ON THE KNESER-HUKUHARA PROPERTY FOR AN INTEGRO-DIFFERENTIAL EQUATION IN BANACH SPACES

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ABSTRACT. In this paper we investigate some topological properties of solutions sets of some integro-differential equations in Banach spaces. Our assumptions and proofs are expressed in terms of the measure of weak noncompactness.

1. Introduction.

Let $I = [0, a]$ be a compact interval in $\mathbb{R}$, $B = \{x \in E: \|x\| \leq b\}$ and let E be a sequentially weakly complete Banach space. Throughout this paper we shall assume that $f: I \times B \mapsto E$ and $g: I^2 \times B \mapsto E$ are functions continuous in the weak—weak sense, that is for every $t \in I$, $x \in B$ and arbitrary weak neighbourhood U of the point $f(t, x)$ there exists an $\varepsilon > 0$ and a weak neighbourhood V of x so that for every $y \in V \cap B$, $s \in I$, $|s - t| < \varepsilon$, $f(s, y) \in U$ is valid.

Consider the Cauchy problem

$$x^{(m)}(t) = f(t, x(t)) + \int_0^t g(t, s, x(s)) ds, \quad (1)$$

$$x(0) = 0, x'(0) = \eta_1, \dots, x^{(m-1)}(0) = \eta_{m-1},$$

where $m \geq 1$ and $\eta_1, \dots, \eta_{m-1} \in E$ and $x^{(m)}$ means the m th order derivative in the weak sense and integral denotes the weak Riemann integral. Let us recall that the weak Riemann integral of a weak continuous function $y(t)$ ($t \in I$) with values in E is defined as the weak limit of Riemann sums (cf. [7]).

In this paper we prove that the set of all weak solutions of this problem, defined on a compact subinterval $J = [0, d]$ of I , for some $d > 0$, is nonempty, compact and connected in the space $C_w(J, E)$ of weakly continuous functions $J \mapsto E$ with the topology of weak uniform convergence.

The method of the proof of our main result is suggested by the paper [9] concerning differential equations. Nevertheless the idea to consider the ε -approximate solutions set of Volterra integral equation

$$x(t) = f(t) + \int_0^t g(t, s, x(s)) ds$$

goes back to H u k u h a r a [6], who proved that this set is connected in $C(J, \mathbb{R}^n)$.

Our approach is to impose a weak compactness type conditions expressed in terms of the measure of weak noncompactness introduced by D e B l a s i [5].

Let A be a nonvoid bounded subset of E . The measure of weak noncompactness $\beta(A)$ is defined by

$\beta(A) = \inf\{\varepsilon > 0 : \text{there exists a weakly compact set } K \text{ such that } A \subset K + \varepsilon B\}$, where B is the norm unit ball.

We make use of the following properties of the measure of weak noncompactness β (for bounded nonvoid subsets A and B of E):

- 1° $A \subset B \Rightarrow \beta(A) \leq \beta(B)$;
- 2° $\beta(\bar{A}^w) = \beta(A)$ where $\bar{A}^w$ denotes the weak closure of A ;
- 3° $\beta(A) = 0 \Leftrightarrow \bar{A}^w$ is weakly compact;
- 4° $\beta(A \cup B) = \max(\beta(A), \beta(B))$;
- 5° $\beta(\text{conv} A) = \beta(A)$;
- 6° $\beta(A + B) \leq \beta(A) + \beta(B)$;
- 7° $\beta(\lambda A) = |\lambda| \beta(A)$, ($\lambda \in \mathbb{R}$);
- 8° $\beta(\bigcup_{|\lambda| \leq h} \lambda A) = h \beta(A)$.

2. Basic lemmas

Let V be a subset of $C_w(J, E)$. Put

$$V(t) = \{u(t) : u \in V\} \quad \text{and} \quad V(T) = \{u(t) : u \in V, t \in T\}.$$

In what follows we shall use the following Ambrosetti-type

LEMMA 1. *If the set V is strongly equicontinuous and uniformly bounded, then*

- (i) the function $t \mapsto \beta(V(t))$ is continuous on J ;
- (ii) for each compact subset T of J

$$\beta(V(T)) = \sup\{\beta(V(t)) : t \in T\},$$

and Krasnoselskii-type.

