ON SOME NEW SUBFAMILIES OF CLASSICAL SPACES OF ABSOLUTELY p -SUMMABLE SEQUENCES

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ABSTRACT. In this paper the properties of some new subfamilies of the spaces $l^p(\mathbb{C})$ are discussed.

1. Introduction

By the first contact with the theory of real series, mathematicians usually get acquainted with the following basic properties of the harmonic series:

$$\begin{cases} \sum_{n=1}^{\infty} \frac{1}{n^x} < \infty & \text{if } x > 1, \\ \lim_{x \rightarrow 1^+} \left(\sum_{n=1}^{\infty} \frac{1}{n^x} \right) = \infty. \end{cases} \quad (1.1)$$

The same property is shared by many other known series $\sum_{n=1}^{\infty} \frac{1}{k_n}$, where $\{k_n\}$ is some subsequence (or its rearrangement) of $\{n\}_{n=1}^{\infty}$, for example: a series $\sum_{n=1}^{\infty} \frac{1}{p_n}$, where $\{p_n\}$ denotes a one-to-one sequence of all prime numbers.

The above property leads almost instantly to the following dual problem: is there any increasing sequence of natural numbers $\{k_n\}$ so that

$$\sum_{n=1}^{\infty} \frac{1}{k_n} < \infty \quad \text{while} \quad \sum_{n=1}^{\infty} \frac{1}{k_n^x} = \infty \quad (1.2)$$

for every $x < 1$?

The above problem is of a delicate nature. Surely, we have

$$\sum_{n=1}^{\infty} \frac{1}{a^{xn}} < \infty$$

for every $a > 1$ and $x > 0$ (i.e., the exponential sequences increase too fast to satisfy condition (1.2)). On the other hand, for sequence $k_n := \lfloor n \ln^y(n+2) \rfloor$, $n \in \mathbb{N}$ and fixed $y > 1$, identity (1.2) holds true (but the sequence $\{k_n\}$ is not strictly increasing). The applicable sequence $\{k_n\}$ should be designated in a more subtle manner. It shall be constructed from the elements of the following auxiliary finite sequences:

$$2^{2^n} + 1, 2^{2^n} + 2, \dots, 2^{2^n} + 2^{2^n - n},$$

where $n \in \mathbb{N}$. It should be noted that $2^{2^{n+1}} > 2^{2^n} + 2^{2^n - n}$,

$$\sum_{k=1}^{2^{2^n} - n} (2^{2^n} + k)^{-1} < \frac{2^{2^n - n}}{2^{2^n} + 1} < 2^{-n}$$

and

$$\sum_{k=1}^{2^{2^n} - n} (2^{2^n} + k)^{-1 + \frac{n}{2^n}} > \frac{2^{2^n - n}}{(2^{2^n} + 2^{2^n - n})^{(1 - \frac{n}{2^n})}} > \frac{2^{2^n - n}}{(2 \cdot 2^{2^n})^{(1 - \frac{n}{2^n})}} = 2^{-1 + \frac{n}{2^n}},$$

for every $n \in \mathbb{N}$. Accordingly, the sequence $\{k_n\}$ determined in the above manner is increasing and certainly satisfies condition (1.2).

Properties (1.1) and (1.2) may be extended to any sequences $\{k_n\}$ of complex numbers. Thus, for the given $p > 0$ we want to find out whether

$$\{k_n\} \in \left(\bigcap_{q>p} l^q \right) \setminus l^p \quad (1.3)$$

(the equivalent of property (1.1)), or, whether

$$\{k_n\} \in l^p \setminus \left(\bigcup_{0 < q < p} l^q \right) \quad (1.4)$$

(the equivalent of property (1.2)). It should be highlighted that if (1.3) holds, the following equality

$$\lim_{q \rightarrow p^+} \sum_{n=1}^{\infty} |k_n|^q = \sum_{n=1}^{\infty} |k_n|^p = \infty$$

must also hold. Hence, relation (1.3) is equivalent to

$$\{k_n\} \in \bigcap_{q>p} l^q \quad \text{and} \quad \lim_{q \rightarrow p^+} \sum_{n=1}^{\infty} |k_n|^q = \infty.$$

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Furthermore, it should be noted that $\{k_n\}$ does not have to meet property (1.3) or (1.4) for any $p > 0$. Nevertheless, it would be interesting to find out if a given sequence $\{k_n\} \subset \mathbb{C}$ satisfies one of the two properties.

In this paper, we consider the properties of the following subfamilies of the classical sequence space l^p (the first one we called **almost** l^p , the second one **exactly** l^p)

$$l^{>p} := \left(\bigcap_{q>p} l^q \right) \setminus l^p$$

and

$$l^{=p} := l^p \setminus \left(\bigcup_{0<q<p} l^q \right)$$

for every $p > 0$. It is our main aim.

We note that “almost all facts” about families $l^{>p}$ and $l^{=p}$, $p > 0$ could be transformed into families $l^{>1}$ and $l^{=1}$, after appropriate rescaling. Thus, our investigations were focused on the properties of families $l^{>1}$ and $l^{=1}$ only.

Typical elements of $l^{>1}$ and $l^{=1}$

There are some typical representatives of both families, $l^{>1}$ and $l^{=1}$:

$$\{k^{-1}\} \in l^{>1}, \quad \left\{ k^{-1-(\ln k)^{-\alpha}} \right\} \in l^{=1}, \quad \alpha > 0, \quad (1.5)$$

$$\left\{ k^{-1-(\ln k)^{-\alpha}} \right\} \in \begin{cases} l^{>1} & \text{if } \alpha \geq 1, \\ l^{=1} & \text{if } 0 < \alpha < 1, \end{cases} \quad (1.6)$$

and

$$\left\{ k^{-1-(\ln k)^{-\alpha}(\ln \ln k)^\beta} \right\} \in \begin{cases} l^{>1} & \text{if either } \alpha > 1 \quad \text{or } \alpha = 1 \wedge \beta \in (0, 1], \\ l^{=1} & \text{if either } 0 < \alpha < 1 \quad \text{or } \alpha = 1 \wedge \beta > 1. \end{cases} \quad (1.7)$$

Proof. Only proof of (1.6) will be given here (proofs of (1.5) and (1.7) run similarly). It is sufficient to consider the problem exactly for which values $\alpha > 0$ the series $\sum_{k=1}^{\infty} k^{-1-(\ln k)^{-\alpha}}$ is convergent. Let us fix $\alpha > 0$. There exists $N_0 \in \mathbb{N}$, such that the function $[N_0, +\infty) \ni x \mapsto x^{-1-(\ln x)^{-\alpha}}$ is decreasing. Hence, if $n \in \mathbb{N}$ and $e^n > N_0$, then we get

$$\sum_{k \in I_n} k^{-1-(\ln k)^{-\alpha}} \geq (e^{n+1} - e^n - 1) e^{-n-1-(n+1)^{1-\alpha}} \geq \left(1 - \frac{2}{e}\right) e^{-(n+1)^{1-\alpha}}$$

and

$$\sum_{k \in I_n} k^{-1-(\ln k)^{-\alpha}} \leq e^{n+1} e^{-n-n^{1-\alpha}} = e e^{-n^{1-\alpha}},$$

where $I_n := \mathbb{N} \cap (e^n, e^{n+1})$.

