], [6]–[11], [13]–[27], and the references cited therein. The objective in this paper is to investigate the convergence and divergence of the solutions of the equation (E) in the case of general delay arguments $\tau_j(n)$, $j = 1, 2, \ldots, w$ and a general deviated argument $\sigma(n)$, depending on real constants c_j, $j = 1, \ldots, w$.

2. Some preliminaries

Assume that $\bigl(x(n)\bigr)_{n\geq -k}$ is a nonoscillatory solution of (E). Then it is either eventually positive or eventually negative. As $\bigl(-x(n)\bigr)_{n\geq -k}$ is also a solution of (E), we can restrict ourselves only to the case where $x(n) > 0$ for all large n. Let $n_1 \geq -k$ be an integer such that $x(n) > 0$, $\forall n \geq n_1$. Then, there exists $n_0 \geq n_1$ such that

$$x\bigl(\sigma(n)\bigr) > 0, \quad x\bigl(\tau_j(n)\bigr) > 0, \quad j = 1, 2, \ldots, w, \quad \forall n \geq n_0.$$

Set

$$z(n) = x(n) + \sum_{j=1}^{w} c_j x\bigl(\tau_j(n)\bigr). \tag{2.1}$$

In view of (2.1), the equation (E) becomes

$$\Delta z(n) - p(n)x\bigl(\sigma(n)\bigr) = 0. \tag{2.2}$$

Taking into account that $p(n) \geq p > 0$, we have

$$\Delta z(n) = p(n)x\bigl(\sigma(n)\bigr) \geq px\bigl(\sigma(n)\bigr) > 0, \qquad \forall n \geq n_0,$$

which means that the sequence $\bigl(z(n)\bigr)$ is eventually strictly increasing, regardless of the values of the real constants c_j.

Let the domain of τ_j be the set $D(\tau_j) = \mathbb{N}_{n_j^*} = \bigl\{n_j^*, n_j^* + 1, n_j^* + 2, \ldots\bigr\}$, where n_j^* is the smallest natural number such that τ_j is defined with. Set

$$n_* = \max_{1\leq j\leq w} n_j^*.$$

Then τ_j, $j = 1, 2, \ldots, w$ is defined in the set $\mathbb{N}_{n_*} = \{n_*, n_* + 1, n_* + 2, \ldots\}$.

Let the subsequence

$$x\big(\tau_{\rho(n)}(n)\big) = \max\Big\{x\big(\tau_1(n)\big), x\big(\tau_2(n)\big), \dots, x\big(\tau_w(n)\big)\Big\}, \tag{2.3}$$

where $\rho(n)$ is a sequence that takes values in the set $\{1, 2, \dots, w\}$. Clearly, condition (1.1) guarantees that $\big(x(\tau_{\rho(n)}(n))\big)$ is a subsequence of $\big(x(n)\big)$.

Notice that

$$\tau_{j_1}\Big(\tau_{j_2}\big(\cdots\ \tau_{j_\ell}(n)\big)\Big) = \tau_{j_1}(n_s), \quad \text{where} \quad n_s = \tau_{j_2}\big(\cdots\ \tau_{j_\ell}(n)\big), \ \ 1 \le j_i \le w. \tag{2.4}$$

The following lemma provides us with some useful tools for establishing the main results.

Lemma 2.1. *Assume that* $\big(x(n)\big)_{n\ge -k}$ *is a positive solution of* (E). *Then the following statements hold:*

(i) *If*

$$\sum_{i=n_0}^{\infty} p(i)x\big(\sigma(i)\big) = S_0 < +\infty,$$

then

$$\lim_{n\to\infty} z(n) = A = \lim_{n\to\infty} \sum_{j=1}^{w} c_j x\Big(\tau_j\big(\sigma(n)\big)\Big), \qquad A \in \mathbb{R}. \tag{2.5}$$

