ON POINTWISE $\mathcal{M}$ -CONTINUITY OF MAPPINGS

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ABSTRACT. Classical Levine's theorem [N. Levine: *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly **70** (1963), 36–41] asserts that for a semi-continuous mapping on a second countable topological space, the discontinuity points form a 1st category set. There are two directions in literature in which this result is generalized: by considering either multi-valued mappings or mappings on some second noncountable spaces (for the latter, see for instance [T. Neubrunn: *Quasi-continuity (topical survey)*, Real Anal. Exchange **14** (1988/89), 259–306]). In this paper, we offer another path, namely, the path of so-called $\mathcal{M}$ -spaces, essentially weaker than the topological ones. Pointwise $\mathcal{M}$ -continuity of a mapping between two $\mathcal{M}$ -spaces is defined and characterized. These characterizations are the basic tool for our generalization.

1. Preliminaries

A subfamily $\mathbf{m}_X$ of the power set 2^X of a nonempty set X is called a *minimal structure* on X [3], [4], [6], if $\emptyset \in \mathbf{m}_X$ and $X \in \mathbf{m}_X$. A pair $(X, \mathbf{m}_X)$ is then called an $\mathcal{M}$ -space. For each subset S of the space X , the $\mathbf{m}_X$ -closure and the $\mathbf{m}_X$ -interior are defined, as in [3], in a usual manner:

1. $\mathbf{m}_X\text{-cl}(S) = \bigcap \{F; S \subset F, X \setminus F \in \mathbf{m}_X\},$
2. $\mathbf{m}_X\text{-int}(S) = \bigcup \{U; S \supset U, U \in \mathbf{m}_X\}.$

A minimal structure $\mathbf{m}_X$ being closed under any unions, is called a *supratopology* while the couple $(X, \mathbf{m}_X)$ is called a *supratopological space* [1]. For an $\mathcal{M}$ -space $(X, \mathbf{m}_X)$, we write $c(\mathbf{m}_X) = \{F : X \setminus F \in \mathbf{m}_X\}$. It is obvious that each topological space is supratopological and each supratopological space is an $\mathcal{M}$ -space. The following two lemmas are necessary for the rest of this paper.

LEMMA 1 ([3]). *Let A and B be any subsets of an $\mathcal{M}$ -space $(X, \mathbf{m}_X)$. The following properties hold:*

1. (a) $\mathfrak{m}_X\text{-cl}(X \setminus A) = X \setminus \mathfrak{m}_X\text{-int}(A)$,
 (b) $\mathfrak{m}_X\text{-int}(X \setminus A) = X \setminus \mathfrak{m}_X\text{-cl}(A)$,
2. (a) if $X \setminus A \in \mathfrak{m}_X$, then $\mathfrak{m}_X\text{-cl}(A) = A$,
 (b) if $A \in \mathfrak{m}_X$ then $\mathfrak{m}_X\text{-int}(A) = A$,
3. (a) $\mathfrak{m}_X\text{-cl}(\emptyset) = \emptyset$, $\mathfrak{m}_X\text{-cl}(X) = X$,
 (b) $\mathfrak{m}_X\text{-int}(\emptyset) = \emptyset$, $\mathfrak{m}_X\text{-int}(X) = X$,
4. (a) if $A \subset B$, then $\mathfrak{m}_X\text{-cl}(A) \subset \mathfrak{m}_X\text{-cl}(B)$,
 (b) if $A \subset B$, then $\mathfrak{m}_X\text{-int}(A) \subset \mathfrak{m}_X\text{-int}(B)$,
5. (a) $A \subset \mathfrak{m}_X\text{-cl}(A)$,
 (b) $A \supset \mathfrak{m}_X\text{-int}(A)$,
6. (a) $\mathfrak{m}_X\text{-cl}(\mathfrak{m}_X\text{-cl}(A)) = \mathfrak{m}_X\text{-cl}(A)$,
 (b) $\mathfrak{m}_X\text{-int}(\mathfrak{m}_X\text{-int}(A)) = \mathfrak{m}_X\text{-int}(A)$.

LEMMA 2 ([6]). Consider an $\mathcal{M}$ -space $(X, \mathfrak{m}_X)$. Let $A \subset X$. Then, $x \in \mathfrak{m}_X\text{-cl}(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in \mathfrak{m}_X$ containing x .