It follows from these inequalities that

$$\sum_{k=2}^{\infty} k^{-1-(\ln k)^{-\alpha}} < \infty \iff \sum_{n=1}^{\infty} e^{-n^{1-\alpha}} < \infty.$$

But we have

$$e^{-n^{1-\alpha}} \xrightarrow{n \rightarrow \infty} \begin{cases} 1 & \text{if } \alpha > 1, \\ e^{-1} & \text{if } \alpha = 1, \end{cases}$$

i.e.,

$$\sum_{k=2}^{\infty} k^{-1-(\ln k)^{-\alpha}} = \infty \quad \text{if } \alpha \geq 1.$$

On the other hand, if $\alpha \in (0, 1)$, then there is $r = r(\alpha) \in \mathbb{N}$ such that $(1-\alpha)r \geq 2$, which implies

$$\sum_{n=1}^{\infty} e^{-n^{1-\alpha}} = \sum_{n=1}^{\infty} \frac{1}{1 + n^{1-\alpha} + \frac{1}{2!}n^{2(1-\alpha)} + \dots + \frac{1}{r!}n^{r(1-\alpha)} + \dots} < \sum_{n=1}^{\infty} \frac{r!}{n^2} < \infty.$$

□

Remark. There are special convergence criteria that can deal with all series (1.5)–(1.7) easily (see Proposition 3.1.3 on page 85 in [8] or criteria (2) and (3) on page 45 in [3]).

Known facts

Let $\{p_n\}_{n=1}^{\infty} \subset \mathbb{R}$, $\lim_{n \rightarrow \infty} p_n = 1$. We define the following operator $\mathbf{p}$ acting from $l^1(\mathbb{C})$ into the space of all real sequences

$$\mathbf{p} \left(\left\{ \sum_{k=1}^n a_k \right\}_{n=1}^{\infty} \right) := \left\{ \sum_{k=1}^n |a_k|^{p_k} \right\}_{n=1}^{\infty}. \quad (1.8)$$

Then the following problem is worthy of consideration:

What additional conditions should the sequence $\{p_n\}_{n=1}^{\infty}$ satisfy to ensure that the inclusion $\mathbf{p}(l^1) \subset c$ holds (where c denotes the space of all convergent real sequences)?

The above problem was approached independently by several authors [2], [5], [7], [9], [10]. The following result was obtained by Biró in [2] and practically, it offers a satisfactory solution.

Let $\{p_n\}_{n=1}^{\infty} \subset \mathbb{R}$ be a strictly increasing sequence and let $\lim_{n \rightarrow \infty} p_n = 1$. Let us put

$$m(x) = \sum_{n=1}^{\infty} x^{\frac{p_n}{1-p_n}}, \quad x \in \mathbb{R}_+,$$

and

$$L := \limsup_{n \rightarrow \infty} n^{1-p_n}.$$

Then the following four conditions are equivalent:

- i) $\mathbf{p}(l^1) \subset c$ ($\mathbf{p}$ is defined in (1.8)).
- ii) There exists a positive x_0 such that $m(x_0) < \infty$.
- iii) $L < \infty$.
- iv) There is a constant α such that $p_n > 1 - \frac{\alpha}{\ln(n)}$ for every $n = 2, 3, \dots$

If these conditions are satisfied, then

$$\sup \{x > 0: m(x) < \infty\} = \frac{1}{L}.$$

For example, for the sequences

$$\left\{ \left(1 + \frac{1}{n} \right)^{-r} \right\}_{n \in \mathbb{N}}, \quad \left\{ \left(1 + \frac{1}{\ln n} \right)^{-r} \right\}_{n \in \mathbb{N}}, \quad \left\{ \left(1 + \frac{\ln n}{n} \right)^{-r} \right\}_{n \in \mathbb{N}},$$

where $r > 1$, we have $L < \infty$; whereas for the sequences

$$\left\{ \left(1 + \frac{\ln \ln n}{\ln n} \right)^{-r} \right\}_{n \in \mathbb{N}}, \quad \left\{ \left(1 + \frac{\ln \ln \ln n}{(\ln \ln n)^k} \right)^{-r} \right\}_{n \in \mathbb{N}},$$

where $k, r > 1$, we have $L = \infty$.

One of the conclusions from the theorem proven in [11] is as follows.

THEOREM 1.1. *Let $(\mathbb{X}, \|\cdot\|)$ be a normed linear space over $\mathbb{F} = \mathbb{R} \vee \mathbb{C}$. Let $\{\alpha_n\} \subset (0, \infty)$ be such that $\alpha_n \rightarrow 0$ and let $\{x_n\} \subset \mathbb{X}$ be a Cauchy sequence. If $\{x_{k_n}\}$ is a subsequence such that $\|x_{k_{n+1}} - x_{k_n}\| < \alpha_n$, $n \in \mathbb{N}$, then there exists a sequence $\{y_n\} \subset \mathbb{X}$ satisfying the conditions*

$$\|x_{k_n} - y_n\| \rightarrow 0$$

and

$$\|y_{n+1} - y_n\| = \alpha_n, \quad n \in \mathbb{N}.$$

Immediately from this theorem, the following proposition can be obtained:

PROPOSITION 1.2. *Let $(\mathbb{X}, \|\cdot\|)$ be an incomplete normed linear space. Then, for every positive real p there exist bounded divergent series $\{\sum_{k=1}^n x_k\}_{n=1}^\infty$ and $\{\sum_{k=1}^n y_k\}_{n=1}^\infty$, with $x_n, y_n \in \mathbb{X}$, such that*

$$\{\|x_n\|\} \in l^p \quad \text{and} \quad \{\|y_n\|\} \in l^{>p}.$$

Proof. Let $\{z_n\} \subset \mathbb{X}$ be a Cauchy nonconvergent sequence. Without loss of generality it can be assumed that

$$\|z_{n+1} - z_n\| < n^{-1-(\ln \ln n)^{-1}}, \quad n \in \mathbb{N}.$$

Then, by using Theorem 1.1 for the sequences

$$\{\alpha_n\} = \{n^{-1}\} \quad \text{and} \quad \{\beta_n\} = \left\{n^{-1-(\ln \ln n)^{-1}}\right\}$$

(see (1.5)), respectively, we obtain two sequences $\{x_n^*\} \subset \mathbb{X}$ and $\{y_n^*\} \subset \mathbb{X}$ such that

$$\|x_n^* - z_n\| \rightarrow 0, \quad \|y_n^* - z_n\| \rightarrow 0, \quad (1.9)$$

$$\|x_{n+1}^* - x_n^*\| = \alpha_n \quad \text{and} \quad \|y_{n+1}^* - y_n^*\| = \beta_n, \quad n \in \mathbb{N}. \quad (1.10)$$

It is sufficient to define

$$\begin{aligned} x_1 &= x_1^*, & x_{n+1} &= x_{n+1}^* - x_n^*, & n &\in \mathbb{N}, \\ y_1 &= y_1^*, & y_{n+1} &= y_{n+1}^* - y_n^*, & n &\in \mathbb{N}. \end{aligned}$$

Hence

$$\sum_{k=1}^n x_k = x_n^* \quad \text{and} \quad \sum_{k=1}^n y_k = y_n^*, \quad n \in \mathbb{N},$$

which, by (1.9) and the fact that $\{z_n\}$ is bounded and nonconvergent, implies that series $\{\sum_{k=1}^n x_k\}_{n=1}^\infty$ and $\{\sum_{k=1}^n y_k\}_{n=1}^\infty$ are also bounded and nonconvergent in $\mathbb{X}$. Moreover, from (1.10) we have

$$\{\|x_k\|\}_{k=2}^\infty = \{\alpha_k\}_{k=1}^\infty \quad \text{and} \quad \{\|y_k\|\}_{k=2}^\infty = \{\beta_k\}_{k=1}^\infty.$$

□

We note that, what is really essential in the proof of Proposition 1.2, both $l^{=p}$ and $l^{>p}$ are nonempty for every $p > 0$.

