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ASYMPTOTIC INTEGRATION OF SOME CLASSES OF FRACTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper we deal with the problem of asymptotic integration of nonlinear higher order fractional differential equations of the Caputo's type. We give some conditions under which all global solutions of these equations behave like linear functions as $t \to \infty$.

1. Introduction

In the asymptotic theory of nth order nonlinear ordinary differential equations

$$y^{(n)} = f\left(t, y, y', \dots, y^{(n-1)}\right) \tag{1}$$

the classical problem is to establish some conditions for the existence of a solution which approaches to a polynomial of degree $1 \le m \le n-1$ as $t \to \infty$. The first paper concerning this problem was published by D. Caligo [5] in 1941. He proved that if

$$|A(t)| < \frac{k}{t^{2+\rho}} \tag{2}$$

for all large t, where $k, \rho > 0$ are given, then any solution $y(t)$ of the linear differential equation

$$y''(t) + A(t)y(t) = 0, \qquad t > 0, \tag{3}$$

satisfying the initial conditions

$$y(1) = c_1, \quad y'(1) = c_2$$

can be represented asymptotically as $y(t) = c + dt + o(1)$ when $t \to +\infty, c, d \in \mathbb{R}$ (see [1]).

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This assertion can easily be proved as follows

$$y(t) = c_1 + c_2(t-1) - \int_1^t (t-s)A(s)y(s)\,\mathrm{d}s, \qquad t \ge 1$$

and therefore

$$\frac{|y(t)|}{t} \le C + k\int_1^t \frac{1}{s^{1+\rho}}\,\frac{|y(s)|}{s}\,\mathrm{d}s,$$

where $C = |c_1| + |c_2|$. From the Gronwall inequality we obtain

$$\frac{|y(t)|}{t} \le K := C\mathrm{e}^{k\int_1^\infty \frac{1}{s^{1+\rho}}\mathrm{d}s} < \infty, \qquad t \ge 1 \tag{4}$$

and

$$|y'(t)| \le K, \qquad t \ge 1. \tag{5}$$

From the equation (4) it follows that

$$\int_1^t |A(s)||y(s)|\mathrm{d}s \le K_1 := K\int_1^\infty |A(s)|s\mathrm{d}s < \infty.$$

This inequality yields the existence of the limit $d := \lim_{t\to\infty} y'(t)$ and using the l'Hôpital rule we obtain

$$\lim_{t\to\infty}\frac{y(t)}{t} = \lim_{t\to\infty} y'(t) = d.$$

Therefore there is a real number c such that $\lim_{t\to\infty}\big[y(t) - (c + dt)\big] = 0$.

The first paper on the nonlinear second order differential equations

$$y''(t) = f\big(t, y(t)\big) \tag{6}$$

was published by W. F. Trench [31] in 1963 and then by D. S. Cohen [7], T. Kusano and W. F. Trench [13] and [14], F. M. Dannan [10], A. Constantin [8] and [9], Y. V. Rogovchenko [27], S. P. Rogovchenko [28], O. G. Mustafa, Y. V. Rogovchenko [23], J. Tong [30], O. Lipovan [15] and others. In the proofs of their results the key role plays the Bihari inequality (see [4]) which is a generalization of the Gronwall inequality. Some results on the existence of solutions of the nth order differential equation approaching to a polynomial function of the degree m with $1 \le m \le n-1$ are proved by Ch. G. Philos, I. K. Purnaras and P. Ch. Tsamatos [25]. Their proofs are based on an application of the Schauder Fixed Point Theorem. The paper by R. P. Agarwal, S. D. Djebali, T. Moussaoui and O. G. Mustafa [1] surveys the literature concerning the topic in the asymptotic integration theory of ordinary differential equations. Several conditions under which all solutions of the one dimensional p-Laplacian equation

$$(|y'|^{p-1}y')' = f(t, y, y'), \qquad p > 1 \tag{7}$$

are asymptotic to $a + bt$ as $t \to \infty$ for some real numbers a, b are proved in [21] and some sufficient conditions for the existence of such solutions of the equation

$$\big(\Phi(y^{(n)})\big)' = f(t, y), \qquad n \geq 1, \tag{8}$$

where $\Phi\colon \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism with a locally Lipschitz inverse satisfying $\Phi(0) = 0$ are given in the paper [19].

