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A GENERALIZED BERNSTEIN APPROXIMATION THEOREM

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ABSTRACT. The present paper is concerned with some generalizations of Bernstein's approximation theorem. One of the most elegant and elementary proofs of the classic result, for a function $f(x)$ defined on the closed interval $[0,1]$, uses the Bernstein's polynomials of f,

$$B_n(x) = B_n^f(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}$$

We shall concern the m-dimensional generalization of the Bernstein's polynomials and the Bernstein's approximation theorem by taking an $(m-1)$-dimensional simplex in cube $[0,1]^m$. This is motivated by the fact that in the field of mathematical biology naturally arouse dynamic systems determined by quadratic mappings of "standard" $(m-1)$-dimensional simplex $\left\{x_i \geq 0,\ i = 1,\dots,m,\ \sum_{i=1}^{m} x_i = 1\right\}$ to self. The last condition guarantees saving of the fundamental simplex. Then there are surveyed some other the m-dimensional generalizations of the Bernstein's polynomials and the Bernstein's approximation theorem.

1. Introduction

For a function $f(x)$ defined on the closed interval $[0,1]$ the expression

$$B_n(x) = B_n^f(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k} \tag{1}$$

is called the Bernstein polynomial of order n of the function $f(x)$. $B_n(x)$ is a polynomial in x of degree $\leq n$. The polynomials $B_n(x)$ were introduced by S. Bernstein (see [1]) to give an especially simple proof of Weierstrass' approximation theorem. Namely, if $f(x)$ is a function continuous on $[0,1]$, then as it can be seen

$$\lim_{n\to\infty} B_n(x) = f(x) \tag{2}$$

uniformly in $[0,1]$.

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A celebrated theorem of Weierstrass says that any continuous real-valued function f defined on the closed interval $[0,1] \subset \mathbb{R}$ is the limit of a uniformly convergent sequence of polynomials. One of the most elegant and elementary proofs of this classic result is that which uses the Bernstein polynomials of f,

$$(B_n f, x) = \sum_{k=0}^{n} \left(\frac{k}{n}\right) f \binom{n}{k} x^k (1-x)^{n-k} \qquad \big(x \in [0,1]\big),$$

one for each integer $n \geq 1$. Bernstein's theorem states that $B_n(f) \to f$ uniformly on $[0,1]$ and, since each $B_n(f)$ is a polynomial of degree at most n, we have as a consequence Weierstrass' theorem. (See, for example, [5]).

The operator B_n defined on the space $C\big([0,1];\mathbb{R}\big)$ with values in the vector subspace of all polynomials of degree at most n has the property that $B_n(f) \geq 0$ whenever $f \geq 0$. Thus Bernstein's theorem also establishes the fact that each positive continuous real-valued function on $[0,1]$ is the limit, of a uniformly convergent sequence of positive polynomials.

The present paper is concerned with some generalizations of Bernstein's theorem.

For the following note that the expressions

$$p_k = p_{nk}(x) = \binom{n}{k} x^k (1-x)^{n-k} \tag{3}$$

contained in (1) are the binomial or Newton probabilities well known in the theory of probability. If $0 \leq x \leq 1$ is the probability of an event E, then $p_{nk}(x)$ is the probability that E will occur exactly k times in n independent trials. Many of the properties of the p_{nk}, and of their sums which we shall need are nothing but theorems of the theory of probability.

As an example, consider Bernoulli's theorem of large numbers. Let $\epsilon > 0$ and $\delta > 0$ be fixed and suppose that among the n independent trials, k is the number of those for which the event E occurs. Then for n sufficiently large, the probability P_δ that $\frac{k}{n}$ differs from x by less than δ is greater than $1-\epsilon$. By the theorem of addition of probabilities, this may be written in the form

$$P_\delta = \sum_{|\frac{k}{n}-x|<\delta} \binom{n}{k} x^k (1-x)^{n-k} \geq 1-\epsilon, \tag{4}$$

for all n sufficiently large. (The last sum is taken for all values $k = 0, 1, \dots, n$ which satisfy the condition

$$\left|\frac{k}{n} - x\right| < \delta;$$

notation of this type is used in the sequel without explanation.) We can see that (2) easily follows from this inequality.