From this observation the next theorem follows (see Corollary 1 of [11]).

THEOREM 1.3. *Let $(\mathbb{X}, \|\cdot\|)$ be a nontrivial normed linear space. Then for every positive real p there exist convergent series $\sum x_n$, $\sum y_n$ and divergent series $\sum z_n$, where $x_n, y_n, z_n \in \mathbb{X}$ such that*

$$\{\|x_n\|\} \in l^{=p} \quad \text{and} \quad \{\|y_n\|\}, \{\|z_n\|\} \in l^{>p}.$$

Proof. Let x be a nonzero element of $\mathbb{X}$ and $p = 1$. It is sufficient to choose $x_n = x \beta_n$, $y_n = (-1)^n x \alpha_n$, $z_n = x \alpha_n$, for every $n \in \mathbb{N}$. □

Notations and terminology

The statement “ $\{a_n\}$ is a sequence” means that all elements of this sequence are positive real numbers. To simplify the notation we will use the bold printed letters **a**, **b**, **u**, ... to denote the sequences $\{a_n\}$, $\{b_n\}$, $\{u_n\}$, ..., respectively.

We introduce the following notation

$$l^{\geq p} = l^{>p} \cup l^{=p},$$

for every $p > 0$. The set $l^{\geq p}$ will be called **exmost** l^p (exactly l^p and almost l^p bringing together).

For any sequences $\{a_n\}$ and $\{b_n\}$ we write $\{b_n\} > \{a_n\}$ if $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$.

Let $\{a_n\}$ be a sequence. We write $\{a_n\} \uparrow 1$ ($\{a_n\} \uparrow\uparrow 1$) if the sequence $\{a_n\}$ is increasing (nondecreasing, respectively) to one. Similarly, the notation $\{a_n\} \downarrow 1$ ($\{a_n\} \downarrow\downarrow 1$) means that the sequence $\{a_n\}$ is decreasing (nonincreasing, respectively) to one.

Moreover, we will denote by $\mathbf{a}^{\mathbf{u}}$ the sequence $\{a_n^{u_n}\}_{n=1}^{\infty}$ for any sequences $\mathbf{a} := \{a_n\}$ and $\mathbf{u} := \{u_n\}$. In the sequel, the symbol $\mathbf{a}^u$ denotes the sequence $\{a_n^u\}_{n=1}^{\infty}$ for every sequence $\mathbf{a} = \{a_n\}$ and a positive real u .

Scope of research

In Section 2 we check when sequence $\{a_n\} \in l^{>p}$ contains a subsequence $\{a_{k(n)}\} \in l^{=p}$. Certain results will be next applied to analyse a series of iterated logarithms (see Remark 2.3). In Section 3 we prove that the relation $>$ is a density relation between the elements of $l^{=p}$ and $l^{>p}$. Finally, in the last section we investigate, among other problems, when, for the given sequence $\mathbf{a} = \{a_n\} \in l^{>p}$, there exists the sequence $\mathbf{x} = \{x_n\}$, $x_n \rightarrow 1$, so that: $\mathbf{a}^{\mathbf{x}} \in l^{=p}$, and vice versa.

2. Subsequences

THEOREM 2.1. *Let $\{a_n\}$ be a sequence of complex numbers convergent to zero. If $\sum |a_n| = \infty$ then, for every $q \in (0, 1]$, there exist subsequences $\{a_{r(n)}\}$ and $\{a_{s(n)}\}$ of $\{a_n\}$ such that*

$$\{a_{r(n)}\} \in l^{>q} \quad \text{and} \quad \{a_{s(n)}\} \in l^{=q}. \quad (2.1)$$

Proof. Since the condition $\sum |a_n| = \infty$ implies $\sum |a_n|^q = \infty$ for every $q \in (0, 1]$, it is sufficient to consider only the case $q = 1$.

Let us choose two increasing sequences $\{u(n)\}$ and $\{v(n)\}$ of positive integers satisfying the conditions

$$u(n) < v(n) < u(n+1), \quad (2.2)$$

$$1 \leq \sum_{i=u(n)}^{v(n)} |a_i| < 2 \quad (2.3)$$

and

$$|a_i| < 2^{-n} \quad \text{for } i \geq u(n), \quad (2.4)$$

for every $n \in \mathbb{N}$. Then, for any $k, n \in \mathbb{N}$ we have

$$\sum_{i=u(n)}^{v(n)} |a_i|^{(k+1)/k} \leq 2^{-n/k} \sum_{i=u(n)}^{v(n)} |a_i| \leq 2^{(-n/k)+1}$$

and since the series $\sum_{n=1}^{\infty} 2^{-n/k}$ is convergent for every $k \in \mathbb{N}$, the sequence $\{r(n)\}$, defined to be an increasing sequence of all members of the set

$$\bigcup_{n=1}^{\infty} \{u(n), 1 + u(n), \dots, v(n)\},$$

has the desired properties.

Now we choose two increasing sequences $\{t(n)\}$ and $\{w(n)\}$ of positive integers for which the following conditions hold true:

$$t(n) < w(n) < t(n+1), \quad (2.5)$$

$$1 \leq \sum_{i=t(n)}^{w(n)} |a_i|^{(n-1)/n} < 2 \quad (2.6)$$

and

$$|a_i| < 2^{-n^2} \quad \text{for } i \geq t(n), \quad (2.7)$$

for every $n \in \mathbb{N}$. Then, for any $k, n \in \mathbb{N}$ we have

$$\sum_{i=t(n)}^{w(n)} |a_i|^{(n-1)/n} > 2^n \sum_{i=t(n)}^{w(n)} |a_i|, \quad (2.8)$$

which implies

$$\sum_{i=t(n)}^{w(n)} |a_i| < 2^{1-n}. \quad (2.9)$$

Let $\{s(n)\}$ be an increasing sequence of all elements of the set

$$\bigcup_{n=1}^{\infty} \{t(n), 1 + t(n), \dots, w(n)\}.$$

Using (2.6) and (2.9) we deduce that $\{a_{s(n)}\} \in l^1$, as required. $\square$

THEOREM 2.2. *Let $\mathbb{K}$ be a nonempty subset of $\mathbb{N}$, $\{a_n^{(k)}\}_n, \{b_n\} \in l^{\geq 1}$ and $\{a_n^{(k)}\}_n > \{b_n\}$, for every $k \in \mathbb{K}$. If $\{b_n\} \in l^{>1}$, then there exists an increasing sequence $\{t(n)\}$ of positive integers such that*

$$\{a_{t(n)}^{(k)}\}_n \in l^{>1}, \quad k \in \mathbb{K}$$

and

$$\{b_{t(n)}\} \in l^1.$$

Proof. Only the case $\mathbb{K} = \mathbb{N}$ will be considered (if $\mathbb{K}$ is finite, it can be defined as $\{a_n^{(k)}\}_n = \{a_n^{(k+1)}\}_n$ for sufficiently large $k \in \mathbb{N}$).

