HALF-LINEAR EULER DIFFERENTIAL EQUATIONS IN THE CRITICAL CASE

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ABSTRACT. We investigate oscillatory properties of the perturbed half-linear Euler differential equation

$$(\Phi(x'))' + \frac{\gamma_p}{t^p} \Phi(x) = 0, \quad \Phi(x) := |x|^{p-2}x, \quad \gamma_p := \left(\frac{p-1}{p}\right)^p.$$

A perturbation is also allowed in the coefficient involving derivative.

1. Introduction

The half-linear Euler differential equation

$$(\Phi(x'))' + \frac{\gamma}{t^p} \Phi(x) = 0, \quad \Phi(x) := |x|^{p-2}x, \quad p > 1, \gamma \in \mathbb{R}, \quad (1)$$

is one of a few half-linear second order differential equations which can be solved explicitly. Similarly to the linear case $p = 2$, if we look for a solution of (1) in the form $x(t) = t^\lambda$ substituting into (1), we find that λ has to be a solution of the algebraic equation

$$(p-1)\Phi(\lambda)(\lambda-1) + \gamma = 0,$$

see also [5]. The function $F(\lambda) := (p-1)\Phi(\lambda)(\lambda-1)$ has a global minimum at $\lambda^* = \frac{p-1}{p}$ and the value of this minimum is $F(\lambda^*) = -\gamma_p := -\left(\frac{p-1}{p}\right)^p$. Consequently, the equation $F(\lambda) + \gamma = 0$ has two real roots if $\gamma < \gamma_p$, one double real root if $\gamma = \gamma_p$, and no real root if $\gamma > \gamma_p$.

Equation (1) is a particular case of the general half-linear second order differential equation

$$(r(t)\Phi(x'))' + c(t)\Phi(x) = 0, \quad (2)$$

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where r, c are continuous functions and $r(t) > 0$. It is known that oscillation theory of (2) is almost the same as this theory for the second order Sturm-Liouville linear differential equation

$$(r(t)x')' + c(t)x = 0 \quad (3)$$

which is a special case $p = 2$ in (2). In particular, all solutions of (2) are either oscillatory or nonoscillatory, see [3]. This means, in particular, that (1) is nonoscillatory if and only if $\gamma \leq \gamma_p$. Also, equation (1) with the critical coefficient $\gamma = \gamma_p$ serves as a comparison equation for the Kneser-type (non)oscillation test which states that (2) with $r(t) = 1$ is oscillatory provided

$$\liminf_{t \rightarrow \infty} t^p c(t) > \gamma_p$$

and nonoscillatory if

$$\limsup_{t \rightarrow \infty} t^p c(t) < \gamma_p.$$

The Kneser test does not apply when $\lim_{t \rightarrow \infty} t^p c(t) = \gamma_p$ and this situation is the principal concern of our paper.

2. Auxiliary results

In a general framework, we suppose that equation (2) is nonoscillatory and we study oscillatory properties of its perturbation

$$\left[(r(t) + \tilde{r}(t))\Phi(x') \right]' + (c(t) + \tilde{c}(t))\Phi(x) = 0. \quad (4)$$

We suppose that the perturbation terms $\tilde{r}, \tilde{c}$ are continuous functions such that $r(t) + \tilde{r}(t) > 0$ for large t .

Let x be a solution of (2) such that $x(t) \neq 0$, then $w = r\Phi\left(\frac{x'}{x}\right)$ is a solution of the Riccati type differential equation

$$w' + c(t) + (p-1)r^{1-q}(t)|w|^q = 0, \quad q := \frac{p}{p-1}.$$

In our research, an important role plays the concept of conditionally oscillatory half-linear equation. Following [3], equation (2) with $\lambda c(t)$ instead of $c(t)$ is said to be *conditionally oscillatory* if there exists a constant λ_0 such that this equation is oscillatory for $\lambda > \lambda_0$ and nonoscillatory for $\lambda < \lambda_0$. The constant λ_0 is called the *oscillation constant* of (2). A typical example of a conditionally oscillatory equation is just Euler equation (1) and its oscillation constant is $\lambda_0 = \gamma_p$.