2. The theorem of Weierstrass

Bernstein polynomials of the function $f(x)$ are linear with respect to the function $f(x)$, i.e.,

$$B_n^f(x) = a_1 B_n^{f_1}(x) + a_2 B_n^{f_2}(x) \tag{5}$$

if $f(x) = a_1 f_1(x) + a_2 f_2(x)$.

Since

$$p_{nk}(x) = \binom{n}{k} x^k (1-x)^{n-k} \geq 0$$

on the interval $0 \leq x \leq 1$ and $\sum_0^n p_{nk} = 1$, we have

$$m \leq B_n^f(x) \leq M \qquad (0 \leq x \leq 1), \tag{6}$$

whenever $m \leq f(x) \leq M$ on this interval.

With the help of the polynomials $B_n(x)$ we may prove the famous theorem of Weierstrass, which asserts that for each function $f(x)$, continuous on a closed interval $[a, b]$, and for each $\epsilon > 0$ there is a polynomial $P(x)$ approximating $f(x)$ uniformly with an error less than ϵ,

$$|f(x) - P(x)| < \epsilon. \tag{7}$$

By a linear substitution, the interval $[a, b]$ may be transformed into $[0, 1]$. The theorem of Weierstrass is therefore a corollary of the following theorem:

Theorem 1 ([1, Bernstein]). *For a function $f(x)$ bounded on $[0, 1]$, the relation*

$$\lim_{n\to\infty} B_n(x) = f(x) \tag{8}$$

holds at each point of continuity x of f and the relation holds uniformly on $[0, 1]$ if $f(x)$ is continuous on this interval.

Proof. We shall compute the value of

$$T = \sum_{k=0}^{n} (k - nx)^2 p_k = \sum_{k=0}^{n} \Big\{ k(k-1) - (2nx-1)k + n^2x^2 \Big\} p_k. \tag{9}$$

Obviously, $\sum_{k=0}^n p_k = 1$, moreover, we have

$$\sum_{k=0}^{n} k p_k = nx \sum_{k=0}^{n-1} \binom{n-1}{m} x^m (1-x)^{n-1-m} = nx.$$

$$\sum_{k=0}^{n} k(k-1) p_k = n(n-1)x^2 \sum_{k=0}^{n-2} \binom{n-2}{m} x^m (1-x)^{n-2-m} = n(n-1)x^2$$

and therefore,

$$T = n^2x^2 - (2nx-1)nx + n(n-1)x^2 = nx(1-x). \tag{10}$$

Since $x(1-x) \leq \frac{1}{4}$ on $[0,1]$, we obtain the inequality

$$\sum_{|\frac{k}{n}-x|\geq\delta} p_k \leq \frac{1}{\delta^2} \sum_{|\frac{k}{n}-x|\geq\delta} \left(\frac{k}{n}-x\right)^2 p_k \leq \frac{1}{n^2\delta^2} T = \frac{x(1-x)}{n\delta^2} \leq \frac{1}{4n\delta^2} \tag{11}$$

Now if the function f is bounded, say $|f(u)| \leq M$ in $0 \leq u \leq 1$ and x a point of continuity, for a given $\epsilon > 0$, we can find a $\delta > 0$ such that $|x - x'| < \delta$ implies $|f(x) - f(x')| < \epsilon$. We have

$$\begin{aligned} |f(x) - B_n(x)| &= \left| \sum_{k=0}^{n} \left\{ f(x) - f\frac{k}{n} \right\} p_k \right| \\ &\leq \sum_{|\frac{k}{n}-x|<\delta} \left| f(x) - f\frac{k}{n} \right| p_k \\ &\quad + \sum_{|\frac{k}{n}-x|\geq\delta} \end{aligned} \tag{12}$$

The first sum is $\leq \epsilon \sum p_k = \epsilon$, the second one is, by (11), $\leq 2M(4n\delta^2)^{-1}$. Therefore,

$$|f(x) - B_n(x)| \leq \epsilon + M\left(2n\delta^2\right)^{-1} \tag{13}$$

and if n is sufficiently large, $|f(x) - B_n(x)| < 2\epsilon$. Finally, if $f(x)$ is continuous in the whole interval $[0,1]$, then (13) holds with a δ independent of x, so that $B_n(x) \to f(x)$ uniformly. This completes the proof. □

3. m-dimensional generalization of the Bernstein theorem

Now we shall discuss m-dimensional generalization of the Bernstein theorem. First we give m-dimensional generalization of the Bernstein polynomials.