2. $\mathcal{M}$ -continuity at a point

DEFINITION 1. Let $(X, \mathfrak{m}_X)$ and $(Y, \mathfrak{m}_Y)$ be two $\mathcal{M}$ -spaces. A mapping $f : (X, \mathfrak{m}_X) \rightarrow (Y, \mathfrak{m}_Y)$ is said to be $\mathcal{M}$ -continuous at $x \in X$ if, for every set $V \in \mathfrak{m}_Y$ with $f(x) \in V$, there exists $U \in \mathfrak{m}_X$ such that $x \in U$ and $U \subset f^{-1}(V)$ (equiv. $f(U) \subset V$). If x is not an $\mathcal{M}$ -continuity point of f , then it is an $\mathcal{M}$ -discontinuity point of f .

THEOREM 1. Let $(X, \mathfrak{m}_X)$, $(Y, \mathfrak{m}_Y)$ be two supratopological spaces and let $x_0 \in X$. For any mapping $f : (X, \mathfrak{m}_X) \rightarrow (Y, \mathfrak{m}_Y)$, the following statements are equivalent:

- (a) f is $\mathcal{M}$ -continuous at x_0 ;
- (b) if $x_0 \in f^{-1}(V)$, then $x_0 \in \mathfrak{m}_X\text{-int}(f^{-1}(V))$ for every $V \in \mathfrak{m}_Y$;
- (c) if $x_0 \in \mathfrak{m}_X\text{-cl}(S)$, then $f(x_0) \in \mathfrak{m}_Y\text{-cl}(f(S))$ for every $S \subset X$;
- (d) if $x_0 \in \mathfrak{m}_X\text{-cl}(f^{-1}(F))$, then $x_0 \in f^{-1}(F)$ for every $F \in c(\mathfrak{m}_Y)$.

For the equivalence (b) $\Leftrightarrow$ (d), it is enough to assume that $(X, \mathfrak{m}_X)$ and $(Y, \mathfrak{m}_Y)$ are $\mathcal{M}$ -spaces.

Proof. (a) $\Rightarrow$ (b). Let $V \in \mathfrak{m}_Y$ be an arbitrarily chosen set such that $x_0 \in f^{-1}(V)$. Then, $f(x_0) \in V$, and by hypothesis, there exists $U \in \mathfrak{m}_X$ such that $x_0 \in U \subset f^{-1}(V)$. Thus $x_0 \in \mathfrak{m}_X\text{-int}(f^{-1}(V))$.

(b) $\Rightarrow$ (c). Suppose there exists a set $S \subset X$ such that $x_0 \in \mathfrak{m}_X\text{-cl}(S)$ and $f(x_0) \notin \mathfrak{m}_Y\text{-cl}(f(S))$. Consider the set $V = Y \setminus \mathfrak{m}_Y\text{-cl}(f(S)) = \mathfrak{m}_Y\text{-int}(Y \setminus f(S))$.

Since $\mathbf{m}_Y$ is a supratopology on Y , $V \in \mathbf{m}_Y$ (as it is a union of some family of sets from $\mathbf{m}_Y$ containing $Y \setminus f(S)$). Moreover, $f(x_0) \in V$. Hence, $x_0 \in f^{-1}(V)$, and by hypothesis, $x_0 \in \mathbf{m}_X\text{-int}(f^{-1}(V))$. Put $U = \mathbf{m}_X\text{-int}(f^{-1}(V))$. Since $\mathbf{m}_X$ is a supratopology $U \in \mathbf{m}_X$, we have what follows:

$$\begin{aligned} U \cap S &= \mathbf{m}_X\text{-int}\left(f^{-1}\left(Y \setminus \mathbf{m}_Y\text{-cl}(f(S))\right)\right) \cap S \subset f^{-1}(Y \setminus f(S)) \cap S \\ &= \left(X \setminus f^{-1}(f(S))\right) \cap S = \emptyset. \end{aligned}$$

Thus, by Lemma 2, $x_0 \notin \mathbf{m}_X\text{-cl}(S)$. It is in contrary to the assumption.