Here we will deal with conditionally oscillatory half-linear equations in a more general sense. We will consider the equation of the form

$$\left[(r(t) + \lambda \tilde{r}(t))\Phi(x') \right]' + (c(t) + \mu \tilde{c}(t))\Phi(x) = 0 \quad (5)$$

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and we say that (5) is conditionally oscillatory if there exist constants $\alpha, \beta, \omega \in \mathbb{R}$, $\alpha \neq 0, \beta \neq 0$, such that (5) is oscillatory for $\alpha\lambda + \beta\mu > \omega$ and nonoscillatory for $\alpha\lambda + \beta\mu < \omega$. A typical example of conditionally oscillatory equation with two parameters is perturbed Euler equation (1) with the critical coefficient γ_p

$$\left[\left(1 + \frac{\lambda}{\log^2 t} \right) \Phi(x') \right]' + \left[\frac{\gamma_p}{t^p} + \frac{\mu}{t^p \log^2 t} \right] \Phi(x) = 0. \quad (6)$$

It is proved in [2] that (6) is oscillatory if $\mu - \lambda\gamma_p > \mu_p := \frac{1}{2} \left(\frac{p-1}{p} \right)^{p-1}$ and nonoscillatory if $\mu - \lambda\gamma_p < \mu_p$. It was conjectured in [2] that (6) is nonoscillatory also in the limiting case

$$\mu - \lambda\gamma_p = \mu_p. \quad (7)$$

In our paper we prove that this conjecture is true. We also discuss some general aspects of the problem.

Next we derive the modified Riccati equation in a more general setting than in [2]. Let $h(t) \neq 0$ be a positive differentiable function, denote

$$G(t) := r(t)h(t)\Phi(h'(t)) \quad (8)$$

and let

$$\Omega(t) := \left(1 + \frac{\tilde{r}(t)}{r(t)} \right) G(t).$$

Define the function

$$\mathcal{G}(t, z) := |z + \Omega(t)|^q - q\Phi^{-1}(\Omega(t))z - |\Omega(t)|^q, \quad (9)$$

where $\Phi^{-1}(x) = |x|^{q-2}x$ is the inverse function of Φ , and put

$$z := h^p(w - w_h) - \frac{\tilde{r}}{r}G = h^p w - G - \tilde{G},$$

where w is a solution of the Riccati equation associated with (4)

$$w' + c(t) + \tilde{c}(t) + (p-1)(r(t) + \tilde{r}(t))^{1-q}|w|^q = 0,$$

$w_h = r\Phi\left(\frac{h'}{h}\right)$, and $\tilde{G} = \tilde{r}h\Phi(h')$. Then z is a solution of the equation

$$z' + C(t) + (p-1)(r(t) + \tilde{r}(t))^{1-q}h^{-q}(t)\mathcal{G}(t, z) = 0, \quad (10)$$

where

$$C(t) = h(t) \left[\left((r(t) + \tilde{r}(t))\Phi(h'(t)) \right)' + (c(t) + \tilde{c}(t))\Phi(h(t)) \right]. \quad (11)$$

Indeed, by a direct computation we have

$$\begin{aligned}
 z' &= p\Phi(h)h'w + h^pw' - r|h'|^p - h(r\Phi(h'))' - \tilde{r}|h'|^p - h(\tilde{r}\Phi(h'))' \\
 &= p\Phi(h)h'w + h^p \left[-(c + \tilde{c}) - (p-1)(r + \tilde{r})^{1-q}|w|^q \right] \\
 &\quad - (r + \tilde{r})|h'|^p - h[(r + \tilde{r})\Phi(h)]' \\
 &= p\frac{h'}{h}(z + G + \tilde{G}) - (r + \tilde{r})|h'|^p - h \left[((r + \tilde{r})\Phi(h'))' + (c + \tilde{c})\Phi(h) \right] \\
 &\quad - (p-1)h^p(r + \tilde{r})^{1-q} \left| h^{-p}z + w_h + \tilde{r}\Phi\left(\frac{h'}{h}\right) \right|^q \\
 &= -h \left[((r + \tilde{r})\Phi(h'))' + (c + \tilde{c})\Phi(h) \right] - (p-1)(r + \tilde{r})^{1-q}h^{-q} \\
 &\quad \times \left\{ |z + \Omega|^q - \frac{p}{p-1}h'(r + \tilde{r})^{q-1}h^{q-1}z - (r + \tilde{r})^qh^q|h'|^p \right\} \\
 &= -h \left[((r + \tilde{r})\Phi(h'))' + (c + \tilde{c})\Phi(h) \right] \\
 &\quad - (p-1)(r + \tilde{r})^{1-q}h^{-q} \left\{ |z + \Omega|^q - q\Phi^{-1}(\Omega)z - |\Omega|^q \right\}.
 \end{aligned}$$

Note that in contrast to [2] we do not suppose that h is a solution of (2), so the extra term

$$h[(r\Phi(h'))' + c\Phi(h)]$$

appears in the definition of the function C in (11).