We recall the multinomial theorem.

Theorem 2. *Multinomial theorem. The following equality is valid*

$$\begin{aligned} &(x_1 + \cdots + x_m)^n \\ &= \sum_{\substack{a_i \geq 0, \\ a_1+\cdots+a_m=n}} \binom{n}{a_1, \ldots, a_m} x_1^{a_1} \cdots x_m^{a_m}, \\ &= \sum_{\substack{a_i \geq 0, \\ a_1+\cdots+a_m=n}} \frac{n!}{a_1! \cdots a_m!} x_1^{a_1} \cdots x_m^{a_m}, \end{aligned}$$

(see [3]).

An important m-dimensional generalization is obtained by taking an $(m-1)$-dimensional simplex $\Delta_p = \{x_i \geq 0,\, i = 1, \ldots, m,\, x_1 + \cdots + x_m = 1\}$. If $f(x_1, \ldots, x_m)$ is defined on Δ_p, we may write with regard to the multinomial theorem

$$B_n^f(x_1, \ldots, x_m) = \sum_{\substack{l_i \geq 0,\\ l_1+\cdots+l_m=n}} f\left(\frac{l_1}{n}, \ldots, \frac{l_m}{n}\right) \frac{n!}{l_1! \ldots l_m!} x_1^{l_1} \ldots x_m^{l_m}, \tag{14}$$

$$p_{l_1,\ldots,l_m;n}(x_1, \ldots, x_m) = \binom{n}{l_1, \ldots, l_m} x_1^{l_1} \ldots x_m^{l_m},$$

$$\binom{n}{l_1, \ldots, l_m} = \frac{n!}{l_1! \ldots l_m!}.$$

Here again we have the convergence $B_n^f \to f$ at a point of continuity of f.

Theorem 3. *Let $f(x_1, \ldots, x_m)$ be a continuous function on Δ_p. Then*

$$\lim_{n\to\infty} B_n^f(x_1, \ldots, x_m) = f(x_1, \ldots, x_m)$$

uniformly on Δ_p.

Proof. The proof is quite similar to that of Theorem 1, and is based on the following properties of the $p_{l_1,\ldots,l_m}$:

(a) the sum of the p's which occur in (14) is equal to 1,

(b) if $\epsilon > 0, \delta > 0$ are given, the sum of those $p_{l_1,\ldots,l_m}$ in (14) for which $|\frac{l_i}{n} - x_i| \geq \delta$ for at least one index i is smaller than ϵ for each sufficiently large n.

To prove (b), we observe that by (11) the sum of p's with $|\frac{l_1}{n} - x_1| \geq \delta$ is equal to

$$\sum_{\substack{|\frac{l_i}{n}-x_i|\geq\delta,\\ l_1+\cdots+l_m=n}} p_{l_1,\ldots,l_m;n}$$

$$= \sum_{l_1} \binom{n}{l_1} x_1^{l_1} \sum_{l_2,\ldots,l_m} \binom{n-l_1}{l_2, \ldots, l_m} x_2^{l_2} \ldots x_m^{l_m},$$

where the summations are extended over $\left|\frac{l_1}{n} - x_1\right| \geq \delta$ and $l_2 + \cdots + l_m \leq n - l_1$, respectively, and this is

$$\sum_{l_1} \binom{n}{l_1} x^{l_1}(1 - x_1)^{n-l_1} \leq (4n\delta^2)^{-1}. \qquad \square$$

Another m-dimensional generalization is obtained by taking a m-dimensional simplex $\Delta = \{x_i \geq 0,\, i = 1, \ldots, m,\, x_1 + \cdots + x_m \leq 1\}$. See [5].

