ANOTHER PROOF OF HUREWICZ THEOREM

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ABSTRACT. A Hurewicz theorem says that every coanalytic non- G_δ set C in a Polish space contains a countable set Q without isolated points such that $\overline{Q} \cap C = Q$. We present another elementary proof of this theorem and generalize it for κ -Suslin sets. As a consequence, under Martin's Axiom, we obtain a characterization of Σ_2^1 sets that are the unions of less than the continuum closed sets.

1. Hurewicz schemes

Several proofs of the Hurewicz theorem are known. The original proof by W. Hurewicz [2] is based on the notion of “Häufungssystem” (which we call Hurewicz scheme below). The proof presented by A. Kechris [3, Theorem 21.18 and Theorem 21.22] is based on the so called separation game. Recently Michal Staš [5] gave another simple and elementary proof of this theorem. Our proof is relative to the proof by M. Staš but, in addition, we introduce a notion of a D -proper mapping between metric spaces and obtain another variant of his characterization. We present a natural generalization to κ -Suslin sets.

DEFINITION 1.1. Let X be a metric space.

- (1) A mapping $\varphi: {}^{<\omega}\omega \rightarrow X$ is an *H-scheme on X* (i.e., *Hurewicz scheme*) if the following two conditions are satisfied:
 - (a) $\varphi(s \smallfrown n) \neq \varphi(s \smallfrown m)$ for all $s \in {}^{<\omega}\omega$ and $n \neq m$,
 - (b) $\varphi(s) = \lim_{n \rightarrow \infty} \varphi(s \smallfrown n)$ for all $s \in {}^{<\omega}\omega$.
- (2) An *H-scheme* φ is *separated* if there is a system of open sets $\{V_s : s \in {}^{<\omega}\omega\}$ such that for all $s \in {}^{<\omega}\omega$,
 - (c) $\varphi(s) \in V_s \setminus \bigcup_{n \in \omega} \overline{V_{s \smallfrown n}}$, and
 - (d) $\{\overline{V_{s \smallfrown n}} : n \in \omega\}$ is a disjoint system of subsets of V_s .

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We say that the system $\{V_s : s \in {}^{<\omega}\omega\}$ is *normal* if

- (e) $\lim_{s \in {}^{<\omega}\omega} \text{diam}(V_s) = 0$, i.e., $(\forall \varepsilon > 0)(\forall^\infty s \in {}^{<\omega}\omega) \text{diam}(V_s) < \varepsilon$.
- (3) An H -scheme $\varphi : {}^{<\omega}\omega \rightarrow X$ is *complete* (in X), if for every $g \in {}^\omega\omega$, the sequence $\{\varphi(g \upharpoonright n)\}_{n \in \omega}$ converges in X .

DEFINITION 1.2. For an H -scheme $\varphi : {}^{<\omega}\omega \rightarrow X$ we denote

$$H(\varphi) = \overline{\text{rng}(\varphi)} \setminus \text{rng}(\varphi),$$

$$G(\varphi) = \{x \in X : (\exists f \in {}^\omega\omega) x = \lim_{n \rightarrow \infty} \varphi(f \upharpoonright n)\}.$$

It is easy to see that $H(\varphi)$ is a G_δ set and, if X is a Polish space, then $G(\varphi)$ is analytic. For every H -scheme φ we can define a (partial) function $f_\varphi : {}^\omega\omega \rightarrow X$ by $f_\varphi(x) = \lim_{n \rightarrow \infty} \varphi(x \upharpoonright n)$, if the limit exists. The set $G(\varphi)$ is the range of f_φ . The function f_φ is total if and only if φ is a complete H -scheme. If φ is an H -scheme on a complete metric space separated by a normal system of open sets, then f_φ is total, injective, and continuous.

By Hurewicz [2], an H -scheme φ is *normal*, if it satisfies condition $\overline{\text{rng}(\varphi)} = G(\varphi) \cup \text{rng}(\varphi)$, i.e., $H(\varphi) \subseteq G(\varphi)$. We need a stronger property because of the next lemma.

