DENSITY TOPOLOGIES ON THE PLANE BETWEEN ORDINARY AND STRONG

ELŻBIETA WAGNER-BOJAKOWSKA — WŁADYSŁAW WILCZYŃSKI

ABSTRACT. Let C_0 denote the set of all non-decreasing continuous functions $f: (0, 1] \rightarrow (0, 1]$ such that $\lim_{x \rightarrow 0^+} f(x) = 0$ and $f(x) \leq x$ for $x \in (0, 1]$ and let A be a measurable subset of the plane. We define the notion of a density point of A with respect to f . This is a starting point to introduce the mapping D_f defined on the family of all measurable subsets of the plane, which is so-called lower density. The mapping D_f leads to the topology $\mathcal{T}_f$, analogously as for the density topology. The properties of the topologies $\mathcal{T}_f$ are considered.

Let C_0 denote the set of all non-decreasing continuous functions

$f: (0, 1] \rightarrow (0, 1]$ such that $\lim_{x \rightarrow 0^+} f(x) = 0$ and $f(x) \leq x$ for $x \in (0, 1]$.

Let $f \in C_0$ and let $A \subset \mathbb{R}^2$ be a measurable subset of the plane.

DEFINITION 1. We will say that $(0, 0)$ is a density point of A with respect to a function f if for each $\epsilon > 0$ there exists $\delta > 0$ such that for each $h, k \in (0, \delta)$ if $f(h) \leq k \leq h$ or $f(k) \leq h \leq k$ then

$$\frac{m_2(A \cap ([-h, h] \times [-k, k]))}{4hk} > 1 - \epsilon. \quad (1)$$

Obviously, the condition (1) is equivalent to the following inequality

$$\frac{m_2(A' \cap ([-h, h] \times [-k, k]))}{4hk} < \epsilon, \quad (2)$$

where $A' = \mathbb{R}^2 \setminus A$.

We say that (x_0, y_0) is a density point of A with respect to a function f if $(0, 0)$ is a density point of $A - (x_0, y_0) = \{(x - x_0, y - y_0) : (x, y) \in A\}$ with respect to f .

Obviously, if $f(x) = x$ for $x \in (0, 1]$, then the notion of a density point with respect to f coincides with the notion of the ordinary density point of a measurable subset of the plane.

THEOREM 2. *Let $f \in C_0$ and let A be a measurable subset of the plane. The point $(0, 0)$ is a density point of A with respect to f if and only if for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all positive integers n, m greater than n_0 if*

$$f\left(\frac{1}{n}\right) \leq \frac{1}{m} \leq \frac{1}{n} \quad \text{or} \quad f\left(\frac{1}{m}\right) \leq \frac{1}{n} \leq \frac{1}{m},$$

then

$$\frac{m_2(A \cap ([-\frac{1}{n}, \frac{1}{n}] \times [-\frac{1}{m}, \frac{1}{m}]))}{\frac{4}{nm}} > 1 - \epsilon. \quad (3)$$

PROOF. “ $\implies$ ” Let ϵ be an arbitrary positive number. From our assumption, there exists $\delta > 0$ such that for each $h, k \in (0, \delta)$ if $f(h) \leq k \leq h$ or $f(k) \leq h \leq k$ then

$$\frac{m_2(A \cap ([-h, h] \times [-k, k]))}{4hk} > 1 - \epsilon.$$

Put $n_0 = \lceil \frac{1}{\delta} \rceil + 1$. Then $n_0 - 1 \leq 1/\delta < n_0$, so $1/n_0 < \delta \leq 1/(n_0 - 1)$. Let n, m be arbitrary positive integers greater than n_0 and suppose that

$$f\left(\frac{1}{n}\right) \leq \frac{1}{m} \leq \frac{1}{n}.$$

Then put $h = \frac{1}{n}$, $k = \frac{1}{m}$. Obviously, $h, k \in (0, \delta)$ and $f(h) \leq k \leq h$; consequently, the inequality (1) holds which means that the condition (3) is fulfilled. If

$$f\left(\frac{1}{m}\right) \leq \frac{1}{n} \leq \frac{1}{m},$$

the proof is analogous.

