ON THE KNESER-HUKUHARA PROPERTY 
FOR AN INTEGRO-DIFFERENTIAL EQUATION 
IN BANACH SPACES

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ABSTRACT. In this paper we investigate some topological properties of solutions sets of some integro-differential equations in Banach spaces. Our assumptions and proofs are expressed in terms of the measure of weak noncompactness.

1. Introduction.

Let \( I = [0, a] \) be a compact interval in \( \mathbb{R} \), \( B = \{ x \in E : \| x \| \leq b \} \) and let \( E \) be a sequentially weakly complete Banach space. Throughout this paper we shall assume that \( f : I \times B \mapsto E \) and \( g : I^2 \times B \mapsto E \) are functions continuous in the weak—weak sense, that is for every \( t \in I, x \in B \) and arbitrary weak neighbourhood \( U \) of the point \( f(t, x) \) there exists an \( \varepsilon > 0 \) and a weak neighbourhood \( V \) of \( x \) so that for every \( y \in V \cap B, s \in I, |s - t| < \varepsilon, f(s, y) \in U \) is valid.

Consider the Cauchy problem

\[
\begin{align*}
  x^{(m)}(t) &= f(t, x(t)) + \int_{0}^{t} g(t, s, x(s)) \, ds, \\
  x(0) &= 0, \quad x'(0) = \eta_1, \ldots, x^{(m-1)}(0) = \eta_{m-1},
\end{align*}
\]

where \( m \geq 1 \) and \( \eta_1, \ldots, \eta_{m-1} \in E \) and \( x^{(m)} \) means the \( m \)th order derivative in the weak sense and integral denotes the weak Riemann integral. Let us recall that the weak Riemann integral of a weak continuous function \( y(t) \) \( (t \in I) \) with values in \( E \) is defined as the weak limit of Riemann sums (cf. [7]).

© 2011 Mathematical Institute, Slovak Academy of Sciences.
2010 Mathematics Subject Classification: 47G20, 45J05.
Keywords: integro-differential equation, topological properties of solution sets, measures of weak noncompactness.
In this paper we prove that the set of all weak solutions of this problem, defined on a compact subinterval $J = [0, d]$ of $I$, for some $d > 0$, is nonempty, compact and connected in the space $C_w(J, E)$ of weakly continuous functions $J \mapsto E$ with the topology of weak uniform convergence.

The method of the proof of our main result is suggested by the paper [9] concerning differential equations. Nevertheless the idea to consider the $\varepsilon$-approximate solutions set of Volterra integral equation

$$x(t) = f(t) + \int_0^t g(t, s, x(s)) \, ds$$

goes back to Hukuhara [6], who proved that this set is connected in $C(J, \mathbb{R}^n)$.

Our approach is to impose a weak compactness type conditions expressed in terms of the measure of weak noncompactness introduced by De Blasi [5].

Let $A$ be a nonvoid bounded subset of $E$. The measure of weak noncompactness $\beta(A)$ is defined by

$$\beta(A) = \inf \{ \varepsilon > 0 : \text{there exists a weakly compact set } K \text{ such that } A \subset K + \varepsilon B \},$$

where $B$ is the norm unit ball.

We make use of the following properties of the measure of weak noncompactness $\beta$ (for bounded nonvoid subsets $A$ and $B$ of $E$):

1° $A \subset B \Rightarrow \beta(A) \leq \beta(B)$;
2° $\beta(\overline{A}^w) = \beta(A)$ where $\overline{A}^w$ denotes the weak closure of $A$;
3° $\beta(A) = 0 \iff \overline{A}^w$ is weakly compact;
4° $\beta(A \cup B) = \max(\beta(A), \beta(B))$;
5° $\beta(\text{conv} A) = \beta(A)$;
6° $\beta(A + B) \leq \beta(A) + \beta(B)$;
7° $\beta(\lambda A) = |\lambda| \beta(A)$, ($\lambda \in \mathbb{R}$);
8° $\beta(\bigcup_{|\lambda| \leq h} \lambda A) = h \beta(A)$.

2. Basic lemmas

Let $V$ be a subset of $C_w(J, E)$. Put

$$V(t) = \{ u(t) : u \in V \} \quad \text{and} \quad V(T) = \{ u(t) : u \in V, t \in T \}.$$ 

In what follows we shall use the following Ambrosetti-type

**Lemma 1.** If the set $V$ is strongly equicontinuous and uniformly bounded, then
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(i) the function \( t \mapsto \beta(V(t)) \) is continuous on \( J \);
(ii) for each compact subset \( T \) of \( J \)

\[
\beta(V(T)) = \sup \{ \beta(V(t)) : t \in T \},
\]

and Krasnoselskii-type.