The problem of asymptotic integration of fractional differential equations of the Riemann-Liouville type is studied in the papers [2], [3], where some conditions for the existence at least one solution of this type of equations approaching to a linear function as $t \to \infty$ are given. In the proofs of these results the fixed point method is applied.

We study asymptotic behavior of differential equations of the Caputo type of the fractional order r, where $n - 1 < r < n, n = [r] + 1, n \in N$. The aim of the paper is to give some more general conditions than in the paper [20] under which for any solution $x(t)$ of the equation there exists a real number c such that

$$x(t) = \frac{c}{(n-1)!t^{n-1}} + o(t^{n-1}) \qquad \text{for} \quad t \to \infty.$$

In the proofs of these results a desingularization method of nonlinear integral inequalities with weakly singular kernels developed in the papers [17], [18] is applied.

2. Preliminaries

In this section, we introduce basic notions, definitions and preliminary facts which are used throughout this paper. They can be found, e.g., in [12], [24] or [29].

Definition 2.1. For a function $x(t)$ of the class C^n on the interval $[a, \infty)$, $a \geq 0$ the Caputo derivative of the fractional order r of this function is defined as

$${}^{c}D_a^r x(t) = \frac{1}{\Gamma(n-r)} \int_a^t (t-s)^{n-r-1} x^{(n)}(s)\, \mathrm{d}s,$$

where $n = [r] + 1$.

Definition 2.2. The Riemann-Liouville integral, or fractional integral of the order q with $n - 1 < r < n$, of the function $h\colon [a, \infty) \to R$, a ≥ 0, is defined as

$$I_a^r h(t) = \frac{1}{\Gamma(r)} \int_a^t (t-s)^{r-1} h(s)\, \mathrm{d}s.$$

Lemma 2.3. *If $r > 0$, $n = [r] + 1$, then the differential equation*

$$^cD_a^r x(t) = 0$$

has a general solution

$$x(t) = c_0 + c_1(t-a) + \cdots + c_{n-1}(t-a)^{n-1},$$

where $c_0, c_1, \dots, c_{n-1} \in \mathbb{R}$ are arbitrary constants.

Lemma 2.4. *If $r > 0$, $n = [r] + 1$, then*

$$I_a^r\big(^cD_a^r x(t)\big) = x(t) - c_0 - c_1(t-a) - \cdots - c_{n-1}(t-a)^{n-1}.$$

for arbitrary $c_0, c_1, \dots, c_{n-1} \in \mathbb{R}$.

As a consequence of Lemma 2.4 we obtain

Lemma 2.5. *If $r > 0$, $n = [r] + 1$ and $g(t)$ is a continuous function on the interval $[a, \infty)$, $a \geq 0$, then the initial value problem*

$$^cD_a^r x(t) = g(t),$$

$$x(a) = c_0, x'(a) = c_1, \dots, x^{(n-1)}(a) = c_{n-1}$$

has the solution

$$x(t) = c_0 + c_1(t-a) + \cdots + \frac{c_{n-1}}{(n-1)!}(t-a)^{n-1} + \frac{1}{\Gamma(r)}\int_a^t (t-s)^{r-1} g(s)\,\mathrm{d}s.$$

3. Asymptotic behavior of fractional differential equations of the order $r \in (1, 2)$

In the paper [20] the fractional differential equation of the Caputo's type

$$^cD_a^{\alpha+1} x(t) = f\big(t, x(t)\big), \qquad a \geq 1, \quad \alpha \in (0,1) \tag{9}$$

is studied. A sufficient condition under which all solutions of this equation are asymptotic to $at + b$, $a, b \in \mathbb{R}$, is proved. The following theorem is proved there.