Let $\{r(n)\}$ and $\{s(n)\}$ be increasing sequences of positive integers chosen in such a way that

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$$r(i) \leq s(i) \leq r(i+1), \quad (2.10)$$

$$b_n \leq \min \left\{ 2^{-i} a_n^{(k)} : k = 1, \dots, i \right\} \quad (2.11)$$

for $n \geq r(i)$,

$$\sum_{n=r(i)}^{s(i)} a_n^{(k)} \geq 1 \quad \text{and} \quad \sum_{n=r(i)}^{s(i)} b_n^{u_i} \geq 1 \quad (2.12)$$

for every $k = 1, \dots, i$.

Either way, there exists $k \in \{1, \dots, i\}$ such that

$$\sum_{n=r(i)}^{s(i)} a_n^{(k)} < 2 \quad \text{or} \quad \sum_{n=r(i)}^{s(i)} b_n^{u_i} < 2$$

and simultaneously,

$$b_n^{1/i} < 2^{-i} \quad \text{for} \quad n \geq r(i), \quad (2.13)$$

for every index $i \in \mathbb{N}$, where $u_i := (i-1)/i$.

If the first condition of (2.13) is fulfilled then, by (2.11), we have

$$\sum_{n=r(i)}^{s(i)} b_n < 2^{-i+1}. \quad (2.14)$$

On the other hand, if the second condition of (2.13) holds true, we have

$$2 > \sum_{n=r(i)}^{s(i)} b_n^{u_i} > 2^i \sum_{n=r(i)}^{s(i)} b_n,$$

i.e., the inequality (2.14) is satisfied again. Therefore, we get

$$\sum_{i=1}^{\infty} \sum_{n=r(i)}^{s(i)} b_n < \infty. \quad (2.15)$$

Let $\{t(n)\}$ be an increasing sequence of all elements of the set

$$\bigcup_{i=1}^{\infty} \{r(i), 1+r(i), \dots, s(i)\}.$$

Then from (2.12) and (2.15) it may be concluded that $\{a_{t(n)}^{(k)}\}_n \in l^{>1}$, $k \in \mathbb{N}$, and $\{b_{t(n)}\} \in l^1$, as claimed. $\square$

Remark 2.3. There is a nice application of Theorem 2.2. First, we need to establish the notation. Let $b \in (1, +\infty)$. Let us set

$$\begin{aligned} a_0(b) &= 1, & a_{n+1}(b) &= b^{a_n(b)}, \\ \log_b^{(0)} x &= x, & \log_b^{(n+1)} x &= \log_b \left(\log_b^{(n)} x \right) \end{aligned}$$

and

$$L_b^n(x) = \left(\prod_{k=0}^n \log_b^{(k)} x \right)^{-1},$$

for every $n \in \mathbb{N}_0$ and $x \in [a_n(b), +\infty)$.

It may be proved that the sequence $\{a_n(b)\}_n$ is always increasing. Moreover, the limit $\lim_{n \rightarrow \infty} a_n(b)$ is finite if and only if $1 < b \leq \sqrt[e]{e}$. Hence, for $b > \sqrt[e]{e}$ and $k \in \mathbb{N}$ there exists $\max\{n \in \mathbb{N} : k \geq a_n(b)\}$; we denote this maximum by $m_{k,b}$. Then we have the following exciting result (see [1], [4], [6]):

$$\{L_b^{m_{k,b}}(k)\}_k \in l^{>1} \quad \text{if and only if} \quad b \geq e.$$

Since

$$\{L_b^s(k)\}_k > \{L_b^{s+1}(k)\}_k > \{L_b^{m_{k,b}}(k)\}_k,$$

for every $b > \sqrt[e]{e}$ and for every $s \in \mathbb{N}$, then from Theorem 2.2 it follows that, for every pair of the indices b and s , there exist increasing sequences $\{p_n\}$ and $\{q_n\}$ of positive integers such that

$$\{L_b^s(p_n)\}_n \in l^{>1} \quad \text{and} \quad \{L_b^{s+1}(p_n)\}_n \in l^{=1},$$

$$\{L_b^t(q_n)\}_n \in l^{>1} \quad \text{for every} \quad t \in \mathbb{N}$$

and

$$\{L_b^{m_{q_n,b}}(q_n)\}_n \in l^{=1}.$$

THEOREM 2.4. Let $\mathbf{a} = \{a_n\}$ and $\mathbf{b} = \{b_n\}$ belong to $l^{\geq 1}$. If $\mathbf{b} > \mathbf{a}$, and simultaneously $\mathbf{a}^u > \mathbf{b}$ for every $u \in (0, 1)$, then for any increasing sequence $\{r_n\}$ of positive integers the following implication holds:

$$\text{if } \{b_{r_n}\} \in l^{=1}, \quad \text{then } \{a_{r_n}\} \in l^{=1}.$$

Proof. Let $\{r_n\}$ be an increasing sequence of positive integers such that $\{b_{r_n}\} \in l^{=1}$. Then, by hypothesis, we have

$$\{a_{r_n}^{uv}\}_n > \{b_{r_n}^v\}_n,$$

for any $u, v \in (0, 1)$, which implies a divergence of the series $\sum_{n=1}^{\infty} a_{r_n}^u$ for every $u \in (0, 1)$. This proves, by the relation $\{b_n\} > \{a_n\}$, that $\{a_{r_n}\} \in l^{=1}$. $\square$

3. Density

THEOREM 3.1. *Let $\{b_n\} \in l^{\geq 1}$. Then there exist sequences $\{a_n\} \in l^{>1}$ and $\{c_n\} \in l^{=1}$ such that*

$$\{a_n\} > \{b_n\} > \{c_n\}.$$

Proof. Case $\{b_n\} \in l^{>1}$. By Theorem 4.5 (ii) (for $f_k(x) = x$, $x \in \mathbb{R}$, $k \in \mathbb{N}$), there exists a sequence $\{u_n\} \downarrow 1$ such that

$$\{b_n^{u_n}\} \in l^{=1}$$

and

$$\lim_{n \rightarrow \infty} b_n^{u_n-1} = 0.$$

Moreover, there exists a sequence $\{v_n\} \uparrow 1$ such that

$$\lim_{n \rightarrow \infty} b_n^{1-v_n} = 0.$$

Indeed, since for every $x > 0$, we have

$$\lim_{n \rightarrow \infty} b_n^x = 0,$$

so the following numbers are correctly determined

$$\widehat{N}_k := \min \left\{ l \in \mathbb{N} : b_n^{1-k^{-1}} \leq k^{-1} \text{ for every } n \geq l \right\},$$

for every $k \in \mathbb{N}$. We define

$$v_n = \begin{cases} \frac{1}{2}, & 1 \leq n < N_1, \\ k^{-1}, & N_k \leq n < N_{k+1}, \quad k = 2, 3, \dots, \end{cases}$$

where $N_1 := \max\{2, \widehat{N}_2\}$, $N_{k+1} := \max\{1 + N_k, \widehat{N}_{k+2}\}$.