LEMMA 2 ([9]). *For any $\varphi \in E^*$, $\varepsilon \geq 0$ and for any weakly continuous function $z: J \mapsto B$ there exists a weak neighbourhood U of 0 in E such that $\|\varphi(f(t, z(t)) - f(t, y(t)))\| \leq \varepsilon$ for $t \in J$ and for every weakly continuous function $y: J \mapsto B$ such that $y(s) - z(s) \in U$ for all $s \in J$.*

In our considerations we apply the following

LEMMA 3 ([8]). *Let $m \geq 1$ be a natural number and let $w: [0, 2b] \mapsto \mathbb{R}_+$ be a continuous nondecreasing function such that $w(0) = 0$, $w(r) > 0$ for $r > 0$ and*

$$\int_{0+} \frac{dr}{\sqrt[m]{r^{m-1}w(r)}} = \infty.$$

If $u: [0, c] \mapsto [0, 2b]$ is a C^m function satisfying the inequalities

$$\begin{aligned} u^{(j)}(t) &\geq 0, & j &= 0, 1, \dots, m, \\ u^{(j)}(0) &= 0, & j &= 0, 1, \dots, m-1, \\ u^{(m)}(t) &\leq w(u(t)), & t &\in [0, c), \end{aligned}$$

then $u = 0$.

3. The main result

Put

$$\begin{aligned} M_1 &= \sup\{\|f(t, x)\| : t \in I, x \in B\}, \\ M_2 &= \sup\{\|g(t, s, x)\| : t, s \in I, x \in B\}. \end{aligned}$$

Choose a positive number d such that $d \leq a$ and

$$\sum_{j=1}^{m-1} \|\eta_j\| \frac{d^j}{j!} + M_1 \frac{d^m}{m!} + M_2 \frac{d^{m+1}}{m!} < b. \quad (2)$$

Let $J = [0, d]$. By $C_w(J, E)$ we denote the space of weakly continuous functions $J \mapsto E$ endowed with the topology of weak uniform convergence and by E^* the space of continuous linear functionals on E .

Our main result is given by the following Kneser-Hukuhara-type

THEOREM 1. *Let $w: \mathbb{R}_+ \mapsto \mathbb{R}_+$ be a continuous nondecreasing function such that $w(0) = 0$ and*

$$\int_{0+} \frac{dr}{\sqrt[m]{r^{m-1}w(r)}} = \infty. \quad (3)$$

If

$$\beta(f(J \times X)) \leq w(\beta(X)) \quad \text{for } X \subset B, \quad (4)$$

and the set $g(I^2 \times B)$ is relatively weakly compact in E , then the set S of all weak solutions of (1) defined on J is nonempty, compact and connected in $C_w(J, E)$.

P r o o f. **1°** Let $\tilde{B}$ denote the set of all weakly continuous functions $J \mapsto B$. We shall consider $\tilde{B}$ as a topological subspace of $C_w(J, E)$. For $t \in J$ and $x \in \tilde{B}$ put

$$\tilde{g}(t, x) = \int_0^t g(t, s, x(s)) ds.$$

Fix $\tau \in J$ and $x \in \tilde{B}$. As the set $J \times x(J)$ is weakly compact, from the weak continuity of g it follows that for each $\varepsilon > 0$ and $\phi \in E^*$ such that $\|\phi\| \leq 1$ there exists $\delta > 0$ such that

$$\phi\left(g(t, s, x(s)) - g(\tau, s, x(s))\right) < \varepsilon \quad \text{for } t, s \in J \text{ with } |t - \tau| < \delta.$$

In view of the inequality

$$\phi\left(\tilde{g}(t, x) - \tilde{g}(\tau, x)\right) \leq M_2|t - \tau| + \int_0^\tau \phi\left(g(t, s, x(s)) - g(\tau, s, x(s))\right) ds,$$

this implies the weak continuity of the function $t \rightarrow \tilde{g}(t, x)$. On the other hand, applying Lemma 2, we can prove that for each fixed $t \in J$ the function $x \rightarrow \tilde{g}(t, x)$ is weakly continuous on $\tilde{B}$. Moreover

$$\|\tilde{g}(t, x)\| \leq M_2 t \quad \text{for } t \in J \text{ and } x \in \tilde{B}.$$

2° The initial value problem (1) is equivalent to the following integral equation

$$x(t) = p(t) + \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} \left[f(s, x(s)) + \tilde{g}(s, x) \right] ds \quad (t \in J), \quad (5)$$

where $p(t) = \sum_{j=1}^{m-1} \eta_j \frac{t^j}{j!}$.

