ON REPRESENTATION OF MULTIMEASURE

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ABSTRACT. We consider a multimeasure with the Radon-Nikodym derivative and apply its Castaing representation to get a representation of the multimeasure.

We assume that X is a real normed space; we denote by $P_o(X)$ the family of nonempty subsets of X , by $P_f(X)$, $P_{fc}(X)$ the families of closed and closed convex members of $P_o(X)$, respectively. In the sequel, $(T, \mathcal{A})$ stands for measurable space, μ for positive real measure.

We say that a multifunction $F: T \rightarrow P_o(X)$ is measurable if for every open set $U \subset X$

$$F^-(U) := \{t \in T : F(t) \cap U \neq \emptyset\} \in \mathcal{A}.$$

The function $f: T \rightarrow X$ is a selection of F if $f(t) \in F(t)$ for $t \in T$.

For measurable multifunctions we have the following characterization (see [3], [2], [5]).

THEOREM 1. *Let X be a complete separable metric space. A multifunction $F: T \rightarrow P_f(X)$ is measurable if and only if there exists a sequence $\{f_n : n \in \mathbb{N}\}$ of measurable selections of F such that*

$$F(t) = \text{cl} \{f_n(t) : n \in \mathbb{N}\}, \quad \text{for } t \in T.$$

The sequence $\{f_n : n \in \mathbb{N}\}$ is called Castaing representation of F .

A measurable multifunction $F: T \rightarrow P_o(X)$ is Aumann integrable if S_F , the set of all Bochner integrable selections of F , is nonempty. Then Aumann integral $\int_A F d\mu := \{\int_A f d\mu : f \in S_F\}$ is the set of all Bochner integrals of functions of S_F .

We say that a multifunction $M: \mathcal{A} \rightarrow P_f(X)$ is a weak multimeasure if for every $x^* \in X^*$, a mapping $A \mapsto \sup\{x^*(x) : x \in M(A)\}$ is a signed measure. M is additive if and only if $M(A \cup B) = \text{cl}(M(A) + M(B))$ for disjoint sets $A, B \in \mathcal{A}$.

EXAMPLE 1 ([6, Ex. 4.6]). If $F: T \rightarrow P_f(X)$ is an Aumann integrable multifunction, then a multifunction $A \mapsto \text{cl} \int_A F d\mu$, $A \in \mathcal{A}$, is a weak multimeasure.

A multifunction $F: T \rightarrow P_f(X)$ is the Radon-Nikodym derivative of a multimeasure $M: \mathcal{A} \rightarrow P_f(X)$ with respect to μ if

$$\int_A F d\mu = M(A) \quad \text{for every } A \in \mathcal{A}.$$

Since the Radon-Nikodym derivative of a multimeasure M has Castaing representation, we can look for a similar representation for M .

A vector measure $m: \mathcal{A} \rightarrow X$ is called a measure selection of M if $m(A) \in M(A)$ for every $A \in \mathcal{A}$. The set of all measure selections of multimeasure M will be denoted by S_M .

Let $\mathcal{M} := \{m: \mathcal{A} \rightarrow X : m \text{ is a vector measure}\}$.

We say that $D \subset \mathcal{M}$ is decomposable if and only if for every measurable set $A \in \mathcal{A}$ and $m, n \in D$

$$m\chi_A + n\chi_{T \setminus A} \in D,$$

where $m\chi_A(B) = m(B \cap A)$.

We say that $A_1, \dots, A_n \in \mathcal{A}$ is a partition of T if the sets are mutually disjoint and $\bigcup_{i=1}^n A_i = T$.

A decomposable hull of $S \subset \mathcal{M}$ is the smallest, in the sense of inclusion, decomposable set containing S . Moreover, (cf. [7]),

$$\begin{aligned} \text{dec } S &:= \bigcap \{D : S \subset D, D \text{ is decomposable}\} \\ &= \left\{ \sum_{i=1}^k m_i \chi_{A_i} : k \in \mathbb{N}, m_i \in S, \{A_1, \dots, A_k\} \text{ is a partition of } T \right\}. \end{aligned}$$

LEMMA 1. *Let $(T, \mathcal{A}, \mu)$ be a measure space, X be a normed space. Let $M: \mathcal{A} \rightarrow P_f(X)$ be a multimeasure with the Radon-Nikodym derivative $F: T \rightarrow P_f(X)$. Then M is an additive weak multimeasure with a decomposable set S_M .*

Proof. According to Example 1, M is a weak multimeasure, since M has closed values and $M(A) = \int_A F d\mu$ for $A \in \mathcal{A}$.