It is sufficient to define

$$a_n = b_n^{v_n} \quad \text{and} \quad c_n = b_n^{u_n}, \quad n \in \mathbb{N}.$$

Case $\{b_n\} \in l^{=1}$. By Theorem 4.5 (iv) (for $f_k(x) = x$, $x \in \mathbb{R}$, $k \in \mathbb{N}$), there exists a sequence $\{v_n\} \uparrow 1$ such that $\{b_n^{v_n}\} \in l^{>1}$ and

$$\lim_{n \rightarrow \infty} b_n^{1-v_n} = 0.$$

Furthermore, there exists a sequence $\{u_n\} \downarrow 1$ such that

$$\lim_{n \rightarrow \infty} b_n^{u_n-1} = 0.$$

The elements a_n and c_n , $n \in \mathbb{N}$, are defined as above. □

THEOREM 3.2. *Let $\mathbf{a}_k = \{a_n^{(k)}\}_n, \mathbf{b}_k = \{b_n^{(k)}\}_n \in l^{\geq 1}, k \in \mathbb{N}$, and let*

$$\left\{a_n^{(1)}\right\}_n > \cdots > \left\{a_n^{(k)}\right\}_n > \cdots > \left\{b_n^{(k)}\right\}_n > \cdots > \left\{b_n^{(1)}\right\}_n.$$

Then there exists a sequence $\{a_n\} \in l^{\geq 1}$ such that

$$\left\{a_n^{(k)}\right\}_n > \{a_n\} > \left\{b_n^{(k)}\right\}_n,$$

for every $k \in \mathbb{N}$.

Furthermore, if $\{a_n^{(k)}\}_n \in l^{>1}$ and $\{b_n^{(k)}\}_n \in l^{=1}$, for every $k \in \mathbb{N}$, then there exist sequences $\{a_n\} \in l^{>1}$ and $\{b_n\} \in l^{=1}$ satisfying

$$\left\{a_n^{(k)}\right\}_n > \{a_n\} > \{b_n\} > \left\{b_n^{(k)}\right\}_n,$$

for every $k \in \mathbb{N}$.

Proof. Only the case $\mathbf{a}_k \in l^{>1}$ and $\mathbf{b}_k \in l^{=1}, k \in \mathbb{N}$, will be considered. Then an increasing sequence $\{u(n)\}$ of positive integers could be found such that

$$a_i^{(k+1)}/a_i^{(k)} \leq 1,$$

$$b_i^{(k)}/b_i^{(k+1)} \leq 1$$

and

$$b_i^{(n)}/a_i^{(n)} < 2^{-3n},$$

for $k = 1, \dots, n-1, i \geq u(n)$ and $n \in \mathbb{N}$. Further, it could be assumed that

$$\sum_{k=u(n)}^{u(n+1)-1} a_i^{(n)} \geq 2^n \quad \text{and} \quad \sum_{i=u(n)}^{\infty} b_i^{(n)} < 2^{-2n},$$

for every $n \in \mathbb{N}$ and $a_i^{(1)} < 1$ for $i \geq u(1)$.

Now let us set

$$a_i = 2^{-n} a_i^{(n)} \quad \text{and} \quad b_i = 2^n b_i^{(n)},$$

for $u(n) \leq i < u(n+1), n \in \mathbb{N}$. By appealing to the above assumptions, the following relations could be obtained

$$a_i/a_i^{(k)} \leq a_i/a_i^{(n)} = 2^{-n},$$

$$b_i^{(k)}/b_i \leq b_i^{(n)}/b_i = 2^{-n},$$

$$b_i/a_i = 2^{2n} b_i^{(n)}/a_i^{(n)} \leq 2^{-n},$$

$$\sum_{k=u(n)}^{u(n+1)-1} a_k \geq 1 \quad \text{and} \quad \sum_{k=u(n)}^{u(n+1)-1} b_k \leq 2^{-n},$$

for every $n \in \mathbb{N}$. This implies $\{a_n\} \in l^{>1}$ and $\{b_n\} \in l^{=1}$, as claimed. $\square$

COROLLARY 3.3. *Let $\{a_n\}, \{d_n\} \in l^{\geq 1}$, $\{a_n\} > \{d_n\}$. Then there exists a sequence $\{b_n\} \in l^{\geq 1}$ such that*

$$\{a_n\} > \{b_n\} > \{d_n\}.$$

Additionally if

$$\{a_n\} \in l^{>1} \quad \text{and} \quad \{d_n\} \in l^{=1},$$

then there exist sequences

$$\{b_n\} \in l^{>1} \quad \text{and} \quad \{c_n\} \in l^{=1}$$

such that

$$\{a_n\} > \{b_n\} > \{c_n\} > \{d_n\}.$$

THEOREM 3.4. *Let $\{a_n^{(k)}\}_n \in l^{\geq 1}$, $k \in \mathbb{N}$. If $\{a_n^{(k)}\}_n > \{a_n^{(k+1)}\}_n$, $k \in \mathbb{N}$, then there exists a sequence $\{a_n\} \in l^{\geq 1}$ such that*

$$\{a_n^{(k)}\}_n > \{a_n\}, \quad k \in \mathbb{N}.$$

Moreover, if all the sequences $\{a_n^{(k)}\}_n$, $k \in \mathbb{N}$, belong to $l^{>1}$, then we may also obtain $\{a_n\} \in l^{>1}$.

If

$$\{a_n^{(k+1)}\}_n > \{a_n^{(k)}\}_n, \quad k \in \mathbb{N},$$

then there exists a sequence

$$\{a_n\} \in l^{=1}$$

such that

$$\{a_n\} > \{a_n^{(k)}\}_n, \quad k \in \mathbb{N}.$$

We can also obtain $\{a_n\} \in l^{=1}$ whenever $\{a_n^{(k)}\}_n \in l^{=1}$, for every $k \in \mathbb{N}$.

P r o o f. We will consider four cases. The scheme of the proof is the same in each case. First we define an auxiliary sequence $\{u(n)\}$ of positive integers and then we obtain the desired sequence $\{a_n\}$ by setting

$$a_i = 2^{-n} a_i^{(n)}, \quad u(n) \leq i < u(n+1), \quad n \in \mathbb{N}.$$

1. Case when $\{a_n^{(k)}\}_n > \{a_n^{(k+1)}\}_n$ and $\{a_n^{(k)}\}_n \in l^{>1}$, $k \in \mathbb{N}$.

A sequence $\{u(n)\}$ is defined in such a way that the following conditions hold true:

$$a_i^{(k+1)} / a_i^{(k)} < 1, \tag{3.1}$$

for $k = 1, 2, \dots, n$ and $i \geq u(n)$, and

$$\sum_{i=u(n)}^{u(n+1)-1} a_i^{(n)} \geq 2^n, \tag{3.2}$$

both of which hold for every $n \in \mathbb{N}$.