**Lemma 2 ([9]).** For any \( \varphi \in E^* \), \( \varepsilon \geq 0 \) and for any weakly continuous function \( z: J \mapsto B \) there exists a weak neighbourhood \( U \) of 0 in \( E \) such that \( \| \varphi(f(t, z(t)) - f(t, y(t))) \| \leq \varepsilon \) for \( t \in J \) and for every weakly continuous function \( y: J \mapsto B \) such that \( y(s) - z(s) \in U \) for all \( s \in J \).

In our considerations we apply the following

**Lemma 3 ([8]).** Let \( m \geq 1 \) be a natural number and let \( w: [0, 2b] \mapsto \mathbb{R}_+ \) be a continuous nondecreasing function such that \( w(0) = 0 \), \( w(r) > 0 \) for \( r > 0 \) and

\[
\int_{0^+} \frac{dr}{\sqrt{r^{m-1}w(r)}} = \infty.
\]

If \( u: [0, c) \mapsto [0, 2b] \) is a \( C^m \) function satisfying the inequalities

\[
\begin{align*}
\quad u^{(j)}(t) & \geq 0, & j = 0, 1, \ldots, m, \\
\quad u^{(j)}(0) & = 0, & j = 0, 1, \ldots, m - 1, \\
\quad u^{(m)}(t) & \leq w(u(t)), & t \in [0, c),
\end{align*}
\]

then \( u = 0 \).

3. The main result

Put

\[
\begin{align*}
M_1 & = \sup \{ \| f(t, x) \| : t \in I, x \in B \}, \\
M_2 & = \sup \{ \| g(t, s, x) \| : t, s \in I, x \in B \}.
\end{align*}
\]

Choose a positive number \( d \) such that \( d \leq a \) and

\[
\sum_{j=1}^{m-1} \| \eta_j \| \frac{d^j}{j!} + M_1 \frac{d^m}{m!} + M_2 \frac{d^{m+1}}{m!} < b. \tag{2}
\]

Let \( J = [0, d] \). By \( C_w(J, E) \) we denote the space of weakly continuous functions \( J \mapsto E \) endowed with the topology of weak uniform convergence and by \( E^* \) the space of continuous linear functionals on \( E \).

Our main result is given by the following Kneser-Hukuhara-type
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Theorem 1. Let \( w : \mathbb{R}_+ \to \mathbb{R}_+ \) be a continuous nondecreasing function such that \( w(0) = 0 \) and
\[
\int_{0^+} \frac{dr}{\sqrt{\frac{m}{r} - 1}} w(r) = \infty. \tag{3}
\]
If
\[
\beta(f(J \times X)) \leq w(\beta(X)) \quad \text{for } X \subset B, \tag{4}
\]
and the set \( g(I^2 \times B) \) is relatively weakly compact in \( E \), then the set \( S \) of all weak solutions of (1) defined on \( J \) is nonempty, compact and connected in \( C_w(J, E) \).

Proof. 1° Let \( \tilde{B} \) denote the set of all weakly continuous functions \( J \mapsto \tilde{B} \). We shall consider \( \tilde{B} \) as a topological subspace of \( C_w(J, E) \). For \( t \in J \) and \( x \in \tilde{B} \) put
\[
\tilde{g}(t, x) = \int_0^t g(t, s, x(s)) \, ds.
\]
Fix \( \tau \in J \) and \( x \in \tilde{B} \). As the set \( J \times x(J) \) is weakly compact, from the weak continuity of \( g \) it follows that for each \( \varepsilon > 0 \) and \( \phi \in E^* \) such that \( \| \phi \| \leq 1 \) there exists \( \delta > 0 \) such that
\[
\phi (g(t, s, x(s)) - g(\tau, s, x(s))) < \varepsilon \quad \text{for } t, s \in J \text{ with } |t - \tau| < \delta.
\]
In view of the inequality
\[
\phi (\tilde{g}(t, x) - \tilde{g}(\tau, x)) \leq M_2 |t - \tau| + \int_0^\tau \phi (g(t, s, x(s)) - g(\tau, s, x(s))) \, ds,
\]
this implies the weak continuity of the function \( t \to \tilde{g}(t, x) \). On the other hand, applying Lemma 2, we can prove that for each fixed \( t \in J \) the function \( x \to \tilde{g}(t, x) \) is weakly continuous on \( \tilde{B} \). Moreover
\[
\| \tilde{g}(t, x) \| \leq M_2 t \quad \text{for } t \in J \text{ and } x \in \tilde{B}.
\]