“ $\impliedby$ ” Let $\epsilon > 0$. From our assumption, there exists $n_0 \in \mathbb{N}$, $n_0 \geq 4/\epsilon$ such that for $n, m \in \mathbb{N}$, $n > n_0$, $m > n_0$, if

$$f\left(\frac{1}{n}\right) \leq \frac{1}{m} \leq \frac{1}{n} \quad \text{or} \quad f\left(\frac{1}{m}\right) \leq \frac{1}{n} \leq \frac{1}{m},$$

then

$$\frac{m_2(A \cap ([-\frac{1}{n}, \frac{1}{n}] \times [-\frac{1}{m}, \frac{1}{m}]))}{\frac{4}{n \cdot m}} > 1 - \frac{\epsilon}{4}.$$

Put $\delta = \frac{1}{n_0}$. Let $h, k \in (0, \delta)$. Then $0 < h < \delta = \frac{1}{n_0}$ and $0 < k < \delta = \frac{1}{n_0}$, so there exist positive integers $n, m \geq n_0$ such that

$$\frac{1}{n+1} \leq h < \frac{1}{n} \quad (4)$$

and

$$\frac{1}{m+1} \leq k < \frac{1}{m}. \quad (5)$$

Suppose that

$$f(h) \leq k \leq h. \quad (6)$$

Obviously,

$$\frac{m_2(A \cap ([-h, h] \times [-k, k]))}{4hk} + \frac{m_2(A' \cap ([-h, h] \times [-k, k]))}{4hk} = 1.$$

We will prove that

$$\frac{m_2(A' \cap ([-h, h] \times [-k, k]))}{4hk} < \epsilon.$$

Assume that $m > n$. Then

$$\frac{1}{m} \leq \frac{1}{n+1}. \quad (7)$$

We have

$$\begin{aligned} & m_2(A' \cap ([-h, h] \times [-k, k])) \\ & \leq m_2\left(A' \cap \left(\left[-\frac{1}{n}, \frac{1}{n}\right] \times \left[-\frac{1}{m}, \frac{1}{m}\right]\right)\right) \\ & \leq m_2\left(A' \cap \left(\left[-\frac{1}{n+1}, \frac{1}{n+1}\right] \times \left[-\frac{1}{m}, \frac{1}{m}\right]\right)\right) + 2\left(\frac{1}{n} - \frac{1}{n+1}\right) \cdot \frac{2}{m} \\ & = m_2\left(A' \cap \left(\left[-\frac{1}{n+1}, \frac{1}{n+1}\right] \times \left[-\frac{1}{m}, \frac{1}{m}\right]\right)\right) + \frac{4}{n \cdot m \cdot (n+1)}. \end{aligned}$$

Hence

$$\begin{aligned} & \frac{m_2(A' \cap ([-h, h] \times [-k, k]))}{4hk} \\ & \leq \frac{m_2\left(A' \cap \left(\left[-\frac{1}{n+1}, \frac{1}{n+1}\right] \times \left[-\frac{1}{m}, \frac{1}{m}\right]\right)\right)}{4 \cdot \frac{1}{n+1} \cdot \frac{1}{m+1}} + \frac{4}{n \cdot m \cdot (n+1)} \cdot \frac{(n+1)(m+1)}{4} \\ & < \frac{m_2\left(A' \cap \left(\left[-\frac{1}{n+1}, \frac{1}{n+1}\right] \times \left[-\frac{1}{m}, \frac{1}{m}\right]\right)\right)}{4 \cdot \frac{1}{n+1} \cdot \frac{1}{m}} \cdot \frac{m+1}{m} + \frac{2}{n}. \end{aligned} \quad (8)$$

From (4), (6), (5) and (7) we have

$$f\left(\frac{1}{n+1}\right) \leq k < \frac{1}{m} \leq \frac{1}{n+1},$$

so, from our assumption,

$$\frac{m_2(A' \cap \left(\left[-\frac{1}{n+1}, \frac{1}{n+1}\right] \times \left[-\frac{1}{m}, \frac{1}{m}\right]\right))}{\frac{4}{(n+1) \cdot m}} < \frac{\epsilon}{4}. \quad (9)$$