To prove the additivity of M , take disjoint sets $A, B \in \mathcal{A}$.

$$x = \int_{A \cup B} f d\mu = \int_A f d\mu + \int_B f d\mu \in M(A) + M(B) \subset \text{cl}(M(A) + M(B)).$$

If $x \in M(A) + M(B)$, then there exist $f, g \in S_F$ such that

$$x = \int_A f d\mu + \int_B g d\mu = \int_{A \cup B} (\chi_A f + \chi_B g) d\mu = \int_{A \cup B} (\chi_A f + \chi_{T \setminus A} g) d\mu.$$

Since S_F is decomposable (see [5]), $\chi_A f + \chi_{T \setminus A} g \in S_F$ and we have

$$x \in \int_{A \cup B} F d\mu = M(A).$$

Therefore,

$$M(A) + M(B) \subset M(A \cup B)$$

and, by the closedness of values of M , we have

$$\text{cl}(M(A) + M(B)) \subset M(A \cup B).$$

Finally, on account of Lemma 4 in [7], the set S_M is decomposable. $\square$

Let $m_n: T \rightarrow X$, $n \in \mathbb{N}$ be vector measures. It is natural to ask whether a multifunction

$$A \mapsto M(A) := \text{cl} \{m_n(A) : n \in \mathbb{N}\}, \quad A \in \mathcal{A}$$

is a weak multimeasure with a decomposable set S_M . The answer is in the following example (see [7]).

EXAMPLE 2. Let $T = [0, 1] \subset \mathbb{R}$, $\mathcal{A}$ be σ -algebra of Lebesgue measurable subsets of T . Let m be Lebesgue measure on $\mathcal{A}$. Then multifunction given by

$$M(A) = \{m(A), 2m(A)\}, \quad A \in \mathcal{A}$$

is a weak multimeasure but S_M is not decomposable. Consequently, M has not the Radon-Nikodym derivative.

THEOREM 2. *Let $(T, \mathcal{A}, \mu)$ be a σ -finite measure space, X be a separable Banach space. Let $M: \mathcal{A} \rightarrow P_f(X)$ be a multimeasure with the Radon-Nikodym derivative $F: T \rightarrow P_f(X)$. Then there exists a sequence $\{m_n : n \in \mathbb{N}\}$ of measure selections of M such that*

$$M(A) = \text{cl} \{m(A) : m \in \text{dec} \{m_n : n \in \mathbb{N}\}\}, \quad A \in \mathcal{A}.$$

PROOF. Let $\{f_n : n \in \mathbb{N}\}$ be Castaing representation of the measurable multifunction F . Define vector measures

$$m_n(A) := \int_A f_n d\mu, \quad \text{for } n \in \mathbb{N}, \quad A \in \mathcal{A}$$

and let $S := \{m_n : n \in \mathbb{N}\}$.

It is clear that for every $A \in \mathcal{A}$, $m_{k_1}, \dots, m_{k_n} \in S$ and a partition $A_1, \dots, A_n$ of T

$$\left(\sum_{i=1}^n m_{k_i} \chi_{A_i} \right)(A) = \sum_{i=1}^n m_{k_i}(A \cap A_i) = \sum_{i=1}^n \int_{A \cap A_i} f_{k_i} d\mu = \int_A \left(\sum_{i=1}^n \chi_{A_i} f_{k_i} \right) d\mu,$$

where $\sum_{i=1}^n \chi_{A_i} f_{k_i} \in \text{dec} \{f_n : n \in \mathbb{N}\}$. On the other hand, by the above equalities, we see that any function $g \in \text{dec} \{f_n : n \in \mathbb{N}\}$ generates measure from $\text{dec } S$.