Theorem 3.1. *Suppose that $\alpha \in (0,1)$, $p > 1$, $p(\alpha - 1) + 1 > 0$, $a \geq 1$, $q = \frac{p}{p-1}$ and the function $f(t,u)$ satisfies the following conditions:*

(i) *$f(t,u)$ is continuous in $D = \big\{(t,u) \colon t \in [0,\infty),\ u \in \mathbb{R}\big\}$;*

(ii) *There are continuous nonnegative functions* $h\colon [0,\infty) \to \mathbb{R}_+$, $g\colon [0,\infty) \to \to \mathbb{R}_+$, *$g$ is nondecreasing and $\gamma > 0$ with $p(\gamma-1)+1>0$ such that*

$$|f(t,x)| \le t^{\gamma-1} h(t) g\left(\frac{|x|}{t}\right), \qquad t>0, \quad (t,x)\in D, \tag{10}$$

where $\gamma = 2-\alpha-\frac{1}{p}$, *i.e.,* $\Theta := p(\alpha+\gamma-2)+1 = 0$ *and*

$$\int_a^\infty h(s)^q \mathrm{d}s < \infty. \tag{11}$$

(iii)

$$\int_a^\infty \frac{\tau^{q-1}\mathrm{d}\tau}{g(\tau)^q} = \infty. \tag{12}$$

Then every solution $x(t)$ of the equation (9) *is asymptotic to $c+dt$ for $t\to\infty$, where $c,d\in\mathbb{R}$.*

We will prove the following generalization of Theorem 3.1 concerning the equation

$${}^cD_a^{\alpha+1}x(t) = f\bigl(t,x(t),x'(t)\bigr), \qquad a\ge 1, \quad \alpha\in(0,1). \tag{13}$$

Theorem 3.2. *Suppose that $\alpha\in(0,1)$, $p>1$, $p(\alpha-1)+1>0$, $a\ge 1$, $q=\frac{p}{p-1}$ and the function $f(t,u,v)$ satisfies the following conditions:*

(i) *$f(t,u,v)$ is continuous in $D=\bigl\{(t,u,v)\colon t\in[0,\infty),\ u,v\in\mathbb{R}\bigr\}$;*

(ii) *There are continuous nonnegative functions $h_i\colon [0,\infty)\to\mathbb{R}_+$, $i=1,2,3$ and continuous nonnegative and nondecreasing functions $g_j\colon \mathbb{R}_+\to\mathbb{R}_+$, $j=1,2$ and $\gamma>0$ with $p(\gamma-1)+1>0$ such that*

$$|f(t,u,v)| \le t^{\gamma-1}\left[h_1(t)g_1\left(\frac{|u|}{t}\right) + h_2(t)g_2(|v|) + h_3(t)\right], \qquad t>0, \quad (t,x)\in D, \tag{14}$$

where $\gamma = 2-\alpha-\frac{1}{p}$, *i.e.,* $\Theta := p(\alpha+\gamma-2)+1 = 0$.

$$\int_a^\infty h_i(s)^q \mathrm{d}s < \infty, \qquad i=1,2,3; \tag{15}$$

(iii)

$$\int_a^\infty \frac{\tau^{q-1}\mathrm{d}\tau}{g_1(\tau)^q + g_2(\tau)^q} = \infty. \tag{16}$$

Then every solution $x(t)$ of the equation (13) *is asymptotic to $c+dt$ for $t\to\infty$, where $c,d\in\mathbb{R}$.*

The proof of this result is based on a desingularization method proposed by the author in the paper [17] (see also [18]) in the study of nonlinear integral inequalities with weakly singular kernels and by the method used in the papers by [27], [28] and by the authors of the papers mentioned above. In the proof the following lemma (see [16], [26]) is used.

Lemma 3.3. *Let β, γ and p be positive constants such that*

$$p(\beta-1)+1>0, \qquad p(\gamma-1)+1>0.$$

Then

$$\int_0^t (t-s)^{p(\beta-1)} s^{p(\gamma-1)} \mathrm{d}s \le t^{\Theta} B, \qquad t \ge 0,$$

where

$$B := B\big[p(\gamma-1)+1, p(\beta-1)+1\big], \quad B[\xi,\eta] = \int_a^1 s^{\xi-1}(1-s)^{\eta-1}\,\mathrm{d}s \quad (\xi>0,\ \eta>0)$$

and

$$\Theta = p(\beta+\gamma-2)+1.$$

We use this lemma also in the next section.