2. Case when $\{a_n^{(k)}\}_n > \{a_n^{(k+1)}\}_n$ and $\{a_n^{(k)}\}_n \in l^=1$, $k \in \mathbb{N}$.

We choose a sequence $\{u(n)\}$ satisfying condition (3.1) and the following one:

$$\sum_{i=u(n)}^{u(n+1)-1} \left(a_i^{(n)}\right)^{(n-1)/n} \geq 2^n. \quad (3.3)$$

3. Case when $\{a_n^{(k+1)}\}_n > \{a_n^{(k)}\}_n$ and $\{a_n^{(k)}\}_n \in l^=1$, $k \in \mathbb{N}$.

Then we can define a sequence $\{u(n)\}$ which satisfies the following two conditions:

$$a_i^{(k)} / a_i^{(k+1)} < 1, \quad (3.4)$$

for $k = 1, 2, \dots, n$ and $i \geq u(n)$, and

$$\sum_{i=u(n)}^{\infty} a_i^{(n)} \leq 2^{-2n}, \quad (3.5)$$

for $n \in \mathbb{N}$.

4. Case when $\{a_n^{(k+1)}\}_n > \{a_n^{(k)}\}_n$ and $\{a_n^{(k)}\}_n \in l^{>1}$, $k \in \mathbb{N}$.

A sequence $\{u(n)\}$ should satisfy condition (3.4) and the following one:

$$\sum_{i=u(n)}^{\infty} \left(a_i^{(n)}\right)^{(n+1)/n} \leq 2^{-2n}, \quad (3.6)$$

for $n \in \mathbb{N}$. □

THEOREM 3.5. Let $\{a_n^{(k)}\}_n$, $k \in \mathbb{N}$, and $\{a_n\}$ all belong to $l^{\geq 1}$. If

$$\left\{a_n^{(k)}\right\}_n > \left\{a_n^{(k+1)}\right\}_n > \{a_n\}, \quad k \in \mathbb{N},$$

then there exists a sequence $\{b_n\} \in l^{\geq 1}$ such that

$$\left\{a_n^{(k)}\right\}_n > \{b_n\} > \{a_n\}, \quad k \in \mathbb{N}.$$

On the other hand, if

$$\{a_n\} > \left\{a_n^{(k+1)}\right\}_n > \left\{a_n^{(k)}\right\}_n, \quad k \in \mathbb{N},$$

then there exists a sequence $\{c_n\} \in l^{\geq 1}$ such that

$$\{a_n\} > \{c_n\} > \left\{a_n^{(k)}\right\}_n, \quad k \in \mathbb{N}.$$

Proof. This proof is simple and will be omitted here. □

4. Exponents

The following (technical and simultaneously basic) theorem is a consequence of the fact that functions $x \mapsto a^x$, where $a > 0$, are continuous on $\mathbb{R}$.

THEOREM 4.1. *Let $\mathbf{a} = \{a_n\}_{n \in \mathbb{N}} \in l^{\geq 1}$. Then the function*

$$(1, \infty) \ni u \mapsto \sum_{n=1}^{\infty} a_n^u$$

is continuous.

Now it will be given a series of results concerning various aspects of raising the elements from exmost l^p to the power.

THEOREM 4.2. *Let $\mathbf{a}_s = \{a_{n,s}\}$ and $\mathbf{b}_t = \{b_{n,t}\}$ belong to $l^{\geq 1}$ and let $\mathbf{b}_t > \mathbf{a}_s$ for $s \in S$ and $t \in T$, where both S and T are non-empty subsets of $\mathbb{N}$.*

- a) *If all sequences $\mathbf{a}_s$, $s \in S$ belong to $l^{>1}$, then there exists a sequence $\mathbf{x} = \{x_n\}$, $x_n \rightarrow 1$, such that $\mathbf{a}_s^{\mathbf{x}} \in l^=1$ and $\mathbf{b}_t^{\mathbf{x}} \in l^{>1}$ for $s \in S$ and $t \in T$.*
- b) *If $\mathbf{b}_t \in l^=1$ for every $t \in T$, then there exists a sequence $\mathbf{y} = \{y_n\}$, $y_n \rightarrow 1$, such that $\mathbf{a}_s^{\mathbf{y}} \in l^=1$ and $\mathbf{b}_t^{\mathbf{y}} \in l^{>1}$ for $s \in S$ and $t \in T$.*

Proof. a) There is no loss of generality in assuming $S = T = \mathbb{N}$. Let us start with choosing an increasing sequence $\{r(n)\}$ of positive integers such that

$$a_{k,s} < 2^{-n} b_{k,t}, \quad k \geq r(n) \quad (4.1)$$

and

$$\sum_{k=r(n)}^{r(n+1)-1} a_{k,t} > 1 > \sum_{k=r(n)}^{r(n+1)-1} b_{k,t}^{(n+1)/n}, \quad (4.2)$$

for any $s, t \in \{1, 2, \dots, n\}$ and $n \in \mathbb{N}$.

Next, we define an auxiliary sequence $\{v(n)\}$ of positive reals satisfying the following properties:

$$v(n) \in (1, (n+1)/n), \quad v(n) < v(k), \quad k = 1, 2, \dots, n-1, \quad (4.3)$$

and

$$\sum_{k=r(n)}^{r(n+1)-1} b_{k,t}^{v(n)} > 1, \quad (4.4)$$

for $t \in \{1, 2, \dots, n\}$. It follows from (4.3) that $\{v(n)\} \downarrow 1$.

We note that, by (4.1) and (4.2)

$$\sum_{k=r(n)}^{r(n+1)-1} a_{k,s}^{(n+1)/n} < 2^{-n},$$

for any $s \in \{1, 2, \dots, n\}$ and $n \in \mathbb{N}$.

Next, by continuity of the function

$$[1, \infty) \ni u \mapsto \sum_{k=r(n)}^{r(n+1)-1} a_{k,s}^u,$$

for $n, s \in \mathbb{N}$, it is possible to define an auxiliary sequence $\{u(n)\}_{n \in \mathbb{N}}$ of positive reals in such a way that

$$u(n) \in (1, 1 + n^{-1}), \quad \max \left\{ \sum_{k=r(n)}^{r(n+1)-1} a_{k,s}^{u(n)} : 1 \leq s \leq n \right\} = 2^{-n},$$

for every $n \in \mathbb{N}$. Then $u(n) \rightarrow 1$ and, by (4.1), we get

$$\sum_{k=r(n)}^{r(n+1)-1} b_{k,t}^{u(n)} > 2^{(u(n)-1)n} \geq 1,$$

for every $t \in \{1, 2, \dots, n\}$ and $n \in \mathbb{N}$.

So, it is sufficient to define

$$x_k := u(n) \quad \text{for } r(n) \leq k < r(n+1), \quad n \in \mathbb{N},$$

and

$$x_k := 1 + u(1) \quad \text{for } 1 \leq k < r(1) \quad \text{if } r(1) > 1,$$

which simultaneously concludes the proof. $\square$

LEMMA 4.3. *If $\{a_n\} \in l^{>1}$, then for any sequence $\{u_n\} \uparrow 1$ of positive reals we have $\{a_n^{u_n}\} \in l^{>1}$. On the other hand, if $\{a_n\} \in l^{=1}$, then for any sequence $\{u_n\} \downarrow 1$ we have $\{a_n^{u_n}\} \in l^{=1}$.*

COROLLARY 4.4. *For any sequence $\{u_n\} \uparrow 1$ of positive reals there exists a sequence $\{a_n\} \in l^{=1}$ such that $\{a_n^{u_n}\} \in l^{=1}$. For any sequence $\{u_n\} \downarrow 1$ of positive reals there exists a sequence $\{a_n\} \in l^{>1}$ such that $\{a_n^{u_n}\} \in l^{>1}$.*

Proof. If $\{u_n\} \uparrow 1$, then $\{1/u_n\} \downarrow 1$ and $\{b_n^{1/u_n}\} \in l^{=1}$, whenever $\{b_n\} \in l^{=1}$. Hence,

$$\{a_n^{u_n}\} \in l^{=1} \quad \text{for } a_n = b_n^{1/u_n}, n \in \mathbb{N}.$$

$\square$

THEOREM 4.5. *Let $\{a_n\} \in l^{\geq 1}$. Let $f_k: \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function at the point 1 and $f_k(1) = 1$ for every $k \in \mathbb{N}$.*

(i) *If $\{a_n\} \in l^{>1}$, then there exists a sequence $\{u_n\} \downarrow 1$ such that*

$$\left\{ a_n^{f_k(u_n)} : n \in \mathbb{N} \right\} \in l^{>1},$$

for every $k \in \mathbb{N}$.

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- (ii) Suppose that there exists a sequence $\{u_n\} \downarrow 1$ such that the inequality $f_k(u_n) > 1$ holds for any $k, n \in \mathbb{N}$ with $n \geq k$. If $\{a_n\} \in l^{>1}$, then there exists a sequence $\{v_n\} \downarrow 1$ such that

$$\left\{a_n^{f_k(v_n)} : n \in \mathbb{N}\right\} \in l^{=1}$$

and

$$\lim_{n \rightarrow \infty} a_n^{f_k(v_n)-1} = 0,$$

for every $k \in \mathbb{N}$. If, in addition, $f_k(x)$ is continuous at every point $x \in \{u_n : n \geq k\}$ for some $k \in \mathbb{N}$, we can also prove that $\{v_n\} \downarrow 1$.

- (iii) If $\{a_n\} \in l^{=1}$, then there exists a sequence $\{u_n\} \uparrow 1$ such that

$$\left\{a_n^{f_k(u_n)} : n \in \mathbb{N}\right\} \in l^{=1},$$

for every $k \in \mathbb{N}$.

- (iv) Suppose that there exists a sequence $\{u_n\} \uparrow 1$ such that the inequality $f_k(u_n) < 1$ holds for every $k, n \in \mathbb{N}$ with $n \geq k$. If $\{a_n\} \in l^{=1}$, then there exists a sequence $\{w_n\} \uparrow 1$ such that

$$\left\{a_n^{f_k(w_n)} : n \in \mathbb{N}\right\} \in l^{>1}$$

and

$$\lim_{n \rightarrow \infty} a_n^{1-f_k(w_n)} = 0,$$

for every $k \in \mathbb{N}$. Moreover, if $f_k(x)$ is continuous at every point $x \in \{u_n : n \geq k\}$ for some $k \in \mathbb{N}$, then we have $\{w_n\} \uparrow 1$.

Proof. First, let us suppose that $\{a_n\} \in l^{>1}$. Then $\sum a_n = \infty$ and by the continuity at the point 1 of all functions $f_k, k \in \mathbb{N}$, we can define an increasing sequence $\{r(n)\}$ of positive integers and a sequence $\{q(n)\} \downarrow 1$ such that

$$\sum_{i=r(n)}^{r(n+1)-1} a_i^{f_k(u)} \geq 1,$$

for every $u \in [1, q(n)]$ and for every $k = 1, 2, \dots, n, n \in \mathbb{N}$. The desired sequence $\{u_i\}$ will be defined in the following way:

$$u_i = q(n) - \frac{i - r(n)}{r(n+1) - r(n)}(q(n) - q(n+1)) \quad (4.5)$$

for every $i = r(n), r(n) + 1, \dots, r(n+1) - 1$, $n \in \mathbb{N}$, and

$$u_i = r(1) + q(1) - i$$

for $i = 1, \dots, r(1) - 1$.

Now suppose that the first assumption of (ii) holds. It is not hard to define an increasing sequence $\{r(n)\}$ of positive integers such that the following inequalities hold true:

$$\sum_{i=r(n)}^{\infty} a_i^{f_k(u_n)} \leq 2^{-n} \quad (4.6)$$

for $k = 1, 2, \dots, n$, and

$$a_i^{f_k(u_n)-1} \leq n^{-1} \quad (4.7)$$

for $i \geq r(n)$, for every $n \in \mathbb{N}$. Then we set $v_i = u_n$ for positive integers i satisfying $r(n) \leq i < r(n+1)$ and for every $n \in \mathbb{N}_0$ where $r(0) = 1$. Since

$$\lim_{n \rightarrow \infty} f_k(u_n) = 1$$

for every $k \in \mathbb{N}$, by (4.6) we get

$$\left\{ a_n^{f_k(v_n)} : n \in \mathbb{N} \right\} \in l^1.$$

Additionally if the second assumption (ii) holds true, then we may choose a sequence $\{r(n)\}$ in such a way that the inequalities (4.6) and (4.7) hold for each $f_k(x)$ with $x \in [u_n - \varepsilon(n), u_n + \varepsilon(n)]$ instead of $f_k(u_n)$, where $\{\varepsilon(n)\}$ is a sequence of positive reals satisfying

$$u_{n+1} + \varepsilon(n+1) < u_n - \varepsilon(n), \quad n \in \mathbb{N}.$$

Then we define

$$v(i) = u_n + \varepsilon(n) - \frac{i - r(n)}{r(n+1) - r(n)} 2\varepsilon(n)$$

for $i = r(n), r(n) + 1, \dots, r(n+1) - 1$ and for $n \in \mathbb{N}$.

Now let $\{a_n\} \in l^1$. Then $\sum a_n < \infty$ and by the continuity at the point 1 of all functions f_k , $k \in \mathbb{N}$, we can construct an increasing sequence $\{r(n)\}$ of positive integers and a sequence $\{q(n)\}$ such that

$$\sum_{i=r(n)}^{r(n+1)-1} a_i^{f_k(u)} < 2^{-n} + \sum_{i=r(n)}^{r(n+1)-1} a_i,$$

for every $u \in [q(n), 1]$, $k = 1, 2, \dots, n$, and $n \in \mathbb{N}$. The sequence $\{u_n\}$ is given by (4.5). □

THEOREM 4.6.