Clearly, $(m+1)/m \leq 2$ and $n_0 \geq 4/\epsilon$, so from (8) and (9),

$$\frac{m_2(A' \cap ([-h, h] \times [-k, k]))}{4hk} < \epsilon.$$

If $f(k) \leq h \leq k$, the proof is analogous. $\square$

Let $\mathcal{S}$ denote the family of all measurable subsets of the plane.

Let $f \in C_0$ and $A \in \mathcal{S}$. $D_f(A)$ will denote the set of all points $(x, y) \in \mathbb{R}^2$ which are the density points of A with respect to f .

Let $f, g \in C_0$ and $f(x) \leq g(x)$ for $x \in (0, 1]$. It is clear that for arbitrary measurable subset A of the plane, $D_f(A) \subset D_g(A)$.

THEOREM 3. *Let $f \in C_0$ and $c \in (0, 1]$. For arbitrary measurable subset A of the plane $D_f(A) = D_{cf}(A)$.*

Proof. Let $A \in \mathcal{S}$. It is clear that $D_{cf}(A) \subset D_f(A)$, because if $c \in (0, 1]$, then $cf(x) \leq f(x)$ for $x \in (0, 1]$.

Now, we will prove that $D_f(A) \subset D_{cf}(A)$. Let $\epsilon > 0$ and let $(x, y) \in D_f(A)$. The Lebesgue measure is invariant with respect to translation, so we may assume that $(x, y) = (0, 0)$. There exists $\delta > 0$ such that for each $h, k \in (0, \delta)$ if $f(h) \leq k \leq h$ or $f(k) \leq h \leq k$, then

$$\frac{m_2(A' \cap ([-h, h] \times [-k, k]))}{4hk} < c \cdot \epsilon. \quad (10)$$

Let $h, k \in (0, \delta)$ be such two numbers that

$$c \cdot f(h) \leq k \leq h. \quad (11)$$

If $f(h) \leq k \leq h$, then the inequality

$$\frac{m_2(A' \cap ([-h, h] \times [-k, k]))}{4hk} < \epsilon \quad (12)$$

follows from (10). In the other case (i.e., $k < f(h)$), let us put $k_1 = f(h)$. Then $k < k_1 = f(h) \leq h < \delta$, so $k_1, h \in (0, \delta)$ and $f(h) \leq k_1 \leq h$. Hence from (10) we obtain

$$\frac{m_2(A' \cap ([-h, h] \times [-k_1, k_1]))}{4hk_1} < c \cdot \epsilon.$$

From (11) we have $c \cdot k_1 \leq k$, so

$$\begin{aligned} & \frac{m_2(A' \cap ([-h, h] \times [-k, k]))}{4hk} \\ & \leq \frac{m_2(A' \cap ([-h, h] \times [-k_1, k_1]))}{4hk_1} \cdot \frac{k_1}{k} \\ & < c \cdot \epsilon \cdot \frac{k_1}{k} \leq \epsilon. \end{aligned} \quad (13)$$

If $c \cdot f(k) \leq h \leq k$, then the proof is analogous. Consequently, $(x, y) \in D_{cf}(A)$. $\square$

DEFINITION 4. We will say that two functions $f, g \in C_0$ are equivalent if for arbitrary measurable set $A \subset \mathbb{R}^2$ we have $D_f(A) = D_g(A)$.

In this case, we will write $f \sim g$. From the previous theorem it follows that $f \sim cf$, for each $f \in C_0$ and $c \in (0, 1]$. Particularly, if $g(x) = c \cdot x$, where $x \in (0, 1]$ and $c \in (0, 1]$, then for arbitrary measurable set $A \subset \mathbb{R}^2$, the set $D_g(A)$ of density points of A with respect to g coincides with the set of ordinary density points of A .