Proof of Theorem 3.2. Let $x(t)$ be a solution of the equation (13) satisfying the initial conditions $x(a) = c_0$, $x'(a) = c_1$. Then

$$x(t) = c_0 + c_1(t-a) + \frac{1}{\Gamma(\alpha+1)} \int_a^t (t-s)(t-s)^{\alpha-1} f\big(s, x(s), x'(s)\big)\,\mathrm{d}s, \qquad t \ge a, \tag{17}$$

$$x'(t) = c_1 + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f\big(s, x(s), x'(s)\big)\,\mathrm{d}s, \qquad t \ge a. \tag{18}$$

From the condition (ii) it follows

$$\frac{|x(t)|}{t} \le z(t) := C + B_1 \int_a^t (t-s)^{\alpha-1} s^{\gamma-1} \left[h_1(s) g_1\left(\frac{|x(s)|}{s}\right) + h_2(s) g_2\big(|x'(s)|\big) + h_3(s) \right] \mathrm{d}s, \qquad t \ge a, \tag{19}$$

where

$$B_1 = \frac{1}{\alpha+1}, \quad C = |c_0| + |c_1| \quad \text{and} \quad |x'(t)| \le z(t), \qquad t \ge a. \tag{20}$$

Since the functions g_1, g_2 are nondecreasing the inequalities (19), (20) yield

$$z(t) \le C + B_1 \int_a^t (t-s)^{\alpha-1} s^{\gamma-1} \Big[h_1(s) g_1\big(z(s)\big) + \\ + h_2(s) g_2\big(z(s)\big) + h_3(s) \Big] \mathrm{d}s, \qquad t \ge a. \quad (21)$$

Using the Lemma 3.3 and Hölder inequality we obtain for $i = 1, 2$

$$\int_a^t (t-s)^{\alpha-1} s^{\gamma-1} h_i(s) g_i\big(z(s)\big) \mathrm{d}s \le B_1 B t^{\frac{\Theta}{p}} \left(\int_a^t h_i(s)^q g_i\big(z(s)\big)^q \mathrm{d}s \right)^{\frac{1}{q}},$$

where

$$B = B\big[p(\alpha-1)+1\big], \quad \big[p(\alpha-1)+1\big], \quad \Theta = p(\alpha+\gamma-2)+1 = 0.$$

From these inequalities it follows

$$z(t) \le C + B_1 B \left[\left(\int_a^t h_1(s)^q g_1\big(z(s)\big)^q \mathrm{d}s \right)^{\frac{1}{q}} + \\ + \left(\int_a^t h_2(s)^q g_2\big(z(s)\big)^q \mathrm{d}s \right)^{\frac{1}{q}} + \left(\int_a^t h_3(s)^q \mathrm{d}s \right)^{\frac{1}{q}} \right].$$

Now, apply the inequality $(a+b+c+d)^q \le 4^{q-1}(a^q+b^q+c^q+d^q)$ for arbitrary nonnegative numbers a, b, c, d we obtain

$$z(t)^q \le 4^{q-1} \left(C^q + (B_1 B)^q \left[\int_a^t h_1(s)^q g_1\big(z(s)\big)^q \mathrm{d}s + \\ + \int_a^t h_2(s)^q g_2\big(z(s)\big)^q \mathrm{d}s + \int_a^t h_3(s)^q \mathrm{d}s \right] \right).$$

If we denote $u(t) = z(t)^q$, then we have

$$u(t) \le A + D \int_a^t \big(h_1(s)^q + h_2(s)^q\big) \omega\big(u(s)\big) \mathrm{d}s,$$

where $A = 4^{q-1} C^q$,

$$D = 4^{q-1}[(B_1 B)^q] + \int_a^\infty h_3(s)^q \mathrm{d}s < \infty, \quad \omega(u) = g_1\left(u^{\frac{1}{q}}\right)^q + g_2\left(u^{\frac{1}{q}}\right)^q.$$

As a consequence of the Bihari inequality we have

$$u(t) \le K_0 := \Omega^{-1}\left[\Omega(A) + D\int_a^\infty \bigl(h_1(s)^q + h_2(s)^q\bigr)\mathrm{d}s\right] < \infty,$$

where

$$\Omega(v) = \int_{v_0}^{v} \frac{\mathrm{d}\sigma}{\omega(\sigma)}, \qquad v \ge v_0 > 0.$$