We will also need the following statement which is a slight modification of [2, Theorem 3] (here we do not require that the function h is a solution of (2)). So we omit its proof since it is the same as that of [2, Theorem 3] and it based on the fact that solvability of (10) implies nonoscillation of (4).

THEOREM 1. *Let h be a positive differentiable function such that $h'(t) \neq 0$ for large t . Denote*

$$R(t) = (r(t) + \tilde{r}(t))h^2(t)|h'(t)|^{p-2}, \quad (12)$$

and suppose that

$$\int_0^\infty \frac{dt}{R(t)} = \infty, \quad \int_0^\infty C(t) dt \text{ is convergent,}$$

where C is given by (11), and

$$\liminf_{t \rightarrow \infty} (r(t) + \tilde{r}(t))h(t)|h'(t)|^{p-1} > 0.$$

If

$$\limsup_{t \rightarrow \infty} \int_0^t \frac{ds}{R(s)} \int_t^\infty C(s) ds < \frac{1}{2q} \quad (13)$$

and

$$\liminf_{t \rightarrow \infty} \int_0^t \frac{ds}{R(s)} \int_t^\infty C(s) ds > -\frac{3}{2q}, \quad (14)$$

then equation (4) is nonoscillatory.

3. Equation (6) in the limiting case

Now we can prove the main result of our paper.

THEOREM 2. *Suppose that (7) holds. Then the perturbed Euler equation with the critical coefficients (6) is nonoscillatory.*

Proof. We rewrite (6) into the form

$$\left[\left(1 + \frac{\lambda}{\log^2 t} \right) \Phi(x') \right]' + \left[\frac{\gamma_p}{t^p} + \frac{\mu_p}{t^p \log^2 t} + \frac{\mu - \mu_p}{t^p \log^2 t} \right] \Phi(x) = 0 \quad (15)$$

and we use the previous computation with

$$r(t) = 1, \quad \tilde{r}(t) = \frac{\lambda}{\log^2 t}, \quad c(t) = \frac{\gamma_p}{t^p} + \frac{\mu_p}{t^p \log^2 t}, \quad \tilde{c}(t) = \frac{\mu - \mu_p}{t^p \log^2 t}, \quad h(t) = t^{\frac{p-1}{p}} \log^{\frac{1}{p}} t.$$

We have

$$h' = \frac{p-1}{p} t^{-\frac{1}{p}} \log^{\frac{1}{p}} t + \frac{1}{p} t^{-\frac{1}{p}} \log^{\frac{1}{p}-1} t = \frac{p-1}{p} t^{-\frac{1}{p}} \log^{\frac{1}{p}} t \left(1 + \frac{1}{(p-1) \log t} \right),$$

$$\Phi(h') = \left(\frac{p-1}{p} \right)^{p-1} t^{-\frac{p-1}{p}} \log^{\frac{p-1}{p}} t \left(1 + \frac{1}{(p-1) \log t} \right)^{p-1},$$

and

$$\begin{aligned} (\Phi(h'))' &= \left(\frac{p-1}{p} \right)^{p-1} \left[-\frac{p-1}{p} t^{-2+\frac{1}{p}} \log^{\frac{p-1}{p}} t \left(1 + \frac{1}{(p-1) \log t} \right)^{p-1} \right. \\ &\quad + \frac{p-1}{p} t^{-2+\frac{1}{p}} \log^{-\frac{1}{p}} t \left(1 + \frac{1}{(p-1) \log t} \right)^{p-1} \\ &\quad \left. - t^{-2+\frac{1}{p}} \log^{\frac{p-1}{p}} t \log^{-2} t \left(1 + \frac{1}{(p-1) \log t} \right)^{p-2} \right] \end{aligned}$$