LEMMA 1.3. Let $\varphi : {}^{<\omega}\omega \rightarrow X$ be an H -scheme on a metric space X separated by a normal system of open sets $\{V_s : s \in {}^{<\omega}\omega\}$. If $\langle s_n : n \in \omega \rangle$ is a sequence in ${}^{<\omega}\omega$, $T = \{s \in {}^{<\omega}\omega : \exists^\infty n s \subseteq s_n\}$, and $x \in X$, then $x = \lim_{n \rightarrow \infty} \varphi(s_n)$ if and only if one of the following conditions holds:

- (i) there is $s \in {}^{<\omega}\omega$ such that $T = \{s \upharpoonright k : k \leq |s|\}$, $\forall^\infty n s \subseteq s_n$, and $x = \varphi(s)$,
or
- (ii) there is $g \in {}^\omega\omega$ such that $T = \{g \upharpoonright k : k \in \omega\}$, $\forall k \forall^\infty n g \upharpoonright k \subseteq s_n$, and $x = \lim_{n \rightarrow \infty} \varphi(g \upharpoonright n) \notin \text{rng}(\varphi)$.

In particular, $G(\varphi) = H(\varphi)$.

Proof. Assume that $x = \lim_{n \rightarrow \infty} \varphi(s_n)$. For every $s \in T$ we have $x \in \overline{V_s}$. By condition (d) it follows that T does not contain any two incomparable elements. If T is finite, let s be its maximal element. There are increasing sequences $\{n_k\}_{k \in \omega}$ and $\{m_k\}_{k \in \omega}$ such that $s \frown m_k \subseteq s_{n_k}$ for all k . Then

$$d(\varphi(s), \varphi(s_{n_k})) \leq d(\varphi(s), \varphi(s \frown m_k)) + \text{diam}(V_{s \frown m_k})$$

and hence

$$\varphi(s) = \lim_{k \rightarrow \infty} \varphi(s_{n_k}) = x.$$

If t is a proper subsequence of s , then $\varphi(t) \in V_t \setminus \overline{V_s}$ and so $\varphi(t) \neq x$. Hence we cannot apply the same argument for any t below s which means that $\forall^\infty n s \subseteq s_n$.

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If T is infinite, then there must be $g \in {}^\omega\omega$ such that $T = \{g \upharpoonright k : k \in \omega\}$. Then for every k , $\varphi(g \upharpoonright k) \in V_{g \upharpoonright k} \setminus \overline{V_{g \upharpoonright (k+1)}}$, hence $\varphi(g \upharpoonright k) \neq x$ and, like in the previous case, $\forall k \forall^\infty n \ g \upharpoonright k \subseteq s_n$. Then $x = \lim_{n \rightarrow \infty} \varphi(g \upharpoonright n)$ because $x \in \bigcap_{n \in \omega} \overline{V_{g \upharpoonright n}}$. $\square$

EXAMPLE 1.4. The Cantor space ${}^\omega 2$ can be represented as a topological closure of an H -scheme separated by a normal system of open sets. By induction let us define $t_s \in {}^{<\omega} 2$ for $s \in {}^{<\omega} \omega$: $t_\emptyset = \emptyset$ and $t_{s \smallfrown n} = t_s \smallfrown 0^n 1$. Then $\varphi_C : {}^{<\omega} \omega \rightarrow {}^\omega 2$ defined by $\varphi_C(s) = t_s \smallfrown \overline{0}$ ($\overline{0}$ is the infinite sequence of 0's) is an H -scheme separated by the normal system of clopen sets $\{[t_s] : s \in {}^{<\omega} \omega\}$. Clearly, $\overline{\text{rng}(\varphi_C)} = {}^\omega 2$ and $G(\varphi_C) = H(\varphi_C) = {}^\omega 2 \setminus \text{rng}(\varphi_C) \simeq {}^\omega \omega$.

THEOREM 1.5. *If X is a metric space and $\varphi : {}^{<\omega} \omega \rightarrow X$ is a complete H -scheme separated by a normal system of open sets, then $H(\varphi) = G(\varphi) \simeq {}^\omega \omega$ and $\overline{\text{rng}(\varphi)}$ is a compact perfect set homeomorphic to the Cantor space.*

PROOF. Let $\psi : {}^{<\omega} \omega \rightarrow Y$ be any other complete H -scheme on a metric space Y separated by a normal system of open sets. There is a homeomorphism $f : \overline{\text{rng}(\varphi)} \rightarrow \overline{\text{rng}(\psi)}$ such that $f(\varphi(s)) = \psi(s)$ and $f(f_\varphi(g)) = f_\psi(g)$ for all $s \in {}^{<\omega} \omega$ and $g \in {}^\omega \omega$. This is possible because the convergence of $\{\varphi(s_n)\}_{n \in \omega}$ and $\{\psi(s_n)\}_{n \in \omega}$ depends on the sequence $\{s_n\}_{n \in \omega}$ in the way described by Lemma 1.3. Apply this fact to the H -scheme on the Cantor space from Example 1.4. $\square$