Let $A \in \mathcal{S}$. $\Phi_0(A)$ ($\Phi_s(A)$) will denote the set of ordinary (strong) density points of A . Obviously, for each function $f \in C_0$ and $A \in \mathcal{S}$

$$\Phi_s(A) \subset D_f(A) \subset \Phi_0(A). \quad (14)$$

Let $A, B \in \mathcal{S}$. We write $A \sim B$ if and only if $m_2(A \triangle B) = 0$ (where $A \triangle B = (A \setminus B) \cup (B \setminus A)$).

Now, let $f \in C_0$ be a fixed function.

THEOREM 5. The mapping $D_f: \mathcal{S} \rightarrow 2^{\mathbb{R}^2}$ has the following properties: for each $A, B \in \mathcal{S}$,

- 1) $A \sim D_f(A)$,
- 2) if $A \sim B$, then $D_f(A) = D_f(B)$,
- 3) $D_f(\emptyset) = \emptyset$, $D_f(\mathbb{R}^2) = \mathbb{R}^2$,
- 4) if $A \subset B$, then $D_f(A) \subset D_f(B)$,
- 5) $D_f(A \cap B) = D_f(A) \cap D_f(B)$.

PROOF. 1) From (14), for arbitrary $A \in \mathcal{S}$ we obtain

$$A \setminus D_f(A) \subset A \setminus \Phi_s(A), \quad (15)$$

so, $m_2(A \setminus D_f(A)) = 0$ (see [S, Theorem 10.2, p. 129]). Simultaneously,

$$D_f(A) \setminus A \subset \Phi_0(A) \setminus A, \quad (16)$$

so, $m_2(D_f(A) \setminus A) = 0$ from Lebesgue density theorem for ordinary density. Consequently, $m_2(A \triangle D_f(A)) = 0$, i.e., $A \sim D_f(A)$.

The conditions 2), 3) and 4) are obvious. 5) Clearly, $A \cap B \subset A$ and $A \cap B \subset B$, so from 4)

$$D_f(A \cap B) \subset D_f(A) \cap D_f(B).$$

To prove the opposite inclusion, use the inequality

$$\frac{m_2(A \cap I)}{m_2(I)} + \frac{m_2(B \cap I)}{m_2(I)} - 1 \leq \frac{m_2(A \cap B \cap I)}{m_2(I)}$$

which holds for arbitrary interval $I \subset \mathbb{R}^2$. $\square$

In fact, $D_f: \mathcal{S} \rightarrow \mathcal{S}$, because from the first condition of the latter theorem, it follows that $D_f(A) \in \mathcal{S}$ for each $A \in \mathcal{S}$.

Let $\mathcal{T}_0$ and $\mathcal{T}_s$ denote the ordinary density topology and the strong density topology on the plane, respectively.

Put

$$\mathcal{T}_f = \{A \in \mathcal{S} : A \subset D_f(A)\}.$$

Obviously, $\mathcal{T}_f \subset \mathcal{S}$ and if $f \sim g$ then $\mathcal{T}_f = \mathcal{T}_g$.

From Theorem 5 it follows that the mapping D_f is a lower density (see [LMZ]). Using Proposition 6.37 in [LMZ] and (14), we obtain the following result:

THEOREM 6. *The family $\mathcal{T}_f$ is a topology on the plane, between ordinary and strong density topologies.*

Observe that there exist functions $f \in C_0$ such that

$$\mathcal{T}_s \subsetneq \mathcal{T}_f \subsetneq \mathcal{T}. \quad (17)$$

Put $f(x) = x^2$ for $x \in (0, 1]$,

$$A = \left\{ (x, y) \in \mathbb{R}^2 : \begin{aligned} &[x > 0 \quad \text{and} \quad (y > x^3 \quad \text{or} \quad y < -x^3)] \\ &\text{or} \quad [x < 0 \quad \text{and} \quad (y > -x^3 \quad \text{or} \quad y < x^3)] \end{aligned} \right\} \cup \{(0, 0)\}$$

and

$$B = \{(x, y) \in \mathbb{R}^2 : y > x^2 \quad \text{or} \quad y < -x^2\} \cup \{(0, 0)\}.$$