(ii) *If*

$$\sum_{i=n_0}^{\infty} p(i)x\big(\sigma(i)\big) = +\infty,$$

then

$$z(n) > 0, \qquad \textit{eventually.} \tag{2.6}$$

Proof. Summing up (2.2) from n_0 to n, $n \ge n_0$, we obtain

$$z(n+1) = z(n_0) + \sum_{i=n_0}^{n} p(i)x\big(\sigma(i)\big). \tag{2.7}$$

For the above relation, exactly one of the following can be true:

$$\sum_{i=n_0}^{\infty} p(i)x\big(\sigma(i)\big) = S_0 < +\infty, \tag{2.7.a}$$

or

$$\sum_{i=n_0}^{\infty} p(i)x\big(\sigma(i)\big) = +\infty. \tag{2.7.b}$$

Assume that (2.7.a) holds. Since $p(n) \geq p > 0$, we have

$$+\infty > S_0 = \sum_{i=n_0}^{\infty} p(i)x\big(\sigma(i)\big) \geq p \sum_{i=n_0}^{\infty} x\big(\sigma(i)\big).$$

The last inequality guarantees that

$$\sum_{i=n_0}^{\infty} x\big(\sigma(i)\big) < +\infty$$

and consequently,

$$\lim_{n\to\infty} x\big(\sigma(n)\big) = 0. \tag{2.8}$$

Also, (2.7.a) guarantees that $\lim_{n\to\infty} z(n)$ exists as a real number. Set

$$\lim_{n\to\infty} z(n) = A \in \mathbb{R}.$$

Since $\big(z(\sigma(n))\big)$ is a subsequence of $\big(z(n)\big)$, we have

$$\lim_{n\to\infty} z\big(\sigma(n)\big) = A,$$

or

$$\lim_{n\to\infty} \left[x\big(\sigma(n)\big) + \sum_{j=1}^{w} c_j x\Big(\tau_j\big(\sigma(n)\big)\Big)\right] = A.$$

Using (2.8), we obtain

$$\lim_{n\to\infty} \sum_{j=1}^{w} c_j x\Big(\tau_j\big(\sigma(n)\big)\Big) = A.$$

Thus

$$\lim_{n\to\infty} z(n) = A = \lim_{n\to\infty} \sum_{j=1}^{w} c_j x\Big(\tau_j\big(\sigma(n)\big)\Big).$$

The proof of the part (i) of the lemma is complete.

Assume that (2.7.b) holds. Then, by taking limits on both sides of (2.7), we obtain

$$\lim_{n\to\infty} z(n) = +\infty$$

which in conjunction with that fact that the sequence $\big(z(n)\big)$ is eventually strictly increasing, means that

$$z(n) > 0, \qquad \text{eventually.}$$

The proof of the part (ii) of the lemma is complete.

The proof of Lemma 2.1 is complete. □

3. Main results

Throughout this section we are going to use the following notation

$$c = \sum_{j=1}^{w} c_j. \tag{3.1}$$

The asymptotic behavior of the solutions of the neutral difference equation (E) is described by the following theorem:

Theorem 3.1. *For every nonoscillatory solution $\big(x(n)\big)$ of the equation* (E) *the following statements hold:*

(I) *If the constants c_j are all nonpositive and $c < -1$, then $\big(x(n)\big)$ either has at least one real accumulation point which is zero or tends to infinity.*

(II) *If the constants c_j are all nonpositive and $c = -1$, then $\big(x(n)\big)$ either tends to zero or it is bounded with more than one real accumulation point besides zero or tends to infinity.*

(III) *If the constants c_j are all nonpositive and $-1 < c < 0$, then $\big(x(n)\big)$ either tends to zero or tends to infinity.*

(IV) *If the constants c_j are all equal to zero, then $\big(x(n)\big)$ tends to infinity.*

(V) *If the constants c_j are all nonnegative and $0 < c < 1$, then $\big(x(n)\big)$ is unbounded.*

(VI) *If the constants c_j are all nonnegative and $c \geq 1$, then $\big(x(n)\big)$ does not converge in $\mathbb{R}$.*

Proof. Assume that $\big(x(n)\big)_{n\geq -k}$ is a nonoscillatory solution of (E). Then it is either eventually positive or eventually negative. As $\big(-x(n)\big)_{n\geq -k}$ is also a solution of (E), we can restrict ourselves only to the case, where $x(n) > 0$ for all large n. We define the sequence $\big(z(n)\big)$ as in (2.1) and reformulate the equation (E) as in (2.2), in the preliminaries. From the preliminaries, we also have that since $p(n) \geq p > 0$, the sequence $\big(z(n)\big)$ is eventually strictly increasing, regardless of the values of the real constants c_j.