(c) $\Rightarrow$ (a). Suppose f is not $\mathcal{M}$ -continuous at x_0 , that is, there exists a set $V \in \mathbf{m}_Y$ with $f(x_0) \in V$ such that, for any $U \in \mathbf{m}_X$ with $x_0 \in U$, one has $U \not\subset f^{-1}(V)$. Set $S = X \setminus f^{-1}(V)$. We shall show that $x_0 \in \mathbf{m}_X\text{-cl}(S)$ and $f(x_0) \notin \mathbf{m}_Y\text{-cl}(f(S))$. Namely, observe that $U \cap S = U \cap (X \setminus f^{-1}(V)) \neq \emptyset$ and by Lemma 2, $x_0 \in \mathbf{m}_X\text{-cl}(S)$. On the other hand, $V \cap f(S) = f(f^{-1}(V) \cap S) = \emptyset$. So, by Lemma 2 once again, we obtain $f(x_0) \notin \mathbf{m}_Y\text{-cl}(f(S))$.

(b) $\Rightarrow$ (d). Let $F \in \mathbf{c}(\mathbf{m}_X)$ be such a set that $x_0 \in \mathbf{m}_X\text{-cl}(f^{-1}(F))$ but $x_0 \notin f^{-1}(F)$. We have $x_0 \notin X \setminus \mathbf{m}_X\text{-cl}(f^{-1}(F)) = \mathbf{m}_X\text{-int}(X \setminus f^{-1}(F)) = \mathbf{m}_X\text{-int}(f^{-1}(V))$, where $V = Y \setminus F \in \mathbf{m}_Y$. Since $x_0 \in f^{-1}(V)$, we get (b) does not hold.

(d) $\Rightarrow$ (b). Suppose there exists $V \in \mathbf{m}_Y$ such that $x_0 \notin \mathbf{m}_X\text{-int}(f^{-1}(V))$ and $x_0 \in f^{-1}(V)$. Then $x_0 \in X \setminus \mathbf{m}_X\text{-int}(f^{-1}(V)) = \mathbf{m}_X\text{-cl}(X \setminus f^{-1}(V)) = \mathbf{m}_X\text{-cl}(f^{-1}(F))$, where $F = Y \setminus V \in \mathbf{c}(\mathbf{m}_Y)$. Observe that $f^{-1}(F) \cap f^{-1}(V) = \emptyset$, and since $x_0 \in f^{-1}(V)$, one gets $x_0 \notin f^{-1}(F)$. Therefore the condition (d) does not hold. This contradiction completes the proof. $\square$

The following corollary clearly follows from the proof of Theorem 1.

THEOREM 2. *Let $(X, \mathbf{m}_X)$ and $(Y, \mathbf{m}_Y)$ be two $\mathcal{M}$ -spaces. If a mapping $f : (X, \mathbf{m}_X) \rightarrow (Y, \mathbf{m}_Y)$ is $\mathcal{M}$ -continuous at a point $x_0 \in X$, then the following hold:*

1. *if $x_0 \in f^{-1}(V)$, then $x_0 \in \mathbf{m}_X\text{-int}(f^{-1}(V))$ for every $V \in \mathbf{m}_Y$,*
2. *if $x_0 \in \mathbf{m}_X\text{-cl}(f^{-1}(F))$, then $x_0 \in f^{-1}(F)$ for every $F \in \mathbf{c}(\mathbf{m}_Y)$.*

Proof. It is enough to analyse the proof of implication (a) $\Rightarrow$ (b) and to use the fact that (b) $\Leftrightarrow$ (d) holds for $\mathcal{M}$ -spaces. $\square$

DEFINITION 2. A subfamily $\beta_{\mathbf{m}_X}$ of the minimal structure $\mathbf{m}_X$ on X is said to be an $\mathbf{m}$ -base for $\mathbf{m}_X$ if for every set $S \in \mathbf{m}_X$ there exists $\beta_S \subset \beta_{\mathbf{m}_X}$ with $S = \bigcup \beta_S$. If $\beta_{\mathbf{m}_X}$ can be found at most countable, then the $\mathcal{M}$ -space $(X, \mathbf{m}_X)$ is called $\mathbf{m}$ -2nd countable.