Define the operator F by the formula

$$F(x)(t) = p(t) + \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} \left[f(s, x(s)) + \tilde{g}(s, x) \right] ds \quad (t \in J, x \in \tilde{B}).$$

For simplicity assume that $m \geq 2$.

Let us remark that if

$$y(t) = \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} f(s, x(s)) ds$$

and

$$z(t) = \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} \tilde{g}(s, x) ds,$$

then

$$y'(t) = \frac{1}{(m-2)!} \int_0^t (t-s)^{m-2} f(s, x(s)) ds$$

and

$$z'(t) = \frac{1}{(m-2)!} \int_0^t (t-s)^{m-2} \tilde{g}(s, x) ds,$$

so that

$$\|y'(t)\| \leq \frac{1}{(m-2)!} \int_0^t (t-s)^{m-2} M_1 ds = M_1 \frac{t^{m-1}}{(m-1)!}$$

and

$$\|z'(t)\| \leq \frac{1}{(m-2)!} \int_0^t (t-s)^{m-2} M_2 t ds = M_2 \frac{t^m}{(m-1)!}.$$

Moreover,

$$\|p'(t)\| \leq \sum_{j=1}^{m-1} \|\eta_j\| \frac{d^{j-1}}{(j-1)!}.$$

By the mean value theorem we obtain

$$\|F(x)(t) - F(x)(\tau)\| \leq K |t - \tau| \quad (x \in \tilde{B}, t, \tau \in J), \quad (6)$$

where

$$K = \sum_{j=1}^{m-1} \|\eta_j\| \frac{d^{j-1}}{(j-1)!} + M_1 \frac{d^{m-1}}{(m-1)!} + M_2 \frac{d^m}{(m-1)!}.$$

Since

$$\|y(t)\| \leq \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} M_1 ds = M_1 \frac{t^m}{m!}$$

and

$$\|z(t)\| \leq \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} M_2 t ds = M_2 \frac{t^{m+1}}{m!},$$

$$\|F(x)(t)\| \leq L \quad (x \in \tilde{B}, t \in J), \quad (7)$$

where $L = \sum_{j=1}^{m-1} \|\eta_j\| \frac{d^j}{j!} + M_1 \frac{d^m}{m!} + M_2 \frac{d^{m+1}}{m!}$.

From (2), (6) and (7) it is clear that $F(\tilde{B}) \subset \tilde{B}$ and the set $F(\tilde{B})$ is strongly equicontinuous. By Lemma 2 we can prove that F is a continuous.

Put $W = \bigcup_{0 \leq \lambda \leq d} \lambda \overline{\text{conv}} g(I^2 \times B)$.

Since for convex subsets of E the closure in the norm topology coincides with the weak closure [4, Th. II. 1], it is clear that

$$\begin{aligned} \int_0^t (t-s)^{m-1} \tilde{g}(s, x) ds &\in t \overline{\text{conv}} \left\{ (t-s)^{m-1} \tilde{g}(s, x) : s \in J, x \in \tilde{B} \right\} \\ &= t \overline{\text{conv}} \left\{ (t-s)^{m-1} W : s \in [0, t] \right\}, \end{aligned}$$

from the above and corresponding properties of β it follows that

$$\begin{aligned} &\beta \left(\left\{ \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} \tilde{g}(s, x) ds : x \in \tilde{B} \right\} \right) \\ &\leq \beta \left(\frac{1}{(m-1)!} t \overline{\text{conv}} \left\{ (t-s)^{m-1} W : s \in J \right\} \right) \\ &\leq \beta \left(\frac{1}{(m-1)!} t \left\{ (t-s)^{m-1} W : s \in J \right\} \right) \\ &= \max_{s \in J} \left(\frac{1}{(m-1)!} t (t-s)^{m-1} \right) \beta(W) \\ &= \frac{1}{(m-1)!} t^m \beta(W) = 0. \end{aligned} \quad (8)$$

3° For given $\varepsilon > 0$ denote by S_ε the set of all $z \in \tilde{B}$ such that

$$\|z(t) - F(z)(t)\| < \varepsilon \quad \text{for all } t \in J.$$

The following lemma is proved in [9].