$$\begin{aligned}
 &= \left(\frac{p-1}{p} \right)^{p-1} \frac{p-1}{p} t^{-2+\frac{1}{p}} \log^{\frac{p-1}{p}} t \left(1 + \frac{1}{(p-1) \log t} \right)^{p-2} \\
 &\quad \times \left[-1 - \frac{1}{(p-1) \log t} + \frac{1}{\log t} \left(1 + \frac{1}{(p-1) \log t} \right) - \frac{p}{p-1} \frac{1}{\log^2 t} \right] \\
 &= \gamma_p t^{-2+\frac{1}{p}} \log^{\frac{p-1}{p}} t \left[1 + \frac{p-2}{(p-1) \log t} + \binom{p-2}{2} \frac{1}{(p-1)^2 \log^2 t} \right. \\
 &\quad \left. + \binom{p-2}{3} \frac{1}{(p-1)^3 \log^3 t} + o(\log^{-3} t) \right] \left[-1 + \frac{1}{\log t} \left(1 - \frac{1}{p-1} \right) \right. \\
 &\quad \left. + \frac{1}{\log^2 t} \left(\frac{1}{p-1} - \frac{p}{p-1} \right) \right] \\
 &= \gamma_p t^{-2+\frac{1}{p}} \log^{\frac{p-1}{p}} t \left[-1 + \frac{1}{\log t} \left(-\frac{p-2}{p-1} + \frac{p-2}{p-1} \right) \right. \\
 &\quad \left. + \frac{1}{\log^2 t} \left(-\frac{(p-2)(p-3)}{2(p-1)^2} + \frac{(p-2)^2}{(p-1)^2} - 1 \right) + o(\log^{-2} t) \right] \\
 &= \gamma_p t^{-2+\frac{1}{p}} \log^{\frac{p-1}{p}} t \left[-1 - \frac{p}{2(p-1) \log^2 t} + O(\log^{-3} t) \right] \\
 &= t^{-2+\frac{1}{p}} \log^{\frac{p-1}{p}} t \left[-\gamma_p - \frac{\mu_p}{\log^2 t} + O(\log^{-3} t) \right], \quad \text{as } t \rightarrow \infty.
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 &\left(\frac{\lambda}{\log^2 t} \Phi(h') \right)' = \lambda \left(\frac{p-1}{p} \right)^{p-1} \left[t^{-\frac{p-1}{p}} \log^{-1-\frac{1}{p}} t \left(1 + \frac{1}{(p-1) \log t} \right)^{p-1} \right]' \\
 &= \lambda \left(\frac{p-1}{p} \right)^{p-1} t^{-2+\frac{1}{p}} \log^{-1-\frac{1}{p}} t \left[1 + \frac{p-2}{(p-1) \log t} + o(\log^{-1} t) \right] \\
 &\quad \times \left[-\frac{p-1}{p} \left(1 + \frac{1}{(p-1) \log t} \right) - \left(1 + \frac{1}{p} \right) \frac{1}{\log t} \left(1 + \frac{1}{(p-1) \log t} \right) - \frac{1}{\log^2 t} \right] \\
 &= \lambda \left(\frac{p-1}{p} \right)^{p-1} t^{-2+\frac{1}{p}} \log^{-1-\frac{1}{p}} t \left[1 + \frac{p-2}{(p-1) \log t} + o(\log^{-1} t) \right] \\
 &\quad \times \left[-\frac{p-1}{p} - \frac{p+2}{p \log t} + o(\log^{-1}) \right] \\
 &= \lambda \left(\frac{p-1}{p} \right)^{p-1} t^{-2+\frac{1}{p}} \log^{-1-\frac{1}{p}} t \left[-\frac{p-1}{p} - \frac{2}{\log t} + o(\log^{-1} t) \right] \quad \text{as } t \rightarrow \infty.
 \end{aligned}$$

Hence, in the limiting case (7) it holds

$$\begin{aligned} & \left[(\tilde{r}\Phi(h'))' + \tilde{c}\Phi(h) \right] \\ &= t^{-2+\frac{1}{p}} \log^{-1-\frac{1}{p}} t \left[-\lambda\gamma_p + \mu - \mu_p - 2\lambda \left(\frac{p-1}{p} \right)^{p-1} \frac{1}{\log t} + o(\log^{-1} t) \right] \\ & \quad - 2\lambda \left(\frac{p}{p-1} \right)^{p-1} t^{-2+\frac{1}{p}} \log^{-2-\frac{1}{p}} t (1 + o(1)) \quad \text{as } t \rightarrow \infty. \end{aligned}$$

Consequently,

$$\begin{aligned} & h \left[(\tilde{r}\Phi(h'))' + \tilde{c}\Phi(h) \right] \\ &= -2\lambda t^{\frac{p-1}{p}} \log^{\frac{1}{p}} t \left(\frac{p}{p-1} \right)^{p-1} t^{-2+\frac{1}{p}} \log^{-2-\frac{1}{p}} t (1 + o(1)) \\ &= O\left(t^{-1} \log^{-2} t\right) \quad \text{as } t \rightarrow \infty. \end{aligned}$$