Assume that the constants c_j are all nonpositive and $c < -1$.

If (2.7.a) holds, then, in view of part (i) of Lemma 2.1, we have

$$\lim_{n\to\infty} z(n) = A = \lim_{n\to\infty} \sum_{j=1}^{w} c_j x\Big(\tau_j\big(\sigma(n)\big)\Big), \qquad A \in \mathbb{R},$$

which means, the sequence $\big(x(n)\big)$ has at least one accumulation point, which is zero, since (2.8) is satisfied.

If (2.7.b) holds, then, in view of (2.7), we have

$$\lim_{n\to\infty} z(n) = +\infty,$$

which guarantees that

$$\lim_{n\to\infty} x(n) = +\infty.$$

The proof of the part (I) of the theorem is complete.

Assume that the constants c_j are all nonpositive and $c = -1$.

If (2.7.a) holds, then, in view of part (i) of Lemma 2.1, we have

$$\lim_{n\to\infty} z(n) = A = \lim_{n\to\infty} \sum_{j=1}^{w} c_j x\Big(\tau_j\big(\sigma(n)\big)\Big), \qquad A \in \mathbb{R}$$

which guarantees that $A \leq 0$.

Since $\big(z(n)\big)$ is eventually strictly increasing, we have

$$z(n) = x(n) + \sum_{j=1}^{w} c_j x\big(\tau_j(n)\big) < A \leq 0.$$

Using (2.3), (2.4) and (3.1), the last inequality becomes

$$x(n) + \left(\sum_{j=1}^{w} c_j\right) x\big(\tau_{\rho_1(n)}(n)\big) < 0,$$

or

$$x(n) < x\big(\tau_{\rho_1(n)}(n)\big),$$

where

$$x\big(\tau_{\rho_1(n)}(n)\big) = \max_{1\leq j\leq w} \Big\{x\big(\tau_j(n)\big)\Big\}.$$

Thus

$$x(n) < x\big(\tau_{\rho_1(n)}(n)\big) < \cdots < x\big(\tau_{\rho_{m(n)}}(n_*)\big),$$

where $m(n)$ is a natural number which determines the number of steps we make in order to reach n_*. This means that the sequence $\big(x(n)\big)$ is bounded.

Let $A < 0$. Set

$$\limsup x(n) = M.$$

Then there exists a subsequence $\big(x(\theta(n))\big)$ of $\big(x(n)\big)$ such that

$$\lim_{n\to\infty} x\big(\theta(n)\big) = M.$$

Therefore

$$\lim_{n\to\infty}\left[x\big(\theta(n)\big)+\sum_{j=1}^{w}c_j x\Big(\tau_j\big(\theta(n)\big)\Big)\right]=A,$$

or

$$-\lim_{n\to\infty}\left[\sum_{j=1}^{w}c_j x\Big(\tau_j\big(\theta(n)\big)\Big)\right]=M-A,$$

or

$$\lim_{n\to\infty}\left[\sum_{j=1}^{w}(-c_j)\,x\Big(\tau_j\big(\theta(n)\big)\Big)\right]=M-A.$$

Consequently,

$$\limsup\left[\sum_{j=1}^{w}(-c_j)\,x\Big(\tau_j\big(\theta(n)\big)\Big)\right]=M-A,$$

or

$$\sum_{j=1}^{w}(-c_j)\limsup x\Big(\tau_j\big(\theta(n)\big)\Big)\ge M-A,$$

or

$$\sum_{j=1}^{w}(-c_j)\,M\ge M-A.$$

Hence

$$M\sum_{j=1}^{w}(-c_j)\ge M-A,$$

or

$$M\ge M-A,\qquad \text{since}\quad \sum_{j=1}^{w}(-c_j)=1$$

which contradicts to our assumption that $A<0$. Therefore

$$A=0,\quad i.e.,\quad \lim_{n\to\infty}z(n)=0.$$

This means that $\big(x(n)\big)$ has at least one real accumulation point which is zero.

If (2.7.b) holds, then, in view of (2.7), we have

$$\lim_{n\to\infty}z(n)=+\infty,$$

which guarantees that

$$\lim_{n\to\infty}x(n)=+\infty.$$

The proof of the part (II) of the theorem is complete.

