MARKOV TYPE POLYNOMIAL INEQUALITY FOR SOME GENERALIZED HERMITE WEIGHT

BRANISLAV FTOREK — MARIANA MARČOKOVÁ

ABSTRACT. In this paper we study some weighted polynomial inequalities of Markov type in L^2 -norm. We use the properties of the system of generalized Hermite polynomials $\{H_n^{(\alpha)}(x)\}_{n=0}^\infty$. The polynomials $H_n^{(\alpha)}(x)$ are orthogonal in $\mathbb{R} = (-\infty, \infty)$ with respect to the weight function

$$W(x) = |x|^{2\alpha} e^{-x^2}, \quad \alpha > -\frac{1}{2}.$$

The classical Hermite polynomials $H_n(x)$ present the special case for $\alpha = 0$.

1. Introduction

Let $\|f\|_\infty = \sup_{a \leq x \leq b} |f(x)|$ and let Π_n be a set of all algebraic polynomials of the degree at most n . For sufficiently smooth function $f(x)$ on $\mathbb{R}$, Kolmogorov in [5] established the inequality

$$\left\| f^{(k)} \right\|_\infty \leq C_{n,k} \left\| f^{(n)} \right\|_\infty^{\frac{k}{n}} \|f\|_\infty^{1-\frac{k}{n}}, \quad 0 < k < n$$

with some constant $C_{n,k}$.

The classical Markov's inequality (cf. [7])

$$\|p'\|_\infty \leq n^2 \|p\|_\infty, \quad -1 \leq x \leq 1, \quad p(x) \in \Pi_n \quad (1.1)$$

and its extension

$$\|p^{(k)}\|_\infty \leq \frac{1}{(2k-1)!!} \prod_{i=0}^{k-1} (n^2 - i^2) \|p\|_\infty, \quad p(x) \in \Pi_n, \quad 1 \leq k \leq n$$

are typical examples of inequalities connecting norms of a polynomial and its derivatives. In (1.1) the equality holds only at $x = \pm 1$ and only when $p(x) = cT_n(x)$, where $T_n(x)$ is the Chebyshev polynomial of the first kind of a degree n

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and c is an arbitrary constant. These facts motivated many authors to investigate polynomial analogues of Kolmogorov's and Markov's inequalities in norms taken with some weight functions (cf. [1], [2], [4], [6], [13]).

Consider now an inner product of the functions $f(x)$, $g(x)$ with the weight function $w(x)$ denoted $(f, g)_w = \int_a^b f(x)g(x)w(x)dx$ and weighted L^2 -norm of $f(x)$ denoted $\|f\|_w^2 = (f, f)_w$.

In the present work we need some properties of the system of generalized Hermite polynomials orthogonal in $\mathbb{R} = (-\infty, \infty)$ with respect to the weight function

$$W(x) = |x|^{2\alpha}e^{-x^2}, \quad \alpha > -\frac{1}{2}. \quad (1.2)$$

This weight function has the singular point $x = 0$ and classical Hermite polynomials $H_n(x)$ are the special case of considered polynomials $H_n^{(\alpha)}(x)$ for $\alpha = 0$. Let $\{\tilde{H}_n^{(\alpha)}(x)\}_{n=0}^\infty$ be a system of polynomials orthonormal with the weight (1.2). These polynomials satisfy the three-term recurrence relations (cf. [3])

$$x\tilde{H}_{2k}^{(\alpha)}(x) = \sqrt{\frac{2k+1+2\alpha}{2}}\tilde{H}_{2k+1}^{(\alpha)}(x) + \sqrt{k}\tilde{H}_{2k-1}^{(\alpha)}(x), \quad (1.3)$$

$$x\tilde{H}_{2k+1}^{(\alpha)}(x) = \sqrt{k+1}\tilde{H}_{2k+2}^{(\alpha)}(x) + \sqrt{\frac{2k+1+2\alpha}{2}}\tilde{H}_{2k}^{(\alpha)}(x) \quad (1.4)$$

and

$$\frac{d}{dx}\tilde{H}_{2k}^{(\alpha)}(x) = 2\sqrt{k}\tilde{H}_{2k-1}^{(\alpha)}(x), \quad (1.5)$$

$$\frac{d}{dx}\tilde{H}_{2k+1}^{(\alpha)}(x) = -\frac{2\alpha}{x}\tilde{H}_{2k+1}^{(\alpha)}(x) + 2\sqrt{\frac{2k+1+2\alpha}{2}}\tilde{H}_{2k}^{(\alpha)}(x), \quad x \neq 0, \quad (1.6)$$