Now we use Theorem 1. In this theorem

$$R = (r + \tilde{r})h^2|h'|^{p-2} = t \log t (1 + o(1)) \sim t \log t$$

(here $f(t) \sim g(t)$ for a pair of functions f, g means $\lim_{t \rightarrow \infty} \frac{f(t)}{g(t)} = 1$),

$$\begin{aligned} & (r + \tilde{r})h\Phi(h') \\ &= \left(\frac{p-1}{p} \right)^{p-1} \left(1 + \frac{\lambda}{\log^2 t} \right) \log t \left(1 + \frac{1}{(p-1) \log t} \right)^{p-1} \rightarrow \infty \quad \text{as } t \rightarrow \infty, \end{aligned}$$

and using the previous computations

$$\begin{aligned} C &= h \left[((r + \tilde{r})\Phi(h'))' + (c + \tilde{c})\Phi(h) \right] \\ &= O\left(t^{-1} \log^{-2} t\right) \quad \text{as } t \rightarrow \infty, \end{aligned}$$

i.e., there exists a constant $M > 0$ such that $|C(t)| \leq Mt^{-1} \log^{-2} t$ for large t . Now, by a direct computation

$$\lim_{t \rightarrow \infty} \left| \int_0^t R^{-1}(s) ds \int_t^\infty C(s) ds \right| \leq M \lim_{t \rightarrow \infty} \frac{\log(\log t)}{\log t} = 0,$$

so by Theorem 1 equation (6) with λ and μ satisfying (7) is nonoscillatory. $\square$

4. Open problems

- (i) In equation (6), the functions $\tilde{r}(t) = \frac{1}{\log^2 t}$, $\tilde{c}(t) = \frac{1}{t^p \log^2 t}$ “match together”, i.e., for $r(t) = 1$ and $c(t) = \gamma_p t^{-p}$ they have such asymptotic growth for $t \rightarrow \infty$ that equation (6) is conditionally oscillatory. This fact is likely a special case of the general situation which is a subject of the present investigation. More precisely, given the functions r, c , we look for functions $\tilde{r}, \tilde{c}$ with such asymptotic growth that equation (5) is conditionally oscillatory. For $\tilde{r} = 0$, this problem has been studied in [4], where conditions on unperturbed equation (2) are found under which its perturbation

$$(r(t)\Phi(x'))' + \left[c(t) + \frac{\mu}{h^p(t)R(t)\left(\int_0^t R^{-1}(s) ds\right)^2} \right] \Phi(x) = 0$$

is conditionally oscillatory (and its oscillation constant is $\mu_0 = \frac{1}{2q}$, where q is the conjugate exponent to p , i.e., $\frac{1}{p} + \frac{1}{q} = 1$). Here h is the so called principal solution of (2) and $R = rh^2|h'|^{p-2}$. The subject of the present investigation is to find an explicit formula for the function $\tilde{r}$ in such a way that together with the function

$$\tilde{c}(t) = \frac{1}{h^p(t)R(t)\left(\int_0^t R^{-1}(s) ds\right)^2}$$

equation (5) is conditionally oscillatory.

- (ii) In [10] the authors establish a “power comparison theorem” for the Riemann-Weber half-linear equation

$$(\Phi(x'))' + \left[\frac{\gamma_p}{t^p} + \frac{\mu}{t^p \log^2 t} \right] \Phi(x) = 0. \quad (16)$$

They proved a (non)oscillation criterion for this equation, where this equation is compared with an equation of the same form, but with a different power in the function Φ and other functions and constants appearing in (16). It suggests to investigate a similar problem for the more general equation (6).

- (iii) In [1], motivated by the linear case treated in [7], [8], [9], we have investigated oscillatory properties of the equation

$$(r(t)\Phi(x'))' + \left[\frac{c(t)}{t^p} + \frac{d(t)}{t^p \log^2 t} \right] \Phi(x) = 0 \quad (17)$$

with positive α -periodic functions r, c, d . It was shown, similarly to the case when these periodic functions are constants, that (17) is conditionally oscillatory and an explicit formula for oscillation constants has been found.

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This result suggests to establish a similar result for (6), where the constants in numerators of the fractions $\frac{\lambda}{\log^2 t}$, $\frac{\gamma_p}{t^p}$, and $\frac{\mu}{t^p \log^2 t}$ are replaced by periodic functions.

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