Assume that the constants c_j are all nonpositive and $-1 < c < 0$.

If (2.7.a) holds, then, in view of part (i) of Lemma 2.1, we have

$$\lim_{n\to\infty} z(n) = A = \lim_{n\to\infty} \sum_{j=1}^{w} c_j x\Big(\tau_j\big(\sigma(n)\big)\Big), \qquad A \in \mathbb{R}$$

which guarantees that $A \le 0$.

Since $\big(z(n)\big)$ is eventually strictly increasing, we have

$$z(n) = x(n) + \sum_{j=1}^{w} c_j x\big(\tau_j(n)\big) < A \le 0.$$

Using (2.3), (2.4) and (3.1) the last inequality becomes

$$x(n) + \left(\sum_{j=1}^{w} c_j\right) x\big(\tau_{\rho_1(n)}(n)\big) < 0,$$

or

$$x(n) < -cx\big(\tau_{\rho_1(n)}(n)\big).$$

Thus

$$x(n) < -cx\big(\tau_{\rho_1(n)}(n)\big) < \cdots < (-c)^{m(n)} x\big(\tau_{\rho_{m(n)}}(n_*)\big) \to 0 \qquad as \quad n \to \infty$$

and consequently,

$$\lim_{n\to\infty} x(n) = 0.$$

If (2.7.b) holds, then, in view of (2.7), we have

$$\lim_{n\to\infty} z(n) = +\infty,$$

which guarantees that

$$\lim_{n\to\infty} x(n) = +\infty.$$

The proof of the part (III) of the theorem is complete.

Assume that the constants c_j are all nonnegative. Then $c \ge 0$. By (2.7) we have

$$z(n+1) = z(n_0) + \sum_{i=n_0}^{n} p(i) x\big(\sigma(i)\big) > 0.$$

Therefore

$$\lim_{n\to\infty} z(n) > 0.$$

Assume that the constants c_j are all equal to zero. Then $c = 0$, and consequently $z(n) = x(n)$. Using (2.7), we take

$$x(n+1) = x(n_0) + \sum_{i=n_0}^{n} p(i) x\big(\sigma(i)\big) > 0,$$

which guarantees that

$$\lim_{n\to\infty} x(n) > 0.$$

Thus $\big(x(\sigma(n))\big)$ cannot tend to zero, and therefore

$$\lim_{n\to\infty} x(n) = +\infty.$$

The proof of the part (IV) of the theorem is complete.

Assume that the constants c_j are all nonnegative and $0 < c < 1$.

If (2.7.a) holds, then, in view part (i) of Lemma 2.1, we have

$$\lim_{n\to\infty} z(n) = A = \lim_{n\to\infty} \sum_{j=1}^{w} c_j x\Big(\tau_j\big(\sigma(n)\big)\Big), \qquad A \in \mathbb{R}.$$

Clearly, $\big(z(n)\big)$ is bounded and therefore $\big(x(n)\big)$ is bounded. Set

$$\limsup x(n) = M.$$

Then there exists a subsequence $\big(x(\theta(n))\big)$ of $\big(x(n)\big)$ such that

$$\lim_{n\to\infty} x\big(\theta(n)\big) = M.$$

Therefore

$$\lim_{n\to\infty} \left[x\big(\theta(n)\big) + \sum_{j=1}^{w} c_j x\Big(\tau_j\big(\theta(n)\big)\Big) \right] = A,$$

or

$$\lim_{n\to\infty} \left[\sum_{j=1}^{w} c_j x\Big(\tau_j\big(\theta(n)\big)\Big) \right] = A - M \geq 0,$$

i.e.,

$$M \leq A. \tag{3.2}$$

On the other hand,

$$\lim_{n\to\infty} z\big(\sigma(n)\big) = A,$$

or

$$\lim_{n\to\infty} \left[x\big(\sigma(n)\big) + \sum_{j=1}^{w} c_j x\Big(\tau_j\big(\sigma(n)\big)\Big) \right] = A.$$