where $\tilde{H}_{-1}^{(\alpha)}(x) = 0$ and $k = 0, 1, 2, \dots$

2. Preliminary lemmas

$$\textbf{LEMMA 2.1.} \quad \left\| \frac{d}{dx}\tilde{H}_{2k}^{(\alpha)} \right\|_W^2 = 4k, \quad (2.1)$$

$$\left\| \frac{d}{dx}\tilde{H}_{2k+1}^{(\alpha)} \right\|_W^2 = 4\alpha^2 \left\| \frac{\tilde{H}_{2k+1}^{(\alpha)}}{x} \right\|_W^2 - 8\alpha + 2(2k+1+2\alpha). \quad (2.2)$$

Proof. Because the interval $\mathbb{R}$ is symmetric and the weight function $W(x)$ is even, we can write

$$\tilde{H}_n^{(\alpha)}(-x) = (-1)^n \tilde{H}_n^{(\alpha)}(x)$$

(cf. [11, p. 15]) and clearly,

$$\frac{\tilde{H}_{2k+1}^{(\alpha)}}{x} \in \Pi_{2k}.$$

Now the proof follows from (1.3), (1.5), (1.6) and from the orthogonal properties

$$\begin{aligned} (\tilde{H}_n^{(\alpha)}, Q_m)_W &= 0, \quad Q_m(x) \in \Pi_{n-1}, \\ (\tilde{H}_i^{(\alpha)}, \tilde{H}_j^{(\alpha)})_W &= \delta_{ij}, \end{aligned}$$

where δ_{ij} is the Kronecker delta. □

LEMMA 2.2.

$$\begin{aligned} \left\| \frac{\tilde{H}_{2k+1}^{(\alpha)}}{x} \right\|_W^2 &= \frac{2}{2k+1+2\alpha} + \frac{2^2 k}{(2k+1+2\alpha)(2k-1+2\alpha)} \\ &+ \frac{2^3 k(k-1)}{(2k+1+2\alpha)(2k-1+2\alpha)(2k-3+2\alpha)} + \dots \\ &\dots + \frac{2^{k+1} k(k-1) \dots 2.1}{(2k+1+2\alpha)(2k-1+2\alpha)(2k-3+2\alpha) \dots (1+2\alpha)}. \end{aligned} \quad (2.3)$$

Proof. We use an induction on k . For $k = 0$ with regard to (1.3) we have

$$\left\| \frac{\tilde{H}_1^{(\alpha)}}{x} \right\|_W^2 = \frac{2}{1+2\alpha} \left\| \tilde{H}_0^{(\alpha)} \right\|_W^2 = \frac{2}{1+2\alpha}.$$

Suppose that for some $k > 0$ there is

$$\begin{aligned} \left\| \frac{\tilde{H}_{2k-1}^{(\alpha)}}{x} \right\|_W^2 &= \frac{2}{2k-1+2\alpha} + \frac{2^2(k-1)}{(2k-1+2\alpha)(2k-3+2\alpha)} \\ &+ \frac{2^3(k-1)(k-2)}{(2k-1+2\alpha)(2k-3+2\alpha)(2k-5+2\alpha)} + \dots \\ &\dots + \frac{2^k(k-1)(k-2) \dots 2.1}{(2k-1+2\alpha)(2k-3+2\alpha)(2k-5+2\alpha) \dots (1+2\alpha)}. \end{aligned}$$

Applying this induction hypothesis, (1.3), and the basic general properties of orthogonal polynomial, we have

$$\begin{aligned} \left\| \frac{\tilde{H}_{2k+1}^{(\alpha)}}{x} \right\|_W^2 &= \frac{2}{2k+1+2\alpha} \left\| \tilde{H}_{2k}^{(\alpha)} - \sqrt{k} \frac{\tilde{H}_{2k-1}^{(\alpha)}}{x} \right\|_W^2 \\ &= \frac{2}{2k+1+2\alpha} \left\{ \left\| \tilde{H}_{2k}^{(\alpha)} \right\|_W^2 + k \left\| \frac{\tilde{H}_{2k-1}^{(\alpha)}}{x} \right\|_W^2 \right\} \\ &= \frac{2}{2k+1+2\alpha} \left\{ 1 + k \left\| \frac{\tilde{H}_{2k-1}^{(\alpha)}}{x} \right\|_W^2 \right\} \end{aligned}$$