Therefore,

$$\text{dec } \{m_n : n \in \mathbb{N}\} = \left\{ \int_{(\cdot)} g d\mu : g \in \text{dec } \{f_n : n \in \mathbb{N}\} \right\} \quad (1)$$

and consequently, for every $A \in \mathcal{A}$,

$$\{m(A) : m \in \text{dec } \{m_n : n \in \mathbb{N}\}\} = \left\{ \int_A g d\mu : g \in \text{dec } \{f_n : n \in \mathbb{N}\} \right\}. \quad (2)$$

We will show that for every $A \in \mathcal{A}$,

$$M(A) = \text{cl } \{m(A) : m \in \text{dec } S\}.$$

Observe that $S \subset S_M$ and on account of Lemma 1 S_M is decomposable. Therefore, $\text{dec } S \subset S_M$ and

$$\{m(A) : m \in \text{dec } S\} \subset M(A), \quad A \in \mathcal{A}.$$

Consequently, by the closedness of values of M ,

$$\text{cl } \{m(A) : m \in \text{dec } S\} \subset M(A).$$

To prove the converse inclusion, take $A \in \mathcal{A}$ and $x \in M(A)$. Then there exists $f \in S_F$ such that $x = \int_A f d\mu$. Fix $\epsilon > 0$. According to Lemma 1.3 in [5], there exist $f_{k_1}, \dots, f_{k_n} \in \{f_n : n \in \mathbb{N}\}$ and $A_1, \dots, A_n$ a partition of T such that

$$\int_T \left\| f - \sum_{i=1}^n \chi_{A_i} f_{k_i} \right\| d\mu < \epsilon.$$

Thus, the function $g := \sum_{i=1}^n \chi_{A_i} f_{k_i} \in \text{dec } \{f_n : n \in \mathbb{N}\}$ and, moreover,

$$\left\| x - \int_A g d\mu \right\| = \left\| \int_A f d\mu - \int_A g d\mu \right\| \leq \int_A \|f - g\| d\mu \leq \int_T \|f - g\| d\mu < \epsilon.$$

Consequently,

$$x \in \text{cl } \left\{ \int_A g d\mu : g \in \text{dec } \{f_n : n \in \mathbb{N}\} \right\}$$

which together with (2) completes the proof. $\square$

According to (1) we have a reformulation of the above theorem.

COROLLARY 1. *Let $(T, \mathcal{A}, \mu)$ be a σ -finite measure space, X be a separable Banach space. Let $M : \mathcal{A} \rightarrow P_f(X)$ be a multimeasure with the Radon-Nikodym*

derivative $F: T \rightarrow P_f(X)$. Then

$$M(A) = \text{cl} \left\{ m(A) : m \in \text{dec} \left\{ \int_{(\cdot)} f_n d\mu : n \in \mathbb{N} \right\} \right\},$$

where $\{f_n : n \in \mathbb{N}\}$ is Castaing representation of F .

A partial converse of Theorem 2 is following.

THEOREM 3. *Let $(T, \mathcal{A}, \mu)$ be a finite measure space, X be a separable Banach space with the Radon-Nikodym property. If $\{m_n : n \in \mathbb{N}\}$ is a sequence of vector measures of bounded variation absolutely continuous with respect to μ , then there exists Aumann integrable multifunction $F: T \rightarrow P_f(X)$ such that*

$$M(A) := \text{cl} \{m(A) : m \in \text{dec} \{m_n : n \in \mathbb{N}\}\} = \text{cl} \int_A F d\mu, \quad A \in \mathcal{A}.$$

Proof. Let $S := \{m_n : n \in \mathbb{N}\}$ and $M(A) := \text{cl} \{m(A) : m \in \text{dec } S\}$ for $A \in \mathcal{A}$. Since X has the Radon-Nikodym property and $m_n \ll \mu$ for $n \in \mathbb{N}$ there exists $f_n := \frac{dm_n}{d\mu}$ for every $n \in \mathbb{N}$. Observe that analogously to the previous theorem, conditions (1) and (2) are satisfied.

Define $F(t) := \text{cl} \{f_n(t) : n \in \mathbb{N}\}$. Observe that each f_n is integrable, therefore $S_F \neq \emptyset$. We will show that $M(A) = \text{cl} \int_A F d\mu$ for $a \in \mathcal{A}$.