Using (2.8), the last relation becomes

$$\lim_{n\to\infty} \sum_{j=1}^{w} c_j x\Big(\tau_j\big(\sigma(n)\big)\Big) = A$$

and consequently

$$\limsup \left[\sum_{j=1}^{w} c_j x\Big(\tau_j\big(\sigma(n)\big)\Big)\right] = A.$$

Hence

$$\sum_{j=1}^{w} c_j \limsup x\Big(\tau_j\big(\sigma(n)\big)\Big) \geq A,$$

or

$$M \sum_{j=1}^{w} c_j \geq A,$$

or

$$M > cM \geq A$$

which contradicts (3.2). Therefore $\big(x(n)\big)$ is unbounded. Thus (2.7.a) is not satisfied, and therefore (2.7.b) holds. By (2.7), we have

$$\lim_{n\to\infty} z(n) = +\infty,$$

which guarantees that

$$\lim_{n\to\infty} x(n) = +\infty.$$

The proof of the part (V) of the theorem is complete.

Assume that the constants c_j are all nonnegative and $c \geq 1$.

If (2.7.a) holds, then

$$\lim_{n\to\infty} z(n) = A \geq 0, \qquad A \in \mathbb{R}.$$

Since $c > 0$, then, in view of part (IV) of theorem, we have

$$\lim_{n\to\infty} z(n) > 0,$$

which means that $A > 0$. Combined with the fact that $\lim_{n\to\infty} x\big(\sigma(n)\big) = 0$, we conclude that $\big(x(n)\big)$ has more than one real accumulation point. Thus $\big(x(n)\big)$ does not converge in $\mathbb{R}$.

If (2.7.b) holds, clearly $\lim_{n\to\infty} z(n) = +\infty$, which means that $\big(x(n)\big)$ is unbounded, and therefore $\big(x(n)\big)$ does not converge in $\mathbb{R}$.

The proof of the part (VI) of the theorem is complete.

The proof of Theorem 3.1 is complete. □

As a consequence of Theorem 3.1, we postulate the following corollary.

Corollary 3.1. *For every nonoscillatory solution* $\big(x(n)\big)$ *of the equation* (E_1) *the following statements hold:*

(i) *If the constants* c_j *are all nonpositive and* $c < 0$*, then* $\big(x(n)\big)$ *either tends to zero or tends to infinity.*

(ii) *If the constants* c_j *are all equal to zero, then* $\big(x(n)\big)$ *tends to infinity.*

(iii) *If the constants* c_j *are all nonnegative and* $c > 0$*, then* $\big(x(n)\big)$ *is unbounded.*

REFERENCES

[1] AGARWAL, R. P.—BOHNER, M.—GRACE, S. R.—O'REGAN, D.: *Discrete Oscillation Theory*. Hindawi Publishing Corporation, New York, 2005.

[2] BAŠTINEC, J.—DIBLÍK, J.—ŠMARDA, Z.: *Oscillation of solutions of a linear second--order discrete-delayed equation,* Adv. Difference Equ. Vol. 2010, Article ID 693867, 12 p., 2010.

[3] BAŠTINEC, J.—DIBLÍK, J.—ŠMARDA, Z.: *Existence of positive solutions of discrete linear equations with a single delay,* J. Difference Equ. Appl. **16** (2010), 1047–1056.

[4] BELLMAN, R.—COOKE, K. L.: *Differential-Difference Equations*. Academic Press, New York, 1963.

[5] BRAYTON, R. K.—WILLOUGHBY, R. A.: *On the numerical integration of a symmetric system of difference-differential equations of neutral type,* J. Math. Anal. Appl. **18** (1967), 182–189.

[6] BRUMLEY, W. E.: *On the asymptotic behavior of solutions of differential-difference equations of neutral type,* J. Difference Equ. **7** (1970), 175–188.

[7] CHATZARAKIS, G. E.—KARAKOSTAS, G. L.—STAVROULAKIS, I. P.: *Convergence of the positive solutions of a nonlinear neutral difference equation,* Nonlinear Oscill. **14** (2011), 407–418.

[8] CHATZARAKIS, G. E.—MILIARAS, G. N.: *Convergence and divergence of the solutions of a neutral difference equation,* J. Appl. Math., Vol. 2011, Article ID 262316, 18 p., 2011.