$$\begin{aligned}
 &= \frac{2}{2k+1+2\alpha} + \frac{2^2k}{(2k+1+2\alpha)(2k-1+2\alpha)} + \cdots \\
 &\quad \cdots + \frac{2^{k+1}k(k-1)\dots 2.1}{(2k+1+2\alpha)(2k-1+2\alpha)(2k-3+2\alpha)\dots(1+2\alpha)}
 \end{aligned}$$

and lemma is proved. $\square$

LEMMA 2.3. *The inequalities*

$$\left\| \frac{\tilde{H}_{2k+1}^{(\alpha)}}{x} \right\|_W^2 \leq \frac{2k+2}{2k+1+2\alpha}, \quad \alpha \geq \frac{1}{2},$$

$$\left\| \frac{\tilde{H}_{2k+1}^{(\alpha)}}{x} \right\|_W^2 < \frac{2^{k+1}(k+1)!}{(2k+1+2\alpha)(2k-1+2\alpha)(2k-3+2\alpha)\dots(1+2\alpha)},$$

$$\alpha \in \left(-\frac{1}{2}, \frac{1}{2} \right)$$

hold.

Proof. In (2.3) let us denote

$$\left\| \frac{\tilde{H}_{2k+1}^{(\alpha)}}{x} \right\|_W^2 = B_0 + B_1 + B_2 + \cdots + B_k.$$

It is obvious that we have a recurrence relation

$$B_j = B_{j-1} \frac{2[k-(j-1)]}{2k-(2j-1)+2\alpha}, \quad (2.4)$$

where $j = 1, 2, \dots, k$.

For $\alpha \geq \frac{1}{2}$ from (2.4) we have $B_{j-1} \geq B_j$, and clearly we get the first inequality

$$\left\| \frac{\tilde{H}_{2k+1}^{(\alpha)}}{x} \right\|_W^2 \leq (k+1)B_0 = \frac{2k+2}{2k+1+2\alpha}.$$

Let $\alpha \in (-\frac{1}{2}, \frac{1}{2})$. Then $B_{j-1} < B_j$, and the inequality

$$\left\| \frac{\tilde{H}_{2k+1}^{(\alpha)}}{x} \right\|_W^2 < (k+1)B_k = \frac{2^{k+1}(k+1)!}{(2k+1+2\alpha)(2k-1+2\alpha)(2k-3+2\alpha)\dots(1+2\alpha)}$$

holds. The proof is complete. $\square$

3. Main theorems

Using the obtained results, we can formulate polynomial inequalities of Markov type in weighted L^2 -norm of polynomials $p_n(x) \in \Pi_n$.

THEOREM 3.1. *Let $W(x)$ be given by (1.2). Then there exists a constant*

$$C = C(\alpha)$$

such that

$$\|p'_n\|_W \leq Cn\|p_n\|_W \quad (3.1)$$

for all polynomials $p_n(x) \in \Pi_n$.

P r o o f. First, using Lemma 2.3 for $\alpha \geq \frac{1}{2}$, we get the relation

$$\left\| \frac{\tilde{H}_{2k+1}^{(\alpha)}}{x} \right\|_W^2 = O(1). \quad (3.2)$$

The case $\alpha \in (-\frac{1}{2}, \frac{1}{2})$ is more complicated. Now we use the definition of Gamma function due to Gauss

$$\Gamma(x) = \lim_{n \rightarrow \infty} \frac{n!n^x}{x(x+1)(x+2)\dots(x+n)}.$$

Then Lemma 2.3 implies

$$\begin{aligned} & \lim_{k \rightarrow \infty} \frac{1}{k^{\frac{1}{2}-\alpha}} (k+1) B_k \\ &= \lim_{k \rightarrow \infty} \frac{1}{k^{\frac{1}{2}-\alpha}} \frac{k+1}{k^{\alpha+\frac{1}{2}}} \frac{k! k^{\alpha+\frac{1}{2}}}{(k+\frac{1}{2}+\alpha)(k-\frac{1}{2}+\alpha)(k-\frac{3}{2}+\alpha)\dots(\frac{1}{2}+\alpha)} \\ &= \Gamma\left(\frac{1}{2}+\alpha\right), \end{aligned}$$

from where we deduce

$$\left\| \frac{\tilde{H}_{2k+1}^{(\alpha)}}{x} \right\|_W^2 = O(k^{\frac{1}{2}-\alpha}). \quad (3.3)$$