This yields

$$z(t) \le K_1 := K_0^{\frac{1}{q}}$$

and therefore

$$\frac{|x(t)|}{t} \le K_1, \qquad |x'(t)| \le K_1, \quad t \ge a. \tag{22}$$

From the condition (ii) and the above estimates it follows that

$$\int_a^t (t-s)^{\alpha-1}\bigl|f\bigl(s, x(s), x'(s)\bigr)\bigr|\,\mathrm{d}s \le K_0^{\frac{1}{q}}.$$

Therefore $d := \lim_{t\to\infty} x'(t)$ exists and by the l'Hôpital rule we obtain

$$\lim_{t\to\infty} \frac{|x(t)|}{t} = \lim_{t\to\infty} x'(t) = d$$

and thus there exists a real number c such that $\lim_{t\to\infty}\bigl(x(t) - (c+dt)\bigr) = 0$. □

If we study the asymptotic properties of solutions of the equation (13) with the initial condition $x(a) = 0$, we need not to assume $a \ge 1$. If $a = 0$ and a solution $x(t)$ satisfies the condition $\lim_{\tau\to 0} x(\tau) = 0$, then it is of the form

$$x(t) = ct + \frac{1}{\Gamma(r)} \int_0^t (t-s)(t-s)^{\alpha-1} f\bigl(s, x(s), x'(s)\bigr)\,\mathrm{d}s, \qquad t \ge 0, \tag{23}$$

and the proof of the inequalities (22) is the same as for the case $a > 1$. Therefore the following theorem holds.

Theorem 3.4. *Let all assumptions of Theorem 3.2 be fulfilled for $a = 0$. Then for any solution $x(t)$ of the equation (13) satisfying the initial condition $\lim_{\tau\to 0} x(\tau) = 0$ there exist numbers $c, d \in \mathbb{R}$ such that*

$$\lim_{t\to\infty}\bigl(x(t) - (c+dt)\bigr) = 0.$$

Example 1. Let $p > 1$, $r = 2 - \frac{1}{2p} = 1 + \alpha$, $\alpha = 1 - \frac{1}{2p}$, $\gamma = 2 - \alpha - \frac{1}{p}$, i.e., $\Theta = p(\alpha + \gamma - 2) + 1 = 0$, $q = \frac{p}{p-1}$, i.e., $\frac{1}{p} + \frac{1}{q} = 1$, $g_1(u) = g_2(u) = u^{\frac{q-1}{q}} \ln(2+u)^{\frac{1}{q}}$, $u \ge 0$, $f(t,u,v) = t^{\gamma-1}\bigl[h_1(t) g_1(\frac{u}{t}) + h_2(t) g_2(|v|) + h_3(t)\bigr]$, where $h_i\colon \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, 2, 3$ are continuous functions with $\int_a^\infty h_i(s)^q \mathrm{d}s < \infty$, $i = 1, 2, 3$. Obviously, $0 < \alpha < 1$, $p(\alpha - 1) + 1 = \frac{1}{2}$, $p(\gamma - 1) + 1 = \frac{1}{2}$ and

$$\int_a^\infty \frac{\tau^{q-1}}{g_1(\tau)^q + g_2(\tau)^q} \mathrm{d}\tau = \frac{1}{2} \int_a^\infty \frac{1}{\ln(2+\tau)} \mathrm{d}\tau = \infty.$$

We have proved that all conditions of Theorem 3.2 are satisfied and by this theorem any solution of the equation (13) is asymptotic to a linear function $c + dt$ as $t \to \infty$.

EXAMPLE 2. Let us consider a fractional analogue of the linear differential equation (3) studied by D. Caligo [5], which we have analysed in Introduction.

$$^cD^{2-\frac{1}{2p}} x(t) + A(t)x(t) = 0, \qquad p > 1,$$

$r = 2 - \alpha - \frac{1}{p}$, $\alpha = 1 - \frac{1}{2p}$.

$$|A(t)| \le kt^{2-\alpha-\frac{1}{p}} \frac{1}{t^{\frac{2+\rho}{q}}} = t^{r-1} h_1(t), \qquad \rho > 0, \quad k > 0,$$

$h_1(t) = k \frac{t^{\frac{1}{q}-1}}{t^{\frac{2+\rho}{q}}}$.