2. Hurewicz theorem

DEFINITION 2.1. Let X be a metric space. An H -scheme $\varphi' : {}^{<\omega} \omega \rightarrow X$ is a *subscheme* of an H -scheme $\varphi : {}^{<\omega} \omega \rightarrow X$, if there is an injective mapping $h : {}^{<\omega} \omega \rightarrow {}^{<\omega} \omega$ such that $|h(s)| = |s|$, $t \subseteq s$ implies $h(t) \subseteq h(s)$, and $\varphi'(s) = \varphi(h(s))$ for all $s \in {}^{<\omega} \omega$.

Subschemes of H -schemes preserve normality, separateness, and completeness. If φ' is a subscheme of an H -scheme φ , then $\text{rng}(\varphi') \subseteq \text{rng}(\varphi)$ and $G(\varphi') \subseteq G(\varphi)$. The notion of a subscheme is similar to the notion of “Restsystem” in [2].

LEMMA 2.2. *Every H -scheme has a subscheme separated by a normal system of open sets.*

PROOF. We define open balls V_s and a function $h : {}^{<\omega} \omega \rightarrow {}^{<\omega} \omega$ by induction on $|s|$ for $s \in {}^{<\omega} \omega$ so that

$$\varphi(h(s)) \in V_s \quad \text{and} \quad \text{diam}(V_s) < 2^{-(n+s(n))} \quad \text{for} \quad s \in {}^{n+1} \omega.$$

Set $h(\emptyset) = \emptyset$ and let $V_\emptyset$ be an open ball with center $\varphi(h(\emptyset))$. Let $s \in {}^{<\omega}\omega$ be arbitrary and let us assume that $h(s)$ and V_s are defined and $\varphi(h(s)) \in V_s$. Let $\{k_n^s\}_{n \in \omega}$ be an increasing sequence such that

$$\varphi(h(s) \smallfrown k_n^s) \in V_s \quad \text{and} \quad d\left(\varphi(h(s) \smallfrown k_{n+1}^s), \varphi(h(s))\right) < d\left(\varphi(h(s) \smallfrown k_n^s), \varphi(h(s))\right).$$

Set $h(s \smallfrown n) = h(s) \smallfrown k_n^s$ and choose open balls $V_{s \smallfrown n} \subseteq V_s \setminus \{\varphi(s)\}$ with centers $\varphi(h(s \smallfrown n))$ for $n \in \omega$ so that their closures are pairwise disjoint, do not contain $\varphi(h(s))$, and their diameters are sufficiently small. $\square$

DEFINITION 2.3. Let $f : Y \rightarrow X$ and $f(Y) \subseteq D \subseteq X$. The function f is said to be *D-proper*, if for every nonempty open set $U \subseteq Y$ the set $\overline{f(U)} \setminus D$ is nonempty and has no isolated points.

LEMMA 2.4. Let X and Y be metric spaces, let $f : Y \rightarrow X$ be continuous, and let $f(Y) \subseteq D \subseteq X$. If Y is complete and f is *D-proper*, then there exists a complete *H-scheme* $\varphi : {}^{<\omega}\omega \rightarrow X \setminus D$ such that $G(\varphi) \subseteq f(Y)$.

Proof. By induction on $|s|$ for $s \in {}^{<\omega}\omega$ we define a sequence $\langle V_s : s \in {}^{<\omega}\omega \rangle$ of nonempty open balls in Y and an *H-scheme* $\varphi : {}^{<\omega}\omega \rightarrow X \setminus D$ such that

- (1) $\text{diam}(V_s) < 2^{-|s|}$ and $\text{diam}(\overline{f(V_s)}) < 2^{-|s|}$,
- (2) $\{\overline{V_{s \smallfrown n}} : n \in \omega\}$ is a disjoint system of subsets of V_s ,
- (3) $\varphi(s) \in \overline{f(V_s)} \setminus D$.

Let $V_\emptyset \subseteq Y$ be any open ball such that $\text{diam}(V_\emptyset) < 1$ and $\text{diam}(\overline{f(V_\emptyset)}) < 1$, and let $\varphi(\emptyset) \in \overline{f(V_\emptyset)} \setminus D$. This is possible because f is continuous and *D-proper*.