LEMMA 3. *Let $(X, \mathfrak{m}_X)$, $(Y, \mathfrak{m}_Y)$ be two $\mathcal{M}$ -spaces, $x_0 \in X$, $f: (X, \mathfrak{m}_X) \rightarrow (Y, \mathfrak{m}_Y)$. If $\beta_{\mathfrak{m}_Y}$ is an $\mathfrak{m}$ -base for $\mathfrak{m}_Y$, then the condition (b) of Theorem 1 is equivalent to the following one:*

(b') *if $x_0 \in f^{-1}(V)$, then $x_0 \in \mathfrak{m}_X\text{-int}(f^{-1}(V))$ for every $V \in \beta_{\mathfrak{m}_Y}$.*

Proof. (b) $\Rightarrow$ (b') is obvious. (b') $\Rightarrow$ (b). Let $x_0 \in f^{-1}(V)$, where V is any member of $\mathfrak{m}_Y$. There is a subfamily $\beta_V \subset \beta_{\mathfrak{m}_Y}$ such that $V = \bigcup \beta_V$. So, $x_0 \in f^{-1}(\bigcup \beta_V) = \bigcup_{U \in \beta_V} f^{-1}(U)$ and thus $x_0 \in f^{-1}(U)$ for some $U \in \beta_V$. By hypothesis and by 4.(b) of Lemma 1, $x_0 \in \mathfrak{m}_X\text{-int}(f^{-1}(U)) \subset \mathfrak{m}_X\text{-int}(f^{-1}(V))$. $\square$

A mapping $f: (X, \mathfrak{m}_X) \rightarrow (Y, \mathfrak{m}_Y)$, where $(X, \mathfrak{m}_X)$ and $(Y, \mathfrak{m}_Y)$ are two $\mathcal{M}$ -spaces, is $\mathcal{M}$ -continuous on $(X, \mathfrak{m}_X)$ [6, Definition 3.3] if it is $\mathcal{M}$ -continuous at each point $x \in X$. The set of all $\mathcal{M}$ -discontinuity points of f will be denoted by $\mathcal{D}_f$. The following result is a consequence of conditions (b), (d) (resp.) of Theorem 1, and condition (b') of Lemma 3.

THEOREM 3. *Let $(X, \mathfrak{m}_X)$, $(Y, \mathfrak{m}_Y)$ be two supratopological spaces, $f: (X, \mathfrak{m}_X) \rightarrow (Y, \mathfrak{m}_Y)$. We have*

1. $\mathcal{D}_f = \bigcup_{V \in \mathfrak{m}_Y} (f^{-1}(V) \setminus \mathfrak{m}_X\text{-int}(f^{-1}(V)))$,
2. $\mathcal{D}_f = \bigcup_{F \in \mathfrak{c}(\mathfrak{m}_Y)} (\mathfrak{m}_X\text{-cl}(f^{-1}(F)) \setminus f^{-1}(F))$,
3. $\mathcal{D}_f = \bigcup_{V \in \beta_{\mathfrak{m}_Y}} (f^{-1}(V) \setminus \mathfrak{m}_X\text{-int}(f^{-1}(V)))$, where $\beta_{\mathfrak{m}_Y}$ is any $\mathfrak{m}$ -base for $\mathfrak{m}_Y$.

3. $\mathcal{M}$ -semi-open sets, $\mathcal{M}$ -semi-continuity and $\mathcal{M}$ -discontinuity points

We start this section with some properties for subsets of any $\mathcal{M}$ -space.

LEMMA 4. *Consider an $\mathcal{M}$ -space $(X, \mathfrak{m}_X)$. Let $\{S_i\}_{i \in \mathcal{J}} \subset 2^X$. Then,*

1. $\mathfrak{m}_X\text{-int}(\bigcap_{i \in \mathcal{J}} S_i) \subset \bigcap_{i \in \mathcal{J}} \mathfrak{m}_X\text{-int}(S_i)$,
2. $\mathfrak{m}_X\text{-cl}(\bigcap_{i \in \mathcal{J}} S_i) \subset \bigcap_{i \in \mathcal{J}} \mathfrak{m}_X\text{-cl}(S_i)$,
3. $\mathfrak{m}_X\text{-int}(\bigcup_{i \in \mathcal{J}} S_i) \supset \bigcup_{i \in \mathcal{J}} \mathfrak{m}_X\text{-int}(S_i)$,
4. $\mathfrak{m}_X\text{-cl}(\bigcup_{i \in \mathcal{J}} S_i) \supset \bigcup_{i \in \mathcal{J}} \mathfrak{m}_X\text{-cl}(S_i)$.