Obviously, for $f(t, u) = -A(t)u$ we have

$$|f(t, x)| = |A(t)|\, t \frac{|x|}{t} \le t^{r-1} h(t) \frac{|x|}{t},$$

$h(t) = th_1(t) = k \frac{t^{\frac{1}{q}}}{t^{\frac{2+\rho}{q}}}$,

$$\int_a^\infty h(s)^q \mathrm{d}s = \int_a^\infty \frac{1}{s^{1+\rho}} \,\mathrm{d}s < \infty.$$

All assumptions of Theorem 3.1 are satisfied and therefore any solution of the equation (9) is asymptotic to a linear function $c + dt$ as $t \to \infty$.

4. Asymptotic integration of higher order fractionally differential equations

Consider the fractional differential equation

$$^cD_a^r x(t) = f\big(t, x(t)\big), \qquad t > 0, \quad 0 < \alpha < 1, \quad a \ge 1, \tag{24}$$

where $r = \alpha + n - 1$, $\alpha \in (0, 1)$, $n \ge 1$ is a natural number, $a \ge 1$ with the initial value conditions

$$x(a) = c_0, \quad x'(a) = c_1, \quad x^{(n-1)}(a) = c_{n-1}. \tag{25}$$

Definition 4.1. A function $u\colon [0,T) \to \mathbb{R}$, $0 < T \le \infty$, is called a solution of the equation (24) if $u \in C^n$ on the interval $(0,T)$, $\lim_{\tau\to 0} u(\tau)$ exists and $u(t)$ satisfies the equation (24) on the interval $(0,T)$. This solution is called global if it exists for all $t \in [0,\infty)$.

We assume the following hypotheses:

(H1) Every solution of the equation (24) is global;

(H2) The function $f(t,u)$ is continuous in $D = \{(t,u) : t \in [a,\infty), u \in \mathbb{R}\}$;

(H3) There exist continuous nonnegative functions $k_1, k_2\colon [a,\infty) \to \mathbb{R}$, a continuous positive nondecreasing function $g\colon [0,\infty) \to \mathbb{R}$ and numbers $q > 1$, $\gamma > 0$ such that

$$k_i = \int_1^\infty k_i^q(s)\,\mathrm{d}s < \infty, \qquad i = 1,2,$$

$$|f(t,u)| \le t^{\gamma-1}\left[k_1(t)g\left(\frac{|u|}{t^{n-1}}\right) + k_2(t)\right], \qquad t \ge 1;$$

(H4)
$$p(\gamma-1)+1>0, \quad p(\alpha-1)+1>0, \quad \gamma = 2-\alpha-\frac{1}{p},$$
$$\text{i.e., } \Theta = p(\alpha+\gamma-2)+1 = 0, \quad \text{where } p = \frac{q}{1-q};$$

(H5)
$$\int_0^\infty \frac{\tau^{q-1}\,\mathrm{d}\tau}{g(\tau)^q} = \infty.$$

Theorem 4.2. *Let $r = \alpha + n - 1$, where $\alpha \in (0,1)$, n is a natural number and let the conditions* (H1)–(H5) *be satisfied. Then for any solution $x(t)$ of the initial value problem* (24), (25), *defined on the interval* $[0,\infty,)$ *there is a number $c \in \mathbb{R}$ such that*

$$x(t) = \frac{c}{(n-1)!}t^{n-1} + o(t^{n-1}) \qquad \text{as} \quad t \to \infty. \tag{26}$$

P r o o f. Let $x(t)$ be a solution of the initial value problem (24), (25). From Lemma 2.4 it follows that it is a continuous solution of the integral equation

$$x(t) = c_0 + c_1(t-a) + \dots + \frac{c_{n-1}}{(n-1)!}(t-a)^{n-1} + \frac{1}{\Gamma(r)}\int_a^t (t-s)^{r-1} f\big(s,x(s)\big)\,\mathrm{d}s. \tag{27}$$

Using the condition (H3) we obtain

$$|x(t)| \le \left(|c_1| + \frac{|c_2|}{2!} + \cdots + \frac{|c_{n-1}|}{(n-1)!}\right) t^{n-1} + \frac{1}{\Gamma(r)} t^{n-1} \int_a^t (t-s)^{\alpha-1} s^{\gamma-1} \left[k_1(s) g\left(\frac{|x(s)|}{s^{n-1}}\right) + k_2(s)\right] \mathrm{d}s. \tag{28}$$