Let us assume that V_s and $\varphi(s) \in \overline{f(V_s)} \setminus D$ have been constructed for a given $s \in {}^m\omega$. Since $\varphi(s)$ is not an isolated point of $\overline{f(V_s)} \setminus D$, we can choose a disjoint sequence $\langle U_n : n \in \omega \rangle$ of open balls in X such that $U_n \cap (\overline{f(V_s)} \setminus D) \neq \emptyset$, $U_n \subseteq B(\varphi(s), 2^{-m-1})$, and $\varphi(s) \notin \overline{U_n}$; hence $\text{diam}(U_n) < 2^{-m-1}$. For each $n \in \omega$ let us choose an open ball $V_{s \smallfrown n} \subseteq f^{-1}(U_n) \cap V_s$ in Y with diameter $< 2^{-m-1}$ such that $\overline{f(V_{s \smallfrown n})} \subseteq U_n$ and let $\varphi(s \smallfrown n) \in \overline{f(V_{s \smallfrown n})} \setminus D$.

Clearly, φ is an *H-scheme* and conditions (1)–(3) are satisfied. We prove that $G(\varphi) \subseteq f(Y)$. Let $g \in {}^\omega\omega$. Since Y is complete there is a unique $y \in \bigcap_{m \in \omega} \overline{V_{g \upharpoonright m}}$. Then $f(y) = \lim_{m \rightarrow \infty} \varphi(g \upharpoonright m)$ because $f(y) \in \overline{f(V_{g \upharpoonright m})} \subseteq \overline{f(V_{g \upharpoonright m})}$ and hence $d(f(y), \varphi(g \upharpoonright m)) < 2^{-m}$. $\square$

Let κ be an infinite cardinal number. For a set $D \subseteq X$ let $I_{<\kappa}(D)$ denote the ideal over D generated by unions of $< \kappa$ closed sets in X which are subsets of D and let $I_\kappa(D) = I_{<\kappa^+}(D)$. Let us recall that $\text{cov}(\mathcal{M})$ denotes the least cardinality of a family of meager sets covering the real line.

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THEOREM 2.5. *Let $\omega \leq \kappa < \text{cov}(\mathcal{M})$. Let X be a metric space, let $A \subseteq D \subseteq X$, and let A be a κ -Suslin set. The following conditions are equivalent:*

- (1) $A \notin I_{<\text{cov}(\mathcal{M})}(D)$.
- (2) $A \notin I_\kappa(D)$.
- (3) *There is a continuous D -proper mapping $f: Y \rightarrow X$ for some complete metric space Y such that $f(Y) \subseteq A$ and the weight of Y is less or equal to κ .*
- (4) *There is a continuous D -proper mapping $f: Y \rightarrow X$ for some complete metric space Y such that $f(Y) \subseteq A$.*
- (5) *There is a complete H -scheme $\varphi: {}^{<\omega}\omega \rightarrow X \setminus D$ such that $G(\varphi) \subseteq A$.*
- (6) *There is a complete H -scheme $\varphi: {}^{<\omega}\omega \rightarrow X \setminus D$ separated by a normal system of open sets such that $G(\varphi) \subseteq A$.*
- (7) *There is a compact perfect set $P \subseteq X$ such that $P \cap D \subseteq A$, $P \setminus D$ is countable, and $\overline{P \setminus D} = P$.*

Proof. The implications (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) are trivial, the implication (4) $\Rightarrow$ (5) is Lemma 2.4 and (5) $\Rightarrow$ (6) follows by Lemma 2.2.

(2) $\Rightarrow$ (3) Let $g: {}^\omega\kappa \rightarrow X$ be a continuous function such that $g({}^\omega\kappa) = A$. As ${}^\omega\kappa$ has a base of open sets of size κ , there is a maximal open set U_0 in ${}^\omega\kappa$ such that $g(U_0) \in I_\kappa(D)$. Denote $Y = {}^\omega\kappa \setminus U_0$ and $f = g|_Y$. The function $f: Y \rightarrow X$ is continuous, Y is a complete metric space, and $f(Y) \subseteq A$. We verify that f is D -proper.