Theorem 4. *If $f(x_1,\dots,x_m)$ is defined on Δ, we write*

$$B_n^f(x_1,\dots,x_m) = \sum_{\substack{l_i\ge 0,\\ l_1+\dots+l_m\le n}} f\left(\frac{l_1}{n},\dots,\frac{l_m}{n}\right) p_{l_1,\dots,l_m;n}(x_1,\dots,x_m), \tag{15}$$

$$p_{l_1,\dots,l_m;n}(x_1,\dots,x_m) = \binom{n}{l_1,\dots,l_m} x_1^{l_1}\dots x_m^{l_m}(1-x_1-\dots-x_m)^{n-l_1-\dots-l_m},$$

$$\binom{n}{l_1,\dots,l_m} = \frac{n!}{l_1!\dots l_m!(n-l_1-\dots-l_m)!}.$$

Here again we have the convergence $B_n^f \to f$ at a point of continuity of f.

Proof. The proof is quite similar to that of Theorem 1, and is based on the following properties of the $p_{l_1,\dots,l_m}$:

(a) the sum of the p's which occur in (15) is equal to 1,

(b) if $\epsilon > 0$, $\delta > 0$ are given, the sum of those $p_{l_1,\dots,l_m}$ in (15) for which $\left|\frac{l_i}{n} - x_i\right| \ge \delta$ for at least one index i is smaller than ϵ for each sufficiently large n. To prove (b), we observe that by (11) the sum of p's with $|\frac{l_1}{n} - x_1| \ge \delta$ is equal to

$$\sum_{\substack{|\frac{l_1}{n}-x_i|\ge\delta,\\ l_1+\dots+l_m\le n}} p_{l_1,\dots,l_m;n}$$

$$= \sum_{l_1}\binom{n}{l_1}x_1^{l_1}\sum_{l_2,\dots,l_m}\binom{n-l_1}{l_2,\dots,l_m}x_2^{l_2}\dots x_m^{l_m}(1-x_1-\dots-x_m)^{n-l_1-\dots-l_m},$$

where the summations are extended over $\left|\frac{l_1}{n} - x_1\right| \ge \delta$ and $l_2+\dots+l_m \le n-l_1$, respectively, and this is

$$\sum_{l_1}\binom{n}{l_1}x^{l_1}(1-x_1)^{n-l_1} \le (4n\delta^2)^{-1}.$$

□

We shall now derive a more m-dimensional generalization of the Bernstein theorem. First we give m-dimensional generalization of the Bernstein polynomials. (See [2], [4]).

Theorem 5. *Let $f(x_1,x_2,\dots,x_m)$ be defined and bounded in the m-dimensional cube $0 \le x_i \le 1$, $i = 1,\dots,m$. Then the Bernstein polynomial defined by*

$$\begin{aligned} &B^f_{n_1,\dots,n_m}(x_1,\dots,x_m)\\ &=\sum_{l_1=0}^{n_1}\dots\sum_{l_m=0}^{n_m}\binom{n_1}{l_1}\dots\binom{n_m}{l_m}f\left(\frac{l_1}{n_1},\dots,\frac{l_m}{n_m}\right)\\ &\quad\times x_1^{l_1}(1-x_1)^{n_1-l_1}\dots x_m^{l_m}(1-x_m)^{n_m-l_m}\end{aligned}$$

converges towards $f(x_1,\dots,x_m)$ *at any point of continuity of this function, as all* $n_i \to \infty$.

4. Bohman-Borovkin theorem

We shall use a modified version of Bohman-Korovkin's Theorem (proved by Păltineanu) to prove a generalized Bernstein theorem. (See Păltineanu [6].) Let X be a compact Hausdorff space containing at least two points.

Proposition 1 (Bohman-Korovkin). *Let there be given* $2m$ *functions* $f_1,\dots,f_m$, $a_1,\dots,a_m \in C(X,\mathbb{R})$, *with the properties:*

$$P(x,y)=\sum_{i=1}^{m} a_i(y)f_i(x) \geq 0, \qquad \text{for all} \quad (x,y)\in X^2$$

and

$$P(x,y)=0 \Leftrightarrow x=y.$$

If H_n *is a sequence of positive linear operators on* $C(X;\mathbb{R})$, *with the property:*

$$H_n(f_i)\to f_i \quad \text{as} \quad n\to\infty \qquad \text{for all} \quad i=1,2,\dots,m;$$

then $H_n(f)\to f$ *as* $n\to\infty$ *for all* $f\in C(X;\mathbb{R})$.