Applying the Hölder inequality and Lemma 3.3 we obtain the estimates

$$\int_a^t (t-s)^{\alpha-1} s^{\gamma-1} k_1(s) g\left(\frac{|x(s)|}{s^{n-1}}\right) \mathrm{d}s \le B^{\frac{1}{p}} \left(\int_a^t k_1(s)^q g\left(\frac{|x(s)|}{s^{n-1}}\right)^q \mathrm{d}s\right)^{\frac{1}{q}},$$

$$\int_a^t (t-s)^{\alpha-1} s^{\gamma-1} k_2(s)\, \mathrm{d}s \le B^{\frac{1}{p}} \left(\int_0^t k_2(s)^q \,\mathrm{d}s\right)^{\frac{1}{q}}$$

and thus the inequality (28) yields

$$u(t) \le M + N \left(\int_a^t k_1(s)^q g\left(\frac{|u(s)|}{s^{n-1}}\right)^q \mathrm{d}s\right)^{\frac{1}{q}}, \tag{29}$$

where $u(t) = \dfrac{|x(t)|}{t^{n-1}}$,

$$M = |c_1| + \frac{|c_2|}{2!} + \cdots + \frac{|c_{n-1}|}{(n-1)!} + \frac{1}{\Gamma(r)} B^{\frac{1}{p}} \left(\int_0^t k_2(s)^q \, ds\right)^{\frac{1}{q}} < \infty, \quad N = \frac{1}{\Gamma(r)} B^{\frac{1}{p}}.$$

Using the inequality $(a+b)^q \le 2^{q-1}(a^q + b^q)$ for any $a \ge 0$, $b \ge 0$, we obtain the integral inequality

$$u(t)^q \le P + Q \int_a^t k_2(s)^q g\big(u(s)\big)^q \,\mathrm{d}s,$$

where $P = 2^{q-1} M^q, Q = 2^{q-1} N^q$.

If we denote $z(t) = u(t)^q$, then we can rewrite this inequality into the form

$$z(t) \le P + Q \int_a^t k_2(s)^q g\left(z(s)^{\frac{1}{q}}\right)^q \mathrm{d}s.$$

From the Bihari inequality we obtain

$$z(t) = u(t)^q \le K := G^{-1}\left(G(P) + Q\int_a^\infty k_1(s)^q \mathrm{d}s\right) < \infty.$$

Since $u(t) = \frac{|x(t)|}{t^{n-1}}$ we have the inequality

$$\frac{|x(t)|}{t^{n-1}} \le K_1 := K^{\frac{1}{q}}, \qquad t \ge a. \tag{30}$$

This inequality and the condition (H3) yields

$$\left|f\big(t, x(t)\big)\right| \le t^{\gamma-1}\left[k_1(t)\sup_{0\le v\le N} g(v) + k_2(t)\right], \qquad t \ge a. \tag{31}$$

Now, by using this inequality, we derive

$$\int_0^t (t-s)^{\alpha-1}\left|f\big(s, x(s)\big)\right|\mathrm{d}s \le L$$
$$:= B^{\frac{1}{q}}\left(\sup_{0\le v\le N} g(v)\int_a^\infty k_1(s)^q \mathrm{d}s + \int_a^\infty k_2(s)^q \mathrm{d}s\right) < \infty.$$

Therefore the integral $\int_0^t (t-s)^{\alpha-1}\left|f\big(s, x(s)\big)\right|\mathrm{d}s$ exists. This yields the existence of the limit

$$c = \lim_{t\to\infty} x^{(n-1)}(t) = c_{n-1} + \frac{1}{\Gamma(\alpha)}\lim_{t\to\infty}\int_0^t (t-s)^{\alpha-1}\left|f\big(s, x(s)\big)\right|\mathrm{d}s < \infty.$$

Therefore using the l'Hôpital rule we obtain

$$\lim_{t\to\infty}\frac{x(t)}{t^{n-1}} = \frac{1}{(n-1)!}\lim_{t\to\infty} x^{(n-1)}(t) = \frac{c}{(n-1)!}$$

and thus the solution $x(t)$ satisfies (26). □

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