LEMMA 4. *For each ε , $0 < \varepsilon < b - L$, the set S_ε is nonempty and connected in $C_w(J, E)$.*

For any positive integer n we define

$$F_n(x)(t) = \begin{cases} p(t) & \text{if } 0 \leq t \leq \frac{d}{n}, \\ p(t) + \frac{1}{(m-1)!} \int_0^{t-\frac{d}{n}} (t-s)^{m-1} \left[f(s, x(s)) + \tilde{g}(s, x) \right] ds & \text{if } \frac{d}{n} \leq t \leq d \end{cases}$$

for $x \in \widetilde{B}$, $t \in J$. Analogously as for F , by inequalities (6) and (7), we can prove that F_n maps continuously $\widetilde{B}$ into itself and

$$\|F_n(x)(t) - F(x)(t)\| \leq K \frac{d}{n} \quad (x \in \widetilde{B}, t \in J). \quad (9)$$

Moreover, there exists a unique $z_n \in \widetilde{B}$ such that $z_n = F_n(z_n)$. It is clear from (9) that $z_n \in S_\varepsilon$ for sufficiently large n .

Next we shall show that the set S is nonempty. From the above it follows that there exists a sequence (u_n) such that $u_n \in \widetilde{B}$ and

$$\lim_{n \rightarrow \infty} \sup_{t \in J} \|u_n(t) - F(u_n)(t)\| = 0. \quad (10)$$

Let $V = \{u_n : n \in N\}$. From (6) and (10) we deduce that the set V is strongly equicontinuous and

$$\beta(V(t)) = \beta(F(V)(t)) \quad \text{for } t \in J. \quad (11)$$

Hence, by Lemma 1, the function $t \mapsto v(t) = \beta(V(t))$ is continuous on J .

Fix $t \in J$ and $\varepsilon > 0$. Choose $\delta > 0$ in such a way that

$$|(t - \tau)^{m-1} w(v(q)) - (t - s)^{m-1} w(v(s))| < \varepsilon \quad (12)$$

if $|\tau - s| < \delta$, $|q - s| < \delta$, $q, s, \tau \in J$. Divide the interval $[0, t]$ into n parts $0 = t_0 < t_1 < \dots < t_n = t$ in such way that $\Delta t_i = t_i - t_{i-1} < \delta$ for $i = 1, \dots, n$. Let $T_i = [t_{i-1}, t_i]$. By Lemma 1 for each i there exists $s_i \in T_i$ such that

$$\beta(V(T_i)) = v(s_i) \quad (i = 1, \dots, n).$$

By (4) we obtain

$$\begin{aligned} & \beta\left(\left\{(t-s)^{m-1} f(s, x(s)) : x \in V, s \in T_i\right\}\right) \\ & \leq (t - t_{i-1})^{m-1} \beta\left(f(T_i \times V(T_i))\right) \\ & \leq (t - t_{i-1})^{m-1} w\left(\beta(V(T_i))\right) \\ & = (t - t_{i-1})^{m-1} w(v(s_i)). \end{aligned}$$