Proof. 1. Let $x \in \mathfrak{m}_X\text{-int}(\bigcap_{i \in \mathcal{J}} S_i)$. Then, there is a $U \in \mathfrak{m}_X$ such that $x \in U \subset \bigcap_{i \in \mathcal{J}} S_i$. Obviously, we get $U \subset S_i$ for each $i \in \mathcal{J}$, that is, $x \in \mathfrak{m}_X\text{-int}(S_i)$ for each $i \in \mathcal{J}$.

2. Let $x \in \mathfrak{m}_X\text{-cl}(\bigcap_{i \in \mathcal{J}} S_i)$. By Lemma 2, one has $U \cap \bigcap_{i \in \mathcal{J}} S_i \neq \emptyset$ for each $U \in \mathfrak{m}_X$ with $x \in U$. Thus, $U \cap S_i \neq \emptyset$, for each $i \in \mathcal{J}$, and consequently,

$x \in \mathfrak{m}_X\text{-cl}(S_i)$. Statement 3. follows by 2. and 1.(b) of Lemma 1. Statement 4. follows by 1. and 1.(a) of Lemma 1. $\square$

DEFINITION 3 (see [3, Definition 2.4(b)] or [4, Definition 2.4(b)]). Let $(X, \mathfrak{m}_X)$ be an $\mathcal{M}$ -space. A set $A \subset X$ is said to be $\mathfrak{m}_X$ -semi-open (or $\mathcal{M}$ -semi-open) if there exists a set $U \in \mathfrak{m}_X$ such that

$$U \subset A \subset \mathfrak{m}_X\text{-cl}(U).$$

It is known (see [3, Theorem 2.7(i)] or [4, Theorem 2.7(i)]) that if A is $\mathfrak{m}_X$ -semi-open, then $A \subset \mathfrak{m}_X\text{-cl}(\mathfrak{m}_X\text{-int}(A))$. Lemmas 5 and 6 slightly reformulate some known results.

LEMMA 5 ([4, Lemma 3.3]). *For an $\mathcal{M}$ -space $(X, \mathfrak{m}_X)$, the following two conditions are equivalent:*

1. $(X, \mathfrak{m}_X)$ is a supratopological space;
2. if $\mathfrak{m}_X\text{-cl}(F) = F$ then $X \setminus F \in \mathfrak{m}_X$ for each $F \subset X$.

It is necessary to prove Lemma 5.

LEMMA 6 ([6, Theorem 2.7(ii)]). *Let $(X, \mathfrak{m}_X)$ be a supratopological space. Then, a subset A of the space X is $\mathfrak{m}_X$ -semi-open if and only if $A \subset \mathfrak{m}_X\text{-cl}(\mathfrak{m}_X\text{-int}(A))$.*

The family of all $\mathfrak{m}_X$ -semi-open subsets of an $\mathcal{M}$ -space $(X, \mathfrak{m}_X)$ will be denoted by $\text{SO}(X, \mathfrak{m}_X)$. According to Lemma 4 (3. and 4.), and Lemma 6, the following result may be easily observed.

LEMMA 7. *If $(X, \mathfrak{m}_X)$ is a supratopological space, then so is $(X, \text{SO}(X, \mathfrak{m}_X))$.*

Proof. It is sufficient to use Lemma 4 (3. and 4.) and Lemma 6. Namely, for any family $\{S_i\}_{i \in \mathcal{I}} \subset \text{SO}(X, \mathfrak{m}_X)$ we have what follows:

$$\begin{aligned} \bigcup_{i \in \mathcal{I}} S_i &\subset \bigcup_{i \in \mathcal{I}} \mathfrak{m}_X\text{-cl}(\mathfrak{m}_X\text{-int}(S_i)) \\ &\subset \mathfrak{m}_X\text{-cl}\left(\bigcup_{i \in \mathcal{I}} \mathfrak{m}_X\text{-int}(S_i)\right) \\ &\subset \mathfrak{m}_X\text{-cl}\left(\mathfrak{m}_X\text{-int}\left(\bigcup_{i \in \mathcal{I}} S_i\right)\right). \end{aligned}$$

Thus, $\bigcup_{i \in \mathcal{I}} S_i \in \text{SO}(X, \mathfrak{m}_X)$. $\square$

LEMMA 8. (a) *In an $\mathcal{M}$ -space $(X, \mathfrak{m}_X)$, $A \in \text{SO}(X, \mathfrak{m}_X)$ implies*

$$\mathfrak{m}_X\text{-cl}(A) = \mathfrak{m}_X\text{-cl}(\mathfrak{m}_X\text{-int}(A)).$$