Then

$$\lim_{h \rightarrow 0^+} \frac{m_2(A' \cap ([-h, h] \times [-h^2, h^2]))}{4h^3} = \lim_{h \rightarrow 0^+} \frac{h}{4} = 0,$$

so $(0, 0) \in D_f(A)$ and $A \in \mathcal{T}_f$. Simultaneously,

$$\lim_{h \rightarrow 0^+} \frac{m_2(A' \cap ([-h, h] \times [-h^3, h^3]))}{4h^4} = \frac{1}{4}.$$

Consequently, $A \notin \mathcal{T}_s$.

Analogously,

$$\lim_{h \rightarrow 0^+} \frac{m_2(B' \cap ([-h, h] \times [-h, h]))}{4h^2} = \lim_{h \rightarrow 0^+} \frac{h}{3} = 0,$$

so $(0, 0) \in \Phi_0(B)$ and $B \in \mathcal{T}_0$. On the other hand,

$$\lim_{h \rightarrow 0^+} \frac{m_2(B' \cap [-h, h] \times [-h^2, h^2])}{4h^3} = \frac{1}{3},$$

so $(0, 0) \notin D_f(B)$ and $B \notin \mathcal{T}_f$.

Essentially in the same way as in [O], [Os], [Sch], [T] and [W], one can obtain the following results (compare also [LMZ, Theorem 6.39]).

THEOREM 7. *If A is a measurable subset of the plane then*

- (a) $\text{Int}_{\mathcal{T}_f}(A) = A \cap D_f(A)$,
- (b) $\bar{A}^{\mathcal{T}_f} = A \cup [\mathbb{R}^2 \setminus D_f(\mathbb{R}^2 \setminus A)]$,
- (c) $A^{d\tau_f} = \mathbb{R}^2 \setminus D_f(\mathbb{R}^2 \setminus A)$.

For arbitrary set $A \subset \mathbb{R}^2$,

$$\text{Int}_{\mathcal{T}_f}(A) = A \cap D_f(B),$$

where B is a measurable kernel of A .

Remark 8. If $m_2(A) = 0$, then A is closed in the topology $\mathcal{T}_f$.

Remark 9. If $A \in \mathcal{S}$ then $A \sim \text{Int}_{\mathcal{T}_f} A$ and $A \sim \bar{A}^{\mathcal{T}_f}$.

THEOREM 10. *The following conditions are equivalent:*

- (a) $m_2(A) = 0$,
- (b) A is a nowhere dense set in the topology $\mathcal{T}_f$,
- (c) A is a set of the first category in the topology $\mathcal{T}_f$,
- (d) $A^{d\tau_f} = \emptyset$.

THEOREM 11. *The following conditions are equivalent:*

- (a) A is measurable on the plane,
- (b) A is a Borel set in the topology $\mathcal{T}_f$,
- (c) A has the Baire property in the topology $\mathcal{T}_f$.

THEOREM 12. *The set $A \subset \mathbb{R}^2$ is compact in the topology $\mathcal{T}_f$ if and only if A is finite.*

Proof. If A is infinite, then there exists a countable set $A_1 \subset A$ which is not compact in the natural topology. Hence, A_1 is not compact in the topology $\mathcal{T}_f$. Consequently, A is not compact in the topology $\mathcal{T}_f$, because A_1 is closed in $\mathcal{T}_f$. The inverse implication is obvious. $\square$

THEOREM 13. *The space $(\mathbb{R}^2, \mathcal{T}_f)$ is neither first countable, nor second countable, nor Lindelöf, nor separable.*

THEOREM 14. *$(\mathbb{R}^2, \mathcal{T}_f)$ is a Baire space.*

Proof. It is obvious because each non-empty set open in the topology $\mathcal{T}_f$ has positive measure. $\square$

THEOREM 15. *$(\mathbb{R}^2, \mathcal{T}_f)$ is a regular space.*

The proof using Theorem 2 is analogous to the proof of Theorem 22.9 in [O].