Since

$$\begin{aligned} F(V)(t) & \subset p(t) + \frac{1}{(m-1)!} \sum_{i=1}^n \Delta t_i \overline{\text{conv}} \left\{ (t-s)^{m-1} f(s, x(s)) : x \in V, s \in T_i \right\} \\ & + \frac{1}{(m-1)!} \left\{ \int_0^t (t-s)^{m-1} \widetilde{g}(s, x) ds : x \in V \right\}, \end{aligned}$$

from (8) and corresponding properties of β we have

$$\begin{aligned}
 \beta(F(V)(t)) &\leq \frac{1}{(m-1)!} \beta \left(\sum_{i=1}^n \Delta t_i \overline{\text{conv}} \{ (t-s)^{m-1} f(s, x(s)) : x \in V, s \in T_i \} \right) \\
 &\quad + \frac{1}{(m-1)!} \beta \left(\left\{ \int_0^t (t-s)^{m-1} \tilde{g}(s, x) ds : x \in V \right\} \right) \\
 &= \frac{1}{(m-1)!} \sum_{i=1}^n \Delta t_i \beta \left\{ (t-s)^{m-1} f(s, x(s)) : x \in V, s \in T_i \right\} \\
 &\leq \frac{1}{(m-1)!} \sum_{i=1}^n \Delta t_i (t-t_{i-1})^{m-1} w(v(s_i)).
 \end{aligned}$$

Furthermore, from (12) we infer that

$$\begin{aligned}
 &\frac{1}{(m-1)!} \sum_{i=1}^n (t-t_{i-1})^{m-1} w(v(s_i)) \Delta t_i \\
 &\leq \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} w(v(s)) ds + \frac{\varepsilon t}{(m-1)!}.
 \end{aligned}$$

Therefore

$$\beta(F(V)(t)) \leq \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} w(v(s)) ds + \frac{\varepsilon t}{(m-1)!}.$$

Because ε is arbitrary

$$\beta(F(V)(t)) \leq \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} w(v(s)) ds.$$

Thus, by (11),

$$v(t) \leq \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} w(v(s)) ds \quad \text{for } t \in J.$$

Putting $u(t) = \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} w(v(s)) ds$ we see that $u \in C^m$, $v(t) \leq u(t)$, $u^{(j)}(t) \geq 0$ for $j = 0, 1, \dots, m$, $u^{(j)}(0) = 0$ for $j = 0, 1, \dots, m-1$ and

$$u^{(m)}(t) = w(v(t)) \leq w(u(t)) \quad \text{for } t \in J.$$

As $u(0) = 0$, from Lemma 3 we deduce that $u(t) = 0$ for $t \in J$. Consequently, $\beta(V(t)) = v(t) = 0$ for $t \in J$, i.e., $V(t)$ is relatively weakly compact for $t \in J$. Hence Ascoli's theorem implies that V is relatively compact in $C_w(J, E)$. Therefore the sequence (u_n) has a limit point x . From (10) and the continuity of F it follows that $x = F(x)$, i.e., $x \in S$.

4° Now we shall prove that the set S is compact and then that it is connected. Since F is continuous, S is closed in $C_w(J, E)$. As $S = F(S)$, we have $\beta(S(t)) = \beta(F(S)(t))$ for $t \in J$. Therefore, repeating the argument from 3°, we can show that S is compact in $C_w(J, E)$. Suppose that S is not connected in $C_w(J, E)$. As S is compact, there are nonempty compact sets S_1, S_2 such that $S = S_1 \cup S_2$ and $S_1 \cap S_2 = \emptyset$, and consequently there are two disjoint open sets U_1, U_2 such that $S_1 \subset U_1, S_2 \subset U_2$. Let $U = U_1 \cup U_2$. We choose n_0 such that $\frac{1}{n_0} < b - L$. Suppose that for each $n \geq n_0$ there exists $u_n \in S_{\frac{1}{n}} \setminus U$. Put $V = \{u_n : n \in N\}$. Because $\lim_{n \rightarrow \infty} \sup_{t \in J} \|u_n(t) - F(u_n)(t)\| = 0$, using once more similar arguments as in 3°, we can prove that there exists $u_0 \in \overline{V}$ such that $u_0 = F(u_0)$, i.e., $u_0 \in S$. Furthermore, $\overline{V} \subset C_w(J, E) \setminus U$, as U is open, so that $u_0 \in S \setminus U$, a contradiction. Therefore there exists $k \in N$ such that $S_{\frac{1}{k}} \subset U$. Since $U_1 \cap S_{\frac{1}{k}} \neq \emptyset \neq U_2 \cap S_{\frac{1}{k}}$, this shows that $S_{\frac{1}{k}}$ is not connected, which contradicts Lemma 4. Hence S is connected. $\square$

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