Let V be an open set in Y , i.e., $V = U \setminus U_0$ for some open set $U \subseteq {}^\omega\kappa$. If $\overline{f(V)} \subseteq D$, then $g(U) = (g(U) \setminus g(U_0)) \cup (g(U) \cap g(U_0)) \subseteq g(V) \cup g(U_0) \subseteq \overline{f(V)} \cup g(U_0) \in I_\kappa(D)$ and hence $V \subseteq U \subseteq U_0$. It follows that if V is a nonempty open set in Y , then $\overline{f(V)} \setminus D \neq \emptyset$. We prove that $\overline{f(V)} \setminus D$ has no isolated points. On the contrary, assume that $x \in \overline{f(V)} \setminus D$ is an isolated point and B is an open ball in X such that $B \cap \overline{f(V)} \setminus D = \{x\}$. The set $F = \overline{f(V)} \cap B \setminus \{x\}$ is an F_σ set in X contained in D , the set $V' = V \cap f^{-1}(B \setminus \{x\})$ is open in Y , and $g(V') \subseteq f(V) \cap B \setminus \{x\} \subseteq F \in I_\kappa(D)$. Let $U' \subseteq {}^\omega\kappa$ be such an open set that $V' = U' \setminus U_0$. Then $g(U') = (g(U') \setminus g(U_0)) \cup (g(U') \cap g(U_0)) \subseteq g(V') \cup g(U_0) \in I_\kappa(D)$. It follows that $V' = \emptyset$ because $V' \subseteq U' \subseteq U_0$. Then $f(V) \cap B \setminus \{x\} = \emptyset$ and $x \in f(V) \subseteq D$ because $x \in \overline{f(V)}$. This is a contradiction.

(6) $\Rightarrow$ (7) If $\varphi: {}^{<\omega}\omega \rightarrow X \setminus D$ is a complete H -scheme separated by a normal system of open sets, then by Theorem 1.5, $P = \text{rng}(\varphi)$ is a compact perfect set, and since $G(\varphi) = H(\varphi) \subseteq A \subseteq D$, then $P \cap D = G(\varphi)$ and $P \setminus D = \text{rng}(\varphi)$.

(7) $\Rightarrow$ (1) Assume that P is a compact perfect set satisfying (7). Then $P \setminus A = P \setminus D$ is a countable dense subset of P . So, if $A \in I_{<\text{cov}(\mathcal{M})}(D)$, then $P \cap A$ and

hence also P is the union of $< \text{cov}(\mathcal{M})$ nowhere dense subsets of P . This is impossible and therefore $A \notin I_{<\text{cov}(\mathcal{M})}(D)$. $\square$

Let us note that the proof of the theorem requires neither that the metric space X is complete nor that X has a countable base. If we use another definition of a κ -Suslin set, then there may be some restrictions (for example, a set in a metric space of weight $\leq \kappa$ is a result of the Suslin operation $\mathcal{A}^\kappa$ applied to closed sets if and only if it is a continuous image of ${}^\omega\kappa$, see [4, Proof of Theorem 2B.1]).

Condition (3) as a variant of (4) was suggested by the referee.

Michal Staš [5] has proved the equivalence of conditions closely related to (2) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7) for analytic sets (i.e., ω_1 -Suslin sets). His proof of the implication (2) $\Rightarrow$ (6) contains a direct construction of an H -scheme by induction using the following lemma:

LEMMA 2.6. *Let X be a space with a countable base of open sets. Let $A \subseteq D \subseteq X$. Then for every open set $U \subseteq X$ such that $A \cap U \notin I_\omega(D)$ there are infinitely many points $p \in U \setminus D$ such that $A \cap V \notin I_\omega(D)$ for all neighbourhoods V of p .* $\square$

Let us look at the meaning of the clause “infinitely many points $p \in U$ ” in the lemma. Let us note that the set A in the lemma need not be analytic. But if A is analytic and $A \cap U \notin I_\omega(D)$, then by Theorem 2.5 there is a continuous D -proper mapping $f: Y \rightarrow X$ defined on a Polish space Y such that $f(Y) \subseteq A \cap U$. For every open set $V \subseteq X$ such that $f(Y) \cap V \neq \emptyset$ the function $f \upharpoonright (f^{-1}(V))$ is again D -proper. Hence for every $p \in \overline{f(Y)} \setminus D$ and for every neighbourhood V of p we have $A \cap V \notin I_\omega(D)$.

It is well-known that every Σ_2^1 set is ω_1 -Suslin. By Theorem 2.5, if $\text{cov}(\mathcal{M}) > \omega_1$, then a Σ_2^1 set A in a (possibly non-complete) metric space is the union of $\leq \omega_1$ closed sets if and only if there is no compact perfect set P such that $P \setminus A$ is countable dense in P . (Take $D = A$ in the theorem.) In particular, we have:

COROLLARY 2.7. *If Martin’s Axiom holds, then a Σ_2^1 set A is the union of $< 2^\omega$ closed sets if and only if there is no compact perfect set P such that $P \setminus A$ is countable dense in P .* $\square$

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