[9] DIBLÍK, J.—RUŽIČKOVÁ, M.—ŠMARDA, Z.—ŠUTÁ, Z.: *Asymptotic convergence of the solutions of a dynamic equation on discrete time scales,* Abstr. Appl. Anal., Vol. 2012, Article ID 580750, 20 p., 2012.

[10] GEORGIOU, D. A.—GROVE, E. A.—LADAS, G.: *Oscillation of neutral difference equations with variable coefficients,* in: Differential Equations, Stability and Control, Lecture Notes in Pure and Appl. Math., Vol. 127, Dekker, New York, 1991, pp. 165–173.

[11] GYŐRI, I.—HORVÁTH, L.: *Asymptotic constancy in linear difference equations: limit formulae and sharp conditions,* Adv. Difference Equ., Vol. 2010, Article ID 789302, 20 p., 2010.

[12] GYŐRI, I.—LADAS, G.: *Oscillation Theory of Delay Differential Equations with Applications*. Clarendon Press, Oxford, 1991.

[13] HUNG, D. C.: *Oscillation and convergence for a neutral difference equation,* J. Sci., Math.-Phys. **24** (2008), 133–143.

[14] JIA, J.—ZHONG, X.—GONG, X.—QUYANG, R.—HAN, H.: *Nonoscillation of first--order neutral difference equation,* Mod. Appl. Sci. **3** (2009), 30–33.

[15] LALLI, B. S.—ZHANG, B. G.: *On existence of positive solutions and bounded oscillations for neutral difference equations,* J. Math. Anal. Appl. **166** (1992), 272–287.

[16] LALLI, B. S.—ZHANG, B. G.—LI, J. Z.: *On the oscillation of solutions of neutral difference equations,* J. Math. Anal. Appl. **158** (1991), 213–233.

[17] MIGDA, M.—ZHANG, G.: *On unstable neutral difference equations with maxima,* Math. Slovaca **56** (2006), 451–463.

[18] PEICS, H.: *Positive solutions of neutral delay difference equation,* Novi Sad J. Math. **35** (2005), 111–122.

[19] SNOW, W.: *Existence, uniqueness and stability for nonlinear differential-difference equations in the neutral case, N. Y. U. Courant*, Inst. Math. Sci., IMM-NYU **328** (February 1965).

[20] TANG, X. H.: *Asymptotic behavior of solutions for neutral difference equations,* Comput. Math. Appl. **44** (2002), 301–315.

[21] TANG, X. H.—CHENG, S. S.: *Positive solutions of a neutral difference equation with positive and negative coefficients,* Georgian Math. J. **11** (2004), 177–185.

[22] THANDAPANI, E.—KUMAR, P. M.: *Oscillation and nonoscillation of nonlinear neutral delay difference equations,* Tamkang J. Math. **38** (2007), 323–333.

[23] THANDAPANI, E.—ARUL, R.—RAJA, P. S.: *The asymptotic behavior of nonoscillatory solutions of nonlinear neutral type difference equations,* Math. Comput. Modelling **39** (2004), 1457–1465.

[24] THANDAPANI, E.—MARIAN, S. L.—GRAEF, J. R.: *Asymptotic behavior of non oscillatory solutions of neutral difference equations,* Comput. Math. Appl. **45** (2003), 1461–1468.

[25] WANG, X.: *Asymptotic behavior of solutions for neutral difference equations,* Comput. Math. Appl. **52** (2006), 1595–1602.

[26] WEI, J.: *Asymptotic behavior results for nonlinear neutral delay difference equations,* Appl. Math. Comput. **217** (2011), 7184–7190.

[27] YU, J. S.—WANG, Z. C.: *Asymptotic behavior and oscillation in neutral delay difference equations,* Funkcial. Ekvac. **37** (1994), 241–248.

Received August 1, 2012

George E. Chatzarakis
Department of Electrical Engineering Educators
School of Pedagogical and
Technological Education (ASPETE)
14121, N. Heraklio
Athens
GREECE
E-mail: geaxatz@otenet.gr
gea.xatz@aspete.gr

George N. Miliaras
American University of Athens
Andrianiou 9, N. Psychico
11525, Athens
GREECE
E-mail: gmiliara@yahoo.gr
gmiliaras@aua.edu