(b) *If $(X, \mathfrak{m}_X)$ is a supratopological space, then, moreover, the above identity implies $A \in \text{SO}(X, \mathfrak{m}_X)$.*

Proof. (a) Let $A \in \text{SO}(X, \mathfrak{m}_X)$. Then, $A \subset \mathfrak{m}_X\text{-cl}(\mathfrak{m}_X\text{-int}(A))$. Using 4.(a) of Lemma 1, we have $\mathfrak{m}_X\text{-cl}(A) \subset \mathfrak{m}_X\text{-cl}(\mathfrak{m}_X\text{-cl}(\mathfrak{m}_X\text{-int}(A)))$ and so, by 6.(a) of the same lemma, $\mathfrak{m}_X\text{-cl}(A) \subset \mathfrak{m}_X\text{-cl}(\mathfrak{m}_X\text{-int}(A))$. On the other hand, $A \supset \mathfrak{m}_X\text{-int}(A)$ obviously implies $\mathfrak{m}_X\text{-cl}(A) \supset \mathfrak{m}_X\text{-cl}(\mathfrak{m}_X\text{-int}(A))$. Thus, $\mathfrak{m}_X\text{-cl}(A) = \mathfrak{m}_X\text{-cl}(\mathfrak{m}_X\text{-int}(A))$.

(b) By 5.(b) of Lemma 1 and our hypothesis, one gets $\mathfrak{m}_X\text{-int}(A) \subset A \subset \mathfrak{m}_X\text{-cl}(\mathfrak{m}_X\text{-int}(A))$. Then, since the space $(X, \mathfrak{m}_X)$ is supratopological, by Definition 3, we directly obtain that $A \in \text{SO}(X, \mathfrak{m}_X)$. $\square$

COROLLARY 1. *If $(X, \mathfrak{m}_X)$ is a supratopological space, $A \in \text{SO}(X, \mathfrak{m}_X)$ if and only if $\mathfrak{m}_X\text{-cl}(A) = \mathfrak{m}_X\text{-cl}(\mathfrak{m}_X\text{-int}(A))$.*

LEMMA 9 ([6, Corollary 3.1]). *Let $(X, \mathfrak{m}_X)$ be a supratopological space and $(Y, \mathfrak{m}_Y)$ be an $\mathcal{M}$ -space. Then, for any mapping $f: (X, \mathfrak{m}_X) \rightarrow (Y, \mathfrak{m}_Y)$, the following two conditions are equivalent:*

1. f is $\mathcal{M}$ -continuous;
2. $f^{-1}(V) \in \mathfrak{m}_X$ for each $V \in \mathfrak{m}_Y$.

DEFINITION 4. Let $(X, \mathfrak{m}_X), (Y, \mathfrak{m}_Y)$ be $\mathcal{M}$ -spaces. A mapping $f: X \rightarrow Y$ is said to be $\mathcal{M}$ -semi-continuous if $f^{-1}(V) \in \text{SO}(X, \mathfrak{m}_X)$ for each $V \in \mathfrak{m}_Y$.

It is worth noticing the following.

THEOREM 4. *Let $(X, \mathfrak{m}_X)$ be a supratopological space and $(Y, \mathfrak{m}_Y)$ be an $\mathcal{M}$ -space. A mapping $f: X \rightarrow Y$ is $\mathcal{M}$ -semi-continuous if and only if f is $\mathcal{M}$ -continuous if defined on $(X, \text{SO}(X, \mathfrak{m}_X))$ instead of $(X, \mathfrak{m}_X)$.*

Proof. As $(X, \text{SO}(X, \mathfrak{m}_X))$ is a supratopological space (Lemma 7), the conclusion follows from Lemma 9. $\square$

DEFINITION 5. Let $(X, \mathfrak{m}_X)$ be an $\mathcal{M}$ -space. A subset A of the space X is said to be $\mathfrak{m}_X$ -nowhere dense if $\mathfrak{m}_X\text{-int}(\mathfrak{m}_X\text{-cl}(A)) = \emptyset$. A is said to be of $\mathfrak{m}_X$ -first category if A can be written as an at most countable union of $\mathfrak{m}_X$ -nowhere dense sets.