(See Păltineanu [6].) We shall formulate an interesting applications of Proposition 1, namely Korovkin theorem.

Theorem 6 (Korovkin). *Let* H_n *be a sequence of positive linear operators on* $C\big([a,b]\big)$ *and* f_1,f_2,f_3 *be the functions defined as*

$$f_1(x)=1, \quad f_2(x)=x, \quad f_3(x)=x^2, \qquad \text{for all} \quad x\in[a,b].$$

If $H_n(f_i)\to f_i$, $i=1,2,3$, *then*

$$H_n(f)\to f, \qquad \text{for all} \quad f\in C\big([a,b],\mathbb{R}\big).$$

Proof. (See Păltineanu [6].) Let $a_1(y)=y^2, a_2(y)=-2y, a_3(y)=1$ and let

$$P(x,y)=\sum_{i=1}^{3} a_i(y)f_i(x)=(y-x)^2.$$

We can see that the conditions of Proposition 1 are satisfied. □

Theorem 7 (S. Bernstein). *Let* B_n *be a sequence of positive linear operators on* $C\big([0,1]\big)$, *defined by*

$$B_n(f)(x)=\sum_{k=0}^{n} f(\frac{k}{n})\binom{n}{k}x^k(1-x)^{n-k}, \qquad \text{for all} \quad x\in[0,1])$$

Then $B_n(f) \to f$ *for all* $f \in C\big([0,1], \mathbb{R}\big)$.

P r o o f. (See P ă l t i n e a n u [6].) It is clear that B_n is a sequence of positive linear operators. Let $f_1(x) = 1$, for all $x \in [0,1]$. Then

$$B_n(f_1) \to f_1.$$

If we denote

$$p_{nk}(x) = \binom{n}{k} x^k (1-x)^{n-k}$$

then we have

$$\sum_{k=1}^{n} k p_{nk}(x) = nx.$$

Further we have

$$\sum_{k=0}^{n} k^2 p_{nk}(x) = n^2x^2 - nx(x-1).$$

Let

$$f_2(x) = x, \quad f_3(x) = x^2, \qquad \text{for all} \quad x \in [0,1].$$

It follows

$$B_n(f_2)(x) = x$$

hence

$$B_n(f_2) \to f_2.$$

Further we obtain

$$B_3(f_3) \to f_3.$$

Since there are satisfied the conditions of Borovkin theorem, we have

$$B_n(f) \to f, \qquad \text{for all} \quad f \in C\big([0,1], \mathbb{R}\big).$$

□

Theorem 8 (Generalized m-dimensional Bernstein theorem)**.** *Let* $X \subset R^m$ *be a compact and let*

$$p_{nk}(x) = \binom{n}{k} x^k (1-x)^{n-k}$$

and

$$B_n(f)(x_1, \dots, x_m) = \sum_{k_1=0}^{n} \dots \sum_{k_m=0}^{n} p_{nk_1}(x_1) \dots p_{nk_m}(x_m) f\left(\frac{k_1}{n}, \dots, \frac{k_m}{n}\right)$$

Then for any $f \in C(X;\mathbb{R}), B_n(f) \to f$ *as* $n \to \infty$*, uniformly on X.*

Proof. (See Păltineanu [6].) To prove this we may put $X = [0,1]^m$. In fact, there exists an m-dimensional cube $Y \supset X$. By Tietze theorem every $f \in C(X,\mathbb{R})$ can be extended to a function $\tilde{f} \in C(Y,\mathbb{R})$ with $\|\tilde{f}\| = \|f\|$. If $B_n(\tilde{f}) \to \tilde{f}$ then $B_n(f) \to f$. Therefore it is enough to prove theorem for the cube Y, since every m-dimensional cube Y is linearly homeomorphic with the cube $X = [0,1]^m$. Consider the following $2m+1$ functions:

$$f_1(x_1,\dots,x_m) = 1, \quad g_i(x_1,\,\dots,x_i,\,\dots,x_m) = x_i,$$
$$h_i(x_1,\,\dots,x_i,\,\dots,x_m) = x_i^2, \qquad i = 1,\dots,m.$$