2° The initial value problem (\( \Pi \)) is equivalent to the following integral equation
\[
x(t) = p(t) + \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} \left[ f(s, x(s)) + \tilde{g}(s, x(s)) \right] ds \quad (t \in J), \tag{5}
\]
where \( p(t) = \sum_{j=1}^{m-1} \eta_j \frac{t^j}{j!} \).

Define the operator \( F \) by the formula
\[
F(x)(t) = p(t) + \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} \left[ f(s, x(s)) + \tilde{g}(s, x(s)) \right] ds \quad (t \in J, x \in \tilde{B}).
\]
For simplicity assume that \( m \geq 2 \).

Let us remark that if
\[
y(t) = \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} f(s, x(s)) \, ds
\]
and
\[
z(t) = \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} \tilde{g}(s, x) \, ds,
\]
then
\[
y'(t) = \frac{1}{(m-2)!} \int_0^t (t-s)^{m-2} f(s, x(s)) \, ds
\]
and
\[
z'(t) = \frac{1}{(m-2)!} \int_0^t (t-s)^{m-2} \tilde{g}(s, x) \, ds,
\]
so that
\[
\|y'(t)\| \leq \frac{1}{(m-2)!} \int_0^t (t-s)^{m-2} M_1 \, ds = M_1 \frac{t^{m-1}}{(m-1)!}
\]
and
\[
\|z'(t)\| \leq \frac{1}{(m-2)!} \int_0^t (t-s)^{m-2} M_2 t \, ds = M_2 \frac{t^m}{(m-1)!}.
\]
Moreover,
\[
\|p'(t)\| \leq \sum_{j=1}^{m-1} \|\eta_j\| \frac{q^{j-1}}{(j-1)!}.
\]
By the mean value theorem we obtain
\[
\|F(x)(t) - F(x)(\tau)\| \leq K \, |t - \tau| \quad (x \in \tilde{B}, \, t, \tau \in J),
\]
where
\[
K = \sum_{j=1}^{m-1} \|\eta_j\| \frac{q^{j-1}}{(j-1)!} + M_1 \frac{t^{m-1}}{(m-1)!} + M_2 \frac{d^m}{(m-1)!}.
\]
Since
\[
\|y(t)\| \leq \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} M_1 \, ds = M_1 \frac{t^m}{m!}
\]
and
\[
\|z(t)\| \leq \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} M_2 t \, ds = M_2 \frac{t^{m+1}}{m!},
\]
moreover.

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\[ \| F(x)(t) \| \leq L \quad (x \in \tilde{B}, \ t \in J), \]  

(7)

where \( L = \sum_{j=1}^{m-1} \| \eta_j \| \frac{d^j}{j!} + M_1 \frac{d^m}{m!} + M_2 \frac{d^{m+1}}{m!} \).

From (2), (6) and (7) it is clear that \( F(\tilde{B}) \subset \tilde{B} \) and the set \( F(\tilde{B}) \) is strongly equicontinuous. By Lemma 2 we can prove that \( F \) is a continuous.

Put \( W = \bigcup_{0 \leq \lambda \leq d} \lambda \text{conv} g(I^2 \times B) \).

Since for convex subsets of \( E \) the closure in the norm topology coincides with the weak closure [4, Th. II. 1], it is clear that

\[
\int_0^t (t-s)^{m-1} \tilde{g}(s, x) \, ds \in t \text{conv} \{ (t-s)^{m-1} \tilde{g}(s, x) : s \in J, \ x \in \tilde{B} \}
\]

from the above and corresponding properties of \( \beta \) it follows that

\[
\beta \left( \left\{ \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} \tilde{g}(s, x) \, ds : x \in \tilde{B} \right\} \right) 
\leq \beta \left( \frac{1}{(m-1)!} t \text{conv} \{ (t-s)^{m-1} W : s \in J \} \right) 
\leq \beta \left( \frac{1}{(m-1)!} t \{ (t-s)^{m-1} W : s \in J \} \right) 
= \max_{s \in J} \left( \frac{1}{(m-1)!} t(t-s)^{m-1} \right) \beta(W) 
= \frac{1}{(m-1)!} t^m \beta(W) = 0. \tag{8}
\]

3° For given \( \varepsilon > 0 \) denote by \( S_\varepsilon \) the set of all \( z \in \tilde{B} \) such that

\[ \| z(t) - F(z)(t) \| < \varepsilon \quad \text{for all } t \in J. \]

The following lemma is proved in [9].