LEMMA 10. *For any subset S of an $\mathcal{M}$ -space $(X, \mathfrak{m}_X)$, the set*

$$\mathfrak{m}_X\text{-cl}(\mathfrak{m}_X\text{-int}(S)) \setminus \mathfrak{m}_X\text{-int}(S)$$

is $\mathfrak{m}_X$ -nowhere dense.

Proof. Set $A = \mathfrak{m}_X\text{-cl}(\mathfrak{m}_X\text{-int}(S)) \setminus \mathfrak{m}_X\text{-int}(S)$. Since $\mathfrak{m}_X\text{-int}(S) \subset \mathfrak{m}_X\text{-cl}(\mathfrak{m}_X\text{-int}(S))$ (5.(a) of Lemma 1), we have

$$A = (X \setminus \mathfrak{m}_X\text{-int}(S)) \cap \mathfrak{m}_X\text{-cl}(\mathfrak{m}_X\text{-int}(S)).$$

Using 2. of Lemma 4, we get

$$\mathfrak{m}_X\text{-cl}(A) \subset \mathfrak{m}_X\text{-cl}(X \setminus \mathfrak{m}_X\text{-int}(S)) \cap \mathfrak{m}_X\text{-cl}(\mathfrak{m}_X\text{-cl}(\mathfrak{m}_X\text{-int}(S))),$$

and by 1.(a) and 6. of Lemma 1,

$$\mathfrak{m}_X\text{-cl}(A) \subset X \setminus \mathfrak{m}_X\text{-int}(S) \cap \mathfrak{m}_X\text{-cl}(\mathfrak{m}_X\text{-int}(S)),$$

respectively. Then by 4.(b) of Lemma 1, 1. of Lemma 4, and 1.(b) of Lemma 1, we obtain

$$\begin{aligned} \mathfrak{m}_X\text{-int}(\mathfrak{m}_X\text{-cl}(A)) &\subset \mathfrak{m}_X\text{-int}(X \setminus \mathfrak{m}_X\text{-int}(S)) \cap \mathfrak{m}_X\text{-int}(\mathfrak{m}_X\text{-cl}(\mathfrak{m}_X\text{-int}(S))) \\ &= (X \setminus \mathfrak{m}_X\text{-cl}(\mathfrak{m}_X\text{-int}(S))) \cap \mathfrak{m}_X\text{-int}(\mathfrak{m}_X\text{-cl}(\mathfrak{m}_X\text{-int}(S))). \end{aligned}$$

However, $\mathfrak{m}_X\text{-int}(\mathfrak{m}_X\text{-cl}(\mathfrak{m}_X\text{-int}(S))) \subset \mathfrak{m}_X\text{-cl}(\mathfrak{m}_X\text{-int}(S))$ (5.(b) of Lemma 1), so $\mathfrak{m}_X\text{-int}(\mathfrak{m}_X\text{-cl}(A)) = \emptyset$. The proof is complete. $\square$

Now, we are ready to prove the main result of this section.

THEOREM 5. *Let $(X, \mathfrak{m}_X)$, $(Y, \mathfrak{m}_Y)$ be supratopological spaces with $(Y, \mathfrak{m}_Y)$ being $\mathfrak{m}_X$ -2nd countable, and let $f: (X, \mathfrak{m}_X) \rightarrow (Y, \mathfrak{m}_Y)$ be $\mathcal{M}$ -semi-continuous. Then, the set $\mathcal{D}_f$ is of $\mathfrak{m}_X$ -first category.*

Proof. Let $\beta_{\mathfrak{m}_Y}$ be a countable $\mathfrak{m}$ -base for $\mathfrak{m}_Y$. By 3. of Theorem 3, we have

$$\mathcal{D}_f \subset \bigcup_{V \in \beta_{\mathfrak{m}_Y}} [\mathfrak{m}_X\text{-cl}(\mathfrak{m}_X\text{-int}(f^{-1}(V))) \setminus \mathfrak{m}_X\text{-int}(f^{-1}(V))].$$

Thus, by Lemma 10, the desired result follows. $\square$

Theorem 5 generalizes the well-known Levine's result. Indeed, putting in the above theorem $\mathfrak{m}_X = \tau$ and $\mathfrak{m}_Y = \sigma$, where τ and σ are topologies on X and Y , respectively, one has

COROLLARY 2 ([2, Theorem 13]). *Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a semi-continuous mapping and let (Y, σ) be a 2nd axiom space. Then, the set of all discontinuity points of f is of first category.*

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Received September 4, 2009

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