From (17) we obtain

$$\mathcal{T}_s \subset \bigcap_{f \in C_0} \mathcal{T}_f \subset \bigcup_{f \in C_0} \mathcal{T}_f \subset \mathcal{T}_0.$$

On the other hand, $\mathcal{T}_0 \subset \bigcup_{f \in C_0} \mathcal{T}_f$, since $\mathcal{T}_0$ is a $\mathcal{T}_f$ topology for a function $f(x) = x$, $x \in (0, 1]$, and $f \in C_0$. Consequently,

$$\bigcup_{f \in C_0} \mathcal{T}_f = \mathcal{T}_0.$$

THEOREM 16. $\bigcap_{f \in C_0} \mathcal{T}_f = \mathcal{T}_s$.

Proof. It is sufficient to prove that

$$\bigcap_{f \in C_0} \mathcal{T}_f \subset \mathcal{T}_s.$$

Let $A \in \bigcap_{f \in C_0} \mathcal{T}_f$ and let $(x_0, y_0) \in A$. We will prove that for arbitrary sequence $\{P_n\}_{n \in \mathbb{N}}$ of rectangles centered at (x_0, y_0) such that $\delta(P_n) \xrightarrow{n \rightarrow \infty} 0$, we have

$$\lim_{n \rightarrow \infty} \frac{m_2(A \cap P_n)}{m_2(P_n)} = 1. \quad (18)$$

Let $\{P_n\}_{n \in \mathbb{N}}$ be an arbitrary sequence of such intervals. We can assume that $(x_0, y_0) = (0, 0)$ and all intervals P_n , $n \in \mathbb{N}$, are contained in $(-1, 1) \times (-1, 1)$. We divide the sequence $\{P_n\}_{n \in \mathbb{N}}$ into two subsequences $\{P_n^1\}_{n \in \mathbb{N}}$ and $\{P_n^2\}_{n \in \mathbb{N}}$ in the following way: in the first sequence there are all intervals $P_n = [-h_n, h_n] \times [-k_n, k_n]$, for which $h_n > k_n$, and in the second sequence there are all the others. We will prove that

$$\lim_{n \rightarrow \infty} \frac{m_2(A \cap P_n^1)}{m_2(P_n^1)} = 1. \quad (19)$$

Let $\{P_n^3\}_{n \in \mathbb{N}}$ be an arbitrary subsequence of $\{P_n^1\}_{n \in \mathbb{N}}$. Put $P_n^3 = [-x_n, x_n] \times [-y_n, y_n]$, $n \in \mathbb{N}$. From the sequence $\{x_n\}_{n \in \mathbb{N}}$ we can choose the subsequence $\{x_{n_k}\}_{k \in \mathbb{N}}$ tending decreasingly to zero, and then, from the sequence $\{y_{n_k}\}_{k \in \mathbb{N}}$ we choose the subsequence $\{y_{n_{k_l}}\}_{l \in \mathbb{N}}$ tending decreasingly to zero. Put

$$P_l^4 = [-x_{n_{k_l}}, x_{n_{k_l}}] \times [-y_{n_{k_l}}, y_{n_{k_l}}]$$

for $l \in \mathbb{N}$. Obviously, $\{P_l^4\}_{l \in \mathbb{N}}$ is a subsequence of $\{P_n^3\}_{n \in \mathbb{N}}$. Let f be a function defined as follows: $f(1) = 1$, $f(x_{n_{k_l}}) = y_{n_{k_l}}$ for $l \in \mathbb{N}$ and f is linear on the intervals $[x_1, 1]$, $[x_{n_{k_l+1}}, x_{n_{k_l}}]$, $l \in \mathbb{N}$. Obviously, $f \in C_0$, so, from our assumption,

$$\lim_{n \rightarrow \infty} \frac{m_2(A \cap P_n^4)}{m_2(P_n^4)} = 1.$$

From Urysohn condition we obtain (19). We can analogously deal with the sequence $\{P_n^2\}_{n \in \mathbb{N}}$ obtaining the function $g \in C_0$. Hence,

$$\lim_{n \rightarrow \infty} \frac{m_2(A \cap P_n^2)}{m_2(P_n^2)} = 1.$$

Consequently, we have (18), so $(0, 0) \in \Phi_s(A)$ and $A \in \mathcal{T}_s$. $\square$