**Lemma 4.** For each \( \varepsilon, \ 0 < \varepsilon < b - L \), the set \( S_\varepsilon \) is nonempty and connected in \( C_w(J, E) \).

For any positive integer \( n \) we define

\[
F_n(x)(t) = \begin{cases} 
p(t) & \text{if } 0 \leq t \leq \frac{d}{n}, \\
p(t) + \frac{1}{(m-1)!} \int_0^{t-s} (t-s)^{m-1} \left[ f(s, x(s)) + \tilde{g}(s, x) \right] ds & \text{if } \frac{d}{n} \leq t \leq d
\end{cases}
\]

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for \( x \in \tilde{B}, t \in J \). Analogously as for \( F \), by inequalities (6) and (7), we can prove that \( F_n \) maps continuously \( \tilde{B} \) into itself and

\[
\|F_n(x)(t) - F(x)(t)\| \leq K\frac{d}{n} \quad (x \in \tilde{B}, t \in J).
\] (9)

Moreover, there exists a unique \( z_n \in \tilde{B} \) such that \( z_n = F_n(z_n) \). It is clear from (9) that \( z_n \in S_\varepsilon \) for sufficiently large \( n \).

Next we shall show that the set \( S \) is nonempty. From the above it follows that there exists a sequence \((u_n)\) such that \( u_n \in \tilde{B} \) and

\[
\lim_{n \to \infty} \sup_{t \in J} \|u_n(t) - F(u_n)(t)\| = 0.
\] (10)

Let \( V = \{u_n : n \in \mathbb{N}\} \). From (6) and (10) we deduce that the set \( V \) is strongly equicontinuous and

\[
\beta(V(t)) = \beta(F(V)(t)) \quad \text{for} \quad t \in J.
\] (11)

Hence, by Lemma 1, the function \( t \mapsto v(t) = \beta(V(t)) \) is continuous on \( J \).

Fix \( t \in J \) and \( \varepsilon > 0 \). Choose \( \delta > 0 \) in such a way that

\[
|\tau - q| < \delta, \quad |q - s| < \delta, \quad q, s, \tau \in J.
\]

if \( |\tau - s| < \delta, |q - s| < \delta, q, s, \tau \in J \). Divide the interval \([0, t]\) into \( n \) parts \( 0 = t_0 < t_1 < \cdots < t_n = t \) in such way that \( \Delta t_i = t_i - t_{i-1} < \delta \) for \( i = 1, \ldots, n \).

Let \( T_i = [t_{i-1}, t_i] \). By Lemma 1 for each \( i \) there exists \( s_i \in T_i \) such that

\[
\beta(V(T_i)) = v(s_i) \quad (i = 1, \ldots, n).
\]

By (4) we obtain

\[
\beta\left(\left\{(t - s)^{m-1}f(s, x(s)): x \in V, s \in T_i\right\}\right)
\]

\[
\leq (t - t_{i-1})^{m-1}\beta\left(f(T_i \times V(T_i))\right)
\]

\[
\leq (t - t_{i-1})^{m-1}w\left(\beta(V(T_i))\right)
\]

\[
= (t - t_{i-1})^{m-1}w\left(v(s_i)\right).
\]

Since

\[
F(V)(t) \subset p(t) + \frac{1}{(m - 1)!} \sum_{i=1}^{n} \Delta t_i \text{conv}\left\{(t - s)^{m-1}f(s, x(s)): x \in V, s \in T_i\right\}
\]

\[
+ \frac{1}{(m - 1)!} \left\{\int_{0}^{t}(t - s)^{m-1}\tilde{g}(s, x) \, ds : x \in V\right\},
\]

