A DECOMPOSITION OF BOUNDED, WEAKLY MEASURABLE FUNCTIONS

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ABSTRACT. Let $(X, \mathcal{A}, \mu)$ be a complete probability space, ρ a lifting, $\mathcal{T}_\rho$ the associated Hausdorff lifting topology on X and E a Banach space. Suppose $F: (X, \mathcal{T}_\rho) \rightarrow E''_\sigma$ be a bounded continuous mapping. It is proved that there is an $A \in \mathcal{A}$ such that $F\chi_A$ has range in a closed separable subspace of E (so $F\chi_A: X \rightarrow E$ is strongly measurable) and for any $B \in \mathcal{A}$ with $\mu(B) > 0$ and $B \cap A = \emptyset$, $F\chi_B$ cannot be weakly equivalent to a E -valued strongly measurable function. Some known results are obtained as corollaries.

1. Introduction and notation

In this paper R stands for the set of real numbers, K will denote the field of real or complex numbers (we will call them scalars). For locally convex spaces, results and notations of [5] will be used; for a locally convex space F with F' its dual, if $x \in F$ and $f \in F'$, $f(x)$ will also be denoted by $\langle f, x \rangle$ and $\langle x, f \rangle$. $(X, \mathcal{A}, \mu)$ is a complete probability space. We fix a lifting ρ for this measure space and taking the lifting topology $\mathcal{T}_\rho$ on X ([8, p. 58]; [3, p. 88]) (the open sets in this topology are $\{\rho(A) \setminus Q : A \in \mathcal{A}, \mu(Q) = 0\}$); this space is a Baire space and the measure μ on this space is τ -smooth ([9], [10]). For a topological space Y with $A \subset Y$, $Y \setminus A$ will also be denoted by A' , E is a Banach space with E' , E'' its dual and bidual. We will denote by E''_σ the space E'' with $\sigma(E'', E')$ topology. A function $f: X \rightarrow E$ is said to be weakly measurable if $g \circ f$ is measurable for every $g \in E'$. For measure theory we will use the results and notations of [1]. For a bounded weakly measurable function $f: X \rightarrow E$, with $\|f\| \leq C$ for some $C > 0$, we get, for every $g \in E'$, a bounded continuous $\rho(g \circ f): (X, \mathcal{T}_\rho) \rightarrow K$ such that $g \circ f = \rho(g \circ f)$ a.e. (μ). So $|\rho(g \circ f)| \leq C$ for all $g \in E'$ with $\|g\| \leq 1$. Define $F: X \rightarrow E'': \langle g, F(x) \rangle = \rho(g \circ f)(x)$. This function is easily checked to

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be well-defined and is continuous as a mapping $F: (X, \mathcal{T}_\rho) \rightarrow E''_\sigma$. In the measure space $(X, \mathcal{A}, \mu)$, two weakly measurable functions f_1, f_2 are said to be weakly equivalent, written as $f_1 \equiv f_2$, if for every $g \in E'$, $g \circ f_1 = g \circ f_2$, *a.e.*(μ), thus the functions f, F are weakly equivalent.

In [7], using Stonian transforms, an interesting result about the decomposition of weakly measurable functions $f: X \rightarrow E$ is proved. Using liftings, we put the result in a different setting. This result is also used to get a simpler method for the decomposition of E -valued measures of finite variations as done in [4].

2. Main theorem

Now we come to the main theorem.

THEOREM 1. *Suppose $F: (X, \mathcal{T}_\rho) \rightarrow E''_\sigma$ is a bounded continuous mapping. Then there is a unique $A \in \mathcal{A}$ (in the sense that $A_1 = A_2$ if $\mu(A_1 \Delta A_2) = 0$) such that $F\chi_A$ has range in a closed separable subspace of E (so $F\chi_A: X \rightarrow E$ is strongly measurable) and for any $B \in \mathcal{A}$ with $\mu(B) > 0$ and $B \cap A = \emptyset$, $F\chi_B$ cannot be weakly equivalent to a strongly measurable function.*

P r o o f. The closed unit ball S of E'' is the intersection of closed cylindrical sets of E''_σ and so $F^{-1}(S)$ is $\mathcal{T}_\rho$ -closed. From this it follows that for any closed separable subspace $Q \subset E$, $F^{-1}(Q) \in \mathcal{A}$. Let $\mathcal{S}$ be the collection of all closed separable subspaces of E . Define an order in $\mathcal{S}$: $E_1 \geq E_2$ if $\rho(F^{-1}(E_1)) \supset \rho(F^{-1}(E_2))$. Let $\mathcal{P}$ is a totally ordered subset of $\mathcal{S}$ and let $c = \sup \mu(\{F^{-1}(P) : P \in \mathcal{P}\})$. Take an increasing sequence $\{P_n : n \in \mathbb{N}\} \subset \mathcal{P}$ such that $\sup \mu(F^{-1}(P_n)) = c$. Thus $P = \overline{\bigcup P_n}$ (closure in E) is a maximal element of $\mathcal{P}$. By Zorn lemma, $\mathcal{S}$ has a maximal element, say E_0 , a closed separable subspace of E . Let $A = F^{-1}(E_0)$. Then $F\chi_A: X \rightarrow E$ is strongly measurable function.