If we use the identities

$$\sum_{k=0}^{n} k p_{nk}(x) = nx, \quad \sum_{k=0}^{n} k^2 p_{nk}(x) = n^2x^2 - nx(x-1),$$

we obtain

$$B_n(f_1) \to f_1, \quad B_n(g_i) \to g_i, \quad B_n(h_i) \to h_i, \qquad i = 1,\dots,m.$$

From the other side if we denote

$$p(x,y) = \sum_{i=1}^{m} (y_i - x_i)^2,$$

we can see that the conditions of Proposition 1 are fulfilled. Hence

$$B_n(f) \to f, \qquad \text{for all} \quad f \in C(X,\mathbb{R}). \qquad \square$$

An earlier version of Bohman-Borovkin‘s theorem was proved by Prolla [7, Theorem 4 and Corollary 7]. However, the version of Păltineanu allows to prove the multidimensional Bernstein‘s theorem.

5. Multinomial theorem

For convenience of the reader we add one of the possibilities of the proofs of the multinomial theorem on the base of (See [3]). If we multiply out the expression

$$(x_1 + \dots + x_m)^n$$

and collect coefficients we get a sum in which each term has the form

$$\binom{n}{a_1,\dots,a_m} x_1^{a_1} x_2^{a_2} \dots x_m^{a_m} \quad \text{with some coefficient} \quad \binom{n}{a_1,\dots,a_m},$$

where a_i are nonnegative integers with

$$a_1 + a_2 + \dots + a_m = n.$$

We shall prove the following proposition. (See [3, p. 18].)

Proposition 2.

$$\binom{n}{a_1,\ldots,a_m} = \frac{n!}{a_1!a_2!\cdots a_m!}, \quad \textit{where } 0! = 1.$$

Proof. The case $m = 2$ is the binomial theorem.

We could now to prove the case $m = 3$ but we shall take any $m > 2$. We shall do it by induction on m. For $m > 2$, we have

$$(x_1 + \cdots + x_m)^n = \sum (x_1 + \cdots + x_{m-1})^{n-a_m} x_m^{a_m} \binom{n}{a_m}$$

Now

$$(x_1 + \cdots + x_{m-1})^{n-a_m} = \sum \binom{n-a_m}{a_1 \cdots a_{m-1}} x_1^{a_1} \cdots x_{m-1}^{a_{m-1}}$$

so the coefficient of

$$x_1^{a_1} \cdots x_{m-1}^{a_m} x_m^{a_m}$$

is

$$\binom{n}{a_1 \cdots a_m} = \binom{n}{a_m} \binom{n-a_m}{a_1 \cdots a_{m-1}}$$

Now use induction and definition of $\binom{n}{a_m}$. □

We have thus proved

Theorem 9. *Multinomial theorem. The following equality is valid*

$$\begin{aligned}
&(x_1 + \cdots + x_m)^n \\
&= \sum_{\substack{a_i \geq 0,\\ a_1 + \cdots + a_m = n}} \binom{n}{a_1,\ldots,a_m} x_1^{a_1} \cdots x_m^{a_m}, \\
&= \sum_{\substack{a_i \geq 0,\\ a_1 + \cdots + a_m = n}} \frac{n!}{a_1! \cdots a_m!} x_1^{a_1} \cdots x_m^{a_m},
\end{aligned}$$

In particular, if $x_1 + \cdots + x_m = 1$, then

$$\begin{aligned}
&(x_1 + \cdots + x_m)^n = 1 \\
&= \sum_{\substack{a_i \geq 0,\\ a_1 + \cdots + a_m = n}} \binom{n}{a_1,\ldots,a_m} x_1^{a_1} \cdots x_m^{a_m}, \\
&= \sum_{\substack{a_i \geq 0,\\ a_1 + \cdots + a_m = n}} \frac{n!}{a_1! \cdots a_m!} x_1^{a_1} \cdots x_m^{a_m},
\end{aligned}$$

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