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from (8) and corresponding properties of $\beta$ we have

$$\beta(F(V)(t)) \leq \frac{1}{(m-1)!} \beta \left( \sum_{i=1}^{n} \Delta t_i \text{conv} \{ (t-s)^{m-1} f(s, x(s)) : x \in V, s \in T_i \} \right)$$

$$+ \frac{1}{(m-1)!} \beta \left( \left\{ \int_{0}^{t} (t-s)^{m-1} g(s, x) \, ds : x \in V \right\} \right)$$

$$= \frac{1}{(m-1)!} \sum_{i=1}^{n} \Delta t_i \beta \left( (t-s)^{m-1} f(s, x(s)) : x \in V, s \in T_i \right)$$

$$\leq \frac{1}{(m-1)!} \sum_{i=1}^{n} \Delta t_i (t-t_{i-1})^{m-1} w(v(s_i)).$$

Furthermore, from (12) we infer that

$$\left( \frac{1}{(m-1)!} \sum_{i=1}^{n} (t-t_{i-1})^{m-1} w(v(s_i)) \right) \Delta t_i$$

$$\leq \frac{1}{(m-1)!} \int_{0}^{t} (t-s)^{m-1} w(v(s)) \, ds + \frac{\varepsilon t}{(m-1)!}.$$ 

Therefore

$$\beta(F(V)(t)) \leq \frac{1}{(m-1)!} \int_{0}^{t} (t-s)^{m-1} w(v(s)) \, ds + \frac{\varepsilon t}{(m-1)!}.$$ 

Because $\varepsilon$ is arbitrary

$$\beta(F(V)(t)) \leq \frac{1}{(m-1)!} \int_{0}^{t} (t-s)^{m-1} w(v(s)) \, ds.$$ 

Thus, by (11),

$$v(t) \leq \frac{1}{(m-1)!} \int_{0}^{t} (t-s)^{m-1} w(v(s)) \, ds \quad \text{for} \quad t \in J.$$ 

Putting $u(t) = \frac{1}{(m-1)!} \int_{0}^{t} (t-s)^{m-1} w(v(s)) \, ds$ we see that $u \in C^m$, $v(t) \leq u(t)$, $u^{(j)}(t) \geq 0$ for $j = 0, 1, \ldots, m$, $u^{(j)}(0) = 0$ for $j = 0, 1, \ldots, m-1$ and

$$u^{(m)}(t) = w(v(t)) \leq w(u(t)) \quad \text{for} \quad t \in J.$$ 

As $u(0) = 0$, from Lemma 3 we deduce that $u(t) = 0$ for $t \in J$. Consequently, $\beta(V(t)) = v(t) = 0$ for $t \in J$, i.e., $V(t)$ is relatively weakly compact for $t \in J$. Hence Ascoli’s theorem implies that $V$ is relatively compact in $C_w(J, E)$. Therefore the sequence $(u_n)$ has a limit point $x$. From (10) and the continuity of $F$ it follows that $x = F(x)$, i.e., $x \in S$. 

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Now we shall prove that the set $S$ is compact and then that it is connected. Since $F$ is continuous, $S$ is closed in $C_w(J, E)$. As $S = F(S)$, we have $\beta(S(t)) = \beta(F(S)(t))$ for $t \in J$. Therefore, repeating the argument from 3o, we can show that $S$ is compact in $C_w(J, E)$. Suppose that $S$ is not connected in $C_w(J, E)$. As $S$ is compact, there are nonempty compact sets $S_1, S_2$ such that $S = S_1 \cup S_2$ and $S_1 \cap S_2 = \emptyset$, and consequently there are two disjoint open sets $U_1, U_2$ such that $S_1 \subset U_1$, $S_2 \subset U_2$. Let $U = U_1 \cup U_2$. We choose $n_0$ such that $\frac{1}{n_0} < b - L$. Suppose that for each $n \geq n_0$ there exists $u_n \in S_1 \setminus U$. Put $V = \{u_n : n \in \mathbb{N}\}$. Because $\lim_{n \to \infty} \sup_{t \in J} \|u_n(t) - F(u_n)(t)\| = 0$, using once more similar arguments as in 3o, we can prove that there exists $u_0 \in V$ such that $u_0 = F(u_0)$, i.e., $u_0 \in S$. Furthermore, $V \subset C_w(J, E) \setminus U$, as $U$ is open, so that $u_0 \in S \setminus U$, a contradiction. Therefore there exists $k \in \mathbb{N}$ such that $S_k \subset U$. Since $U_1 \cap S_k \neq \emptyset \neq U_2 \cap S_k$, this shows that $S_k$ is not connected, which contradicts Lemma 4. Hence $S$ is connected.

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