Take a $B \in \mathcal{A}$, $\mu(B) > 0$ such that $A \cap B = \emptyset$. Assume $F\chi_B$ is weakly equivalent to a strongly measurable function f_0 . Since f_0 is strongly measurable, by Egoroff's theorem, there is a $C \subset B$, $C \in \mathcal{A}$ with $\mu(C) > 0$, and a sequence of measurable simple functions $\{f_n: X \rightarrow (E, \|\cdot\|)\}$ such that $f_n \rightarrow f_0$ uniformly on C with norm topology on E . From this it follows that there is an absolutely convex compact subset $Q \subset E$ such that $f_0(C) \subset Q$. Put $f = f_0\chi_C$. Define $\bar{f}: X \rightarrow E''$, $\langle g, \bar{f}(x) \rangle = \rho(g \circ f)(x)$. This function $\bar{f}: (X, \mathcal{T}_\rho) \rightarrow E''_\sigma$ is easily seen to be well-defined and continuous. We claim $\bar{f}$ is Q -valued. If this is not true, there is an $x \in X$, a $g \in E'$ such that $Rl(\langle g, \bar{f}(x) \rangle) > \sup Rl g(Q) \geq \sup |g(f(X))|$. Since $\sup |(\rho(g \circ f))(X)| = \sup |((g \circ f))(X)|$, this is a contradiction.

Thus $F\chi_C \equiv \bar{f}$ and so $F\chi_{\rho(C)} \equiv \bar{f}$. Because of the continuity of $F\chi_{\rho(C)}$ and $\bar{f}$, we get $F\chi_{\rho(C)} = \bar{f}$. So the range of $F\chi_C$ is contained in a closed separable

subspace G_0 of E . Let $E_1 = \overline{\text{span}(E_0 \cup G_0)}$ (closure in E). Now $F^{-1}(E_1) \supset (A \cup C)$ and $\mu(A \cup C) = \mu(A) + \mu(C) > \mu(A)$, a contradiction.

To prove the uniqueness of A , suppose $A_1 \in \mathcal{A}$ also satisfies the conditions of the theorem and $\mu(A_1 \Delta A) > 0$. Suppose $\mu(A_1 \setminus A) > 0$. Put $B = A_1 \setminus A$. Then $\mu(B) > 0$ and the range of $F\chi_B$ is contained in a closed separable subspace of E . Thus $F\chi_B$ is weakly equivalent to a strongly measurable. This contradicts the definition of A . In a similar way, we can prove that $\mu(A \setminus A_1) > 0$ leads to a contradiction. $\square$

Remark 2. A can be obtained in another way which gives a more insight about its connection to E :

Since μ is τ -smooth, the measure μF^{-1} on E''_σ will also be τ -smooth. Because of boundedness of F , this will be a Radon measure. Since $(E, \sigma(E, E'))$ is universally measurable ([6, Theorem 3.4, p. 8]), E is measurable in E''_σ relative to the measure μF^{-1} . Thus $F^{-1}(E) = A_0 \in \mathcal{A}$. Put $A = A_0 \cap \rho(A_0)$. Thus A is an open set in X and $\mu(A_0 \Delta A) = 0$. If for some $g \in E'$, $g \circ F\chi_A = 0$, a.e. (μ), then, using the openness of A and continuity of F , one could easily verify that $g \circ F\chi_A = 0$ on X . From [2, Theorem 5, p. 390] it follows that range of $F\chi_A$ is contained in a separable subspace of E . This gives additional insight about A relative to E . Also $(F\chi_{A'})^{-1}(E) = A' \cap A_0 = A' \cap (A \cup Q) = A' \cap Q$ (Q a null set), is a closed null set and so is of first category (cf. [7, Definition 203, p. 430]).

The main result of [7] can be put in the following form.

COROLLARY 3. *Suppose $f: X \rightarrow E$ is a bounded weakly measurable function. Then there is an $A \in \mathcal{A}$ such that $f\chi_A$ is weakly equivalent to a strongly measurable function and for any $B \in \mathcal{A}$ with $B \cap A = \emptyset$ and $\mu(B) > 0$, $f\chi_B$ is not weakly equivalent to a strongly measurable function. This A is unique in the sense that if there is another A_1 with this property, then $\mu(A \Delta A_1) = 0$.*

Proof. Using liftings, we get a bounded, continuous $F: (X, \mathcal{T}_\rho) \rightarrow E''_\sigma$. Using Theorem 1, it is a routine verification. $\square$

Remark 4. This result is proved in [7, Theorem 2.5, p. 430] without the introduction of the set A . If we decompose $f = \chi_A f + \chi_{A'} f$, then trivially $(g \circ (\chi_A f))(g \circ (\chi_{A'} f)) = 0$ on X .

In [4] a result about the decomposition of vector measures with finite variations is proved. Using Theorem 1, this result comes easily.

COROLLARY 5. *Let $\nu: \mathcal{A} \rightarrow E$ be a measure of bounded variation. Putting $\mu = |\nu|$, we assume that $(X, \mathcal{A}, \mu)$ is a complete probability space. Then there is an $A \in \mathcal{A}$ such that $\chi_A \nu$ has RN derivative and for any $B \in \mathcal{A}$ with $B \cap A = \emptyset$ and $\mu(B) > 0$, $\chi_B \mu$ does not have RN derivative. This A is unique in the sense that if there is another A_1 with this property, then $\mu(A \Delta A_1) = 0$.*

P r o o f. Using liftings and the RN property of scalar-valued measures, we get a bounded, continuous $F: (X, \mathcal{T}_\rho) \rightarrow E''_\sigma$. Using Theorem 1, the proof is a routine procedure. $\square$

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