Take $A \in \mathcal{A}$. By (2)

$$\{m(A) : m \in \text{dec} \{m_n : n \in \mathbb{N}\}\} = \left\{ \int_A g d\mu : g \in \text{dec} \{f_n : n \in \mathbb{N}\} \right\} \subset \int_A F d\mu$$

hence,

$$\text{cl} \{m(A) : m \in \text{dec} \{m_n : n \in \mathbb{N}\}\} \subset \text{cl} \int_A F d\mu.$$

Now, let $A \in \mathcal{A}$ and $x \in \int_A F d\mu$. There exists $f \in S_F$ such that $x = \int_A f d\mu$.

Fix $\epsilon > 0$. According to Lemma 1.3 in [5], there exist $f_{k_1}, \dots, f_{k_n} \in \{f_n : n \in \mathbb{N}\}$ and $A_1, \dots, A_n$ a partition of T such that

$$\int_T \left\| f - \sum_{i=1}^n \chi_{A_i} f_{k_i} \right\| d\mu < \epsilon.$$

Define $g := \sum_{i=1}^n \chi_{A_i} f_{k_i}$. It follows that $g \in \text{dec} \{f_n : n \in \mathbb{N}\}$ and

$$\left\| x - \int_A g d\mu \right\| = \left\| \int_A f d\mu - \int_A g d\mu \right\| \leq \int_T \|f - g\| d\mu < \epsilon$$

which means that (see (2))

$$x \in \text{cl} \left\{ \int_A g d\mu : g \in \text{dec} \{f_n : n \in \mathbb{N}\} \right\} = \text{cl} \{m(A) : m \in \text{dec} \{m_n : n \in \mathbb{N}\}\}.$$

Thus, $\int_A F d\mu \subset M(A)$ which with closedness of $M(A)$ completes the proof. $\square$

By the above proof and by Theorem 4.5 in [5], we have what follows.

COROLLARY 2. *Let $(T, \mathcal{A})$ be a measurable space and μ be finite measure with no atoms, let X be a separable reflexive Banach space. If $\{m_n : n \in \mathbb{N}\}$ is a sequence of vector measures of bounded variation absolutely continuous with respect to μ , then there exists Aumann integrable multifunction $F : T \rightarrow P_{fc}(X)$ such that*

$$\text{cl} \{m(A) : m \in \text{dec} \{m_n : n \in \mathbb{N}\}\} = \int_A F d\mu, \quad A \in \mathcal{A}.$$

Moreover, $\{\frac{dm_n}{d\mu} : n \in \mathbb{N}\}$ is Castaing representation of F .

Proof. According to the above theorem, there exists $F : T \rightarrow P_f(X)$ such that

$$\text{cl} \{m(A) : m \in \text{dec} \{m_n : n \in \mathbb{N}\}\} = \text{cl} \int_A F d\mu, \quad A \in \mathcal{A}.$$

By Corollary 4.3 in [5], for $A \in \mathcal{A}$

$$\text{cl} \int_A F d\mu = \text{cl} \int_A \text{cl conv } F d\mu.$$

However, on account of Theorem 4.5 in [5], the integral $\int_A \text{cl conv } F d\mu$ is closed, so $M(A) = \int_A \text{cl conv } F d\mu$ for, $A \in \mathcal{A}$ and the proof is complete. $\square$

Finally, we have a characterization of multimeasures with a representation generated by Castaing representation of their Radon-Nikodym derivative.

THEOREM 4. *Let $(T, \mathcal{A})$ be a measurable space and μ be finite measure with no atoms, let X be a separable reflexive Banach space. A multimeasure $M : \mathcal{A} \rightarrow P_f(X)$ has the Radon-Nikodym derivative $F : T \rightarrow P_{fc}(X)$ if and only if there exists a sequence of vector measures $\{m_n : n \in \mathbb{N}\}$ of bounded variation absolutely continuous with respect to μ such that*

$$M(A) = \text{cl} \{m(A) : m \in \text{dec} \{m_n : n \in \mathbb{N}\}\}.$$

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