THEOREM 17. *Let $f, g \in C_0$ and $f(x) \leq g(x)$ for $x \in (0, 1]$. If there exists a number $c \in (0, 1)$ such that*

$$\liminf_{x \rightarrow 0^+} \frac{f(x)}{g(x)} = c,$$

then $\mathcal{T}_f = \mathcal{T}_g$.

Proof. Let $f, g \in C_0$ and $f(x) \leq g(x)$ for $x \in (0, 1]$. From our assumption there exists $\delta_1 > 0$ such that

$$\frac{f(x)}{g(x)} > \frac{c}{2},$$

for $x \in (0, \delta_1)$. We will prove that $D_{\frac{c}{2}g}(A) \subset D_f(A)$ for arbitrary $A \in \mathcal{S}$.

Let $A \in \mathcal{S}$ and let $(0, 0) \in D_{\frac{c}{2}g}(A)$. There exists $\delta_2 > 0$ such that for each $h, k \in (0, \delta_2)$ if $\frac{c}{2}g(h) \leq k \leq h$ or $\frac{c}{2}g(k) \leq h \leq k$, then

$$\frac{m_2(A' \cap [-h, h] \times [-k, k])}{4hk} < \epsilon. \quad (20)$$

Put $\delta_0 = \min(\delta_1, \delta_2)$. Let $h, k \in (0, \delta_0)$ be two numbers such that $f(h) \leq k \leq h$. Then $h \in (0, \delta_1)$, so

$$\frac{c}{2}g(h) \leq f(h) \leq k \leq h$$

and $h, k \in (0, \delta_2)$. Consequently, the condition (20) is fulfilled. Analogously, if $f(k) \leq h \leq k$. Hence, $(0, 0) \in D_f(A)$.

Obviously, $D_f(A) \subset D_g(A)$ for $A \in \mathcal{S}$. From Theorem 3 we have $D_{\frac{c}{2}g}(A) = D_g(A)$ for $A \in \mathcal{S}$. Consequently, $D_f(A) = D_g(A)$ for each $A \in \mathcal{S}$. Hence, $f \sim g$ and $\mathcal{T}_f = \mathcal{T}_g$. $\square$

COROLLARY 18. *If $f \in C_0$ and $\liminf_{x \rightarrow 0^+} \frac{f(x)}{x} > 0$, then $\mathcal{T}_f = \mathcal{T}_0$.*

Now, let A be a subset of a real line and let $x \in \mathbb{R}$. Let $p^+(A, x)$ denote the right-hand porosity of A at x , i.e.,

$$p^+(A, x) = \limsup_{h \rightarrow 0^+} \frac{\lambda(A, (x, x+h))}{h},$$

where $\lambda(A, I) = \sup\{m(J) : J \subset I \setminus A\}$.

Let $f, g \in C_0$ and $c \in (0, 1)$. Put

$$M_c = \{x \in (0, 1] : g(x) \geq c \cdot f(x)\}.$$

DEFINITION 19. We say that the pair (f, g) fulfills the condition $(*)$ if for each decreasing sequence $\{c_n\}_{n \in \mathbb{N}}$ convergent to zero

$$\lim_{n \rightarrow \infty} p^+(M_{c_n}, 0) = 0.$$

Observe that if $c_1, c_2 \in (0, 1]$, $c_1 < c_2$ then $M_{c_1} \supset M_{c_2}$. Consequently, the condition $(*)$ is equivalent to the following

$$\text{there exists } \{c_n\} \searrow 0 \text{ such that } \lim_{n \rightarrow \infty} p^+(M_{c_n}, 0) = 0. \quad (**)$$

The implication $(*) \implies (**)$ is obvious. Now, assume $(**)$ and let $\{d_n\}_{n \in \mathbb{N}}$ be an