Second, every $p_n(x) \in \Pi_n$ can be uniquely represented as a linear combination of the system $\{\tilde{H}_i^{(\alpha)}(x)\}_{i=0}^n$ (cf. [11])

$$p_n(x) = \sum_{i=0}^n c_i \tilde{H}_i^{(\alpha)}(x),$$

where $c_i = (\tilde{H}_i^{(\alpha)}, p_n)_W$. From this we have

$$p'_n(x) = \sum_{i=0}^n c_i \frac{d}{dx} \tilde{H}_i^{(\alpha)}(x)$$

and by Schwarz inequality,

$$\begin{aligned} p_n'^2(x) &= \left[\sum_{i=0}^n c_i \frac{d}{dx} \tilde{H}_i^{(\alpha)}(x) \right]^2 \\ &\leq \sum_{i=0}^n c_i^2 \sum_{i=0}^n \left[\frac{d}{dx} \tilde{H}_i^{(\alpha)}(x) \right]^2 \\ &= \|p_n\|_W^2 \sum_{i=0}^n \left[\frac{d}{dx} \tilde{H}_i^{(\alpha)}(x) \right]^2. \end{aligned}$$

Now, we multiply the last inequality by $W(x)$ and after integrating on $\mathbb{R}$ we obtain

$$\|p_n'\|_W^2 \leq \|p_n\|_W^2 \sum_{i=0}^n \left\| \frac{d}{dx} \tilde{H}_i^{(\alpha)} \right\|_W^2. \quad (3.4)$$

From (2.1) we have

$$\left\| \frac{d}{dx} \tilde{H}_{2k}^{(\alpha)} \right\|_W^2 = O(2k). \quad (3.5)$$

From (2.2) with regard to (3.2) and (3.3) we get

$$\left\| \frac{d}{dx} \tilde{H}_{2k+1}^{(\alpha)} \right\|_W^2 = O(2k+1). \quad (3.6)$$

Using (3.4), (3.5) and (3.6), we have

$$\|p_n'\|_W^2 \leq \|p_n\|_W^2 \sum_{i=0}^n \left\| \frac{d}{dx} \tilde{H}_i^{(\alpha)} \right\|_W^2 \leq C_1 \|p_n\|_W^2 \sum_{i=0}^n i \leq C_2 n^2 \|p_n\|_W^2, \quad (3.7)$$

where C_1, C_2 are positive constants independent of n and of x , but dependent of α . Finally, from (3.7), we obtain

$$\|p_n'\|_W \leq Cn \|p_n\|_W. \quad \square$$

Now, we will deal with the some generalization of the weight function (1.2). Let

$$W_\delta(x) = \delta(x)|x|^{2\alpha}e^{-x^2} = \delta(x)W(x) \quad (3.8)$$

be the weight function, where the factor $\delta(x)$ satisfies the condition

$$0 < m \leq \delta(x) \leq M, \quad x \in \mathbb{R}. \quad (3.9)$$

Using Theorem 3.1 and (3.9), we have basic estimates

$$\|p_n'\|_{W_\delta}^2 \leq M \|p_n'\|_W^2 \leq MC^2 n^2 \|p_n\|_W^2 \leq \frac{M}{m} C^2 n^2 \|p_n\|_{W_\delta}^2 \leq C_3 n^2 \|p_n\|_{W_\delta}^2$$

with some positive constant C_3 . Based on the above considerations, we state the following theorem.

MARKOV TYPE POLYNOMIAL INEQUALITY

THEOREM 3.2. *For all polynomials $p_n(x) \in \Pi_n$ and for some constant $C^* = C^*(\alpha)$ the inequality*

$$\|p'_n\|_{W_\delta} \leq C^* n \|p_n\|_{W_\delta}$$

holds. The weight function $W_\delta(x)$ is given by (3.8).

In [9] Mirsky considered the case with an arbitrary weight function

$$w : (a, b) \rightarrow \mathbb{R}_+, \quad -\infty \leq a < b \leq \infty$$

for which all moments are finite, i.e., $\mu_k = \int_a^b x^k w(x) dx < \infty$. His general result is, however, qualitative. For the simplest and nonsingular Hermite weight function

$$w(x) = e^{-x^2}$$

($\alpha = 0$ in (1.2)) the estimate of the type (3.1) becomes (cf. [8, p. 570])

$$\frac{\|p'_n\|_w}{\|p_n\|_w} = O(n^{\frac{3}{2}}).$$

The contrast between this estimate, our results, and the classical result of Schmidt [10] and Turán [12] $\|p'_n\|_w \leq \sqrt{2n} \|p_n\|_w$ is evident.

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