- (i) Let $\{a_n\} \in l^{>1}$ and let $f_n: \mathbb{R} \rightarrow \mathbb{R}$, $n \in \mathbb{N}$, be functions such that $f_{n+1}(x) \leq f_n(x)$ for $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Suppose that there exist sequences $\{\varepsilon_n\}$ and $\{d(n)\}$ such that $[1, 1 + d(n)] \subset \text{cl}(f_n([1, 1 + \varepsilon_n]))$ and $\lim_{n \rightarrow \infty} \varepsilon_n = 0$. Then there exists a sequence $\{u_n\}$ with $u_n > 1$, which is convergent to one and such that

$$\{a_n^{f_k(u_n)} : n \in \mathbb{N}\} \in l^{>1}$$

and

$$\lim_{n \rightarrow \infty} a_n^{f_k(u_n)-1} = 0,$$

for every $k \in \mathbb{N}$.

- (ii) Let $\{a_n\} \in l^{=1}$ and let $f_n: \mathbb{R} \rightarrow \mathbb{R}$, $n \in \mathbb{N}$, be functions such that $f_n(x) \leq f_{n+1}(x)$ for $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Suppose that there exist sequences $\{\varepsilon_n\}$ and $\{d(n)\} \subset (0, 1)$ such that $[1 - d(n), 1] \subset \text{cl}(f_n([1 - \varepsilon_n, 1]))$ and $\lim_{n \rightarrow \infty} \varepsilon_n = 0$. Then there exists a sequence $\{v_n\} \subset (0, 1)$ convergent to one and such that

$$\{a_n^{f_k(v_n)} : n \in \mathbb{N}\} \in l^{=1}$$

and

$$\lim_{n \rightarrow \infty} a_n^{1-f_k(v_n)} = 0,$$

for every $k \in \mathbb{N}$.

P r o o f. (i) Let us define an increasing sequence $\{r(n)\}$ of positive integers such that

$$na_i^{d(n)} < 1, \quad i \geq r(n) \quad (4.8)$$

and

$$\sum_{i=r(n)}^{r(n+1)-1} a_i \geq 1, \quad (4.9)$$

for every $n \in \mathbb{N}$. Then, by (4.8) and by the inclusion

$$[1, 1 + d(n)] \subset \text{cl}(f_n([1, 1 + \varepsilon_n])), \quad n \in \mathbb{N},$$

for every positive integer i satisfying $r(n) \leq i < r(n+1)$ there exist $u_i \in [1, 1 + \varepsilon_n]$ and $\xi_i \in [\frac{1}{2}, 2]$ such that

$$f_n(u_i) \in [1, 1 + d_n] \quad \text{and} \quad \xi_i na_i^{f_n(u_i)} = a_i,$$

which, by (4.9), implies

$$\sum_{n=1}^{\infty} \sum_{i=r(n)}^{r(n+1)-1} a_i^{f_n(u_i)} \geq \sum_{n=1}^{\infty} 0.5n^{-1} = \infty.$$

Hence, together with the monotonicity of sequence $\{f_n(x)\}$ and the convergence to zero of $\{\varepsilon_n\}$, we conclude that the sequence $\{u_n\}$ has the desired properties.

(ii) Let us fix an increasing sequence $\{r(n)\}$ of positive integers satisfying condition (4.8) and the following one

$$\sum_{i=r(n)}^{\infty} a_i < n^{-3}, \quad (4.10)$$

for every $n \in \mathbb{N}$. Similarly as in the case (i), we can choose positive reals $v_i \in [1 - \varepsilon_n, 1]$ and $\xi_i \in [\frac{1}{2}, 2]$ for $i, n \in \mathbb{N}$, $r(n) \leq i < r(n+1)$ in such a way that

$$f_n(v_i) \in [1 - d_n, 1] \quad \text{and} \quad \xi_i n a_i = a_i^{f_n(v_i)}.$$

This forces (by (4.10)) that

$$\sum_{n=1}^{\infty} \sum_{i=r(n)}^{r(n+1)-1} a_i^{f_n(v_i)} \leq \sum_{n=1}^{\infty} 2n^{-2} < \infty.$$

The replacement of the arguments of the last part of (i) terminates the proof. $\square$

COROLLARY 4.7.

(i) Let $\{a_n\} \in l^{>1}$. Let $\{f_n(x)\}$ be a sequence of real functions having one of the following forms:

$$\begin{aligned} &\{\sqrt[n]{x}\}, \quad \{1 + \ln^n(x)\}, \quad \{\sin^{-n}(0.5\pi x)\}, \\ &\{1 + (g(x) - 1)^n\} \quad \text{and} \quad \{1 + \sigma_n(x)\}, \end{aligned}$$

where

$$\sigma_n(x) = \underbrace{g \cdot g \cdot \dots \cdot g(x)}_{n \text{ times}},$$

$n \in \mathbb{N}$, and $g: \mathbb{R} \rightarrow \mathbb{R}$ is a function right-continuous at the point 1 and satisfying the inequalities $1 < g(x) < x$ for $x > 1$. For example, let $g(x) = 1 + \sin(x - 1)$ or, more generally, have the form

$$g(x) = x + \sum_{n=s}^{\infty} a_n (x - 1)^n,$$

where $a_s < 0$ and the radius of convergence of this power series is > 0 . Then, by Theorem 4.6, there exists a sequence $\{u_n\}$ with $u_n > 1$ convergent to one and such that

$$\{a_n^{f_k(u_n)} : n \in \mathbb{N}\} \in l^{>1}$$

and

$$\lim_{n \rightarrow \infty} a_n^{f_k(u_n)-1} = 0,$$

for every $k \in \mathbb{N}$.

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- (ii) Let $\{a_n\} \in l^{=1}$ and let $g_k(x) = 2 - f_k(x)$, $k \in \mathbb{N}$, where $\{f_k(x)\}$ is one of the sequences considered in (i). Then, by Theorem 4.6, there exists a sequence $\{v_n\}$ with $v_n < 1$, which is convergent to one and such that

$$\{a_n^{f_k(v_n)} : n \in \mathbb{N}\} \in l^{=1}$$

and

$$\lim_{n \rightarrow \infty} a_n^{f_k(v_n)-1} = 0,$$

for every $k \in \mathbb{N}$.

THEOREM 4.8. For any $\{a_n\} \in l^{>1}$, there exist sequences $\{b_n\}$ and $\{c_n\}$ of positive reals, both convergent to zero and such that

$$\{a_n b_n\} \in l^{>1}, \quad \{a_n c_n\} \in l^{=1},$$

$$\{(a_n b_n)^u\} > \{a_n\} \quad \text{and} \quad \{a_n c_n\} > \{a_n^v\}$$

for every $u \in (0, 1)$ and $v > 1$.

Proof. By Theorems 4.5 and 4.6, there exist sequences $\{u_n\}$ and $\{v_n\}$ with $u_n, v_n > 1$, $n \in \mathbb{N}$, both convergent to one and such that

$$\{a_n^{u_n}\} \in l^{>1}, \quad \{a_n^{v_n}\} \in l^{=1}$$

and

$$\lim_{n \rightarrow \infty} a_n^{u_n-1} = \lim_{n \rightarrow \infty} a_n^{v_n-1} = 0.$$

It is sufficient to define $b_n = a_n^{u_n-1}$ and $c_n = a_n^{v_n-1}$ for every $n \in \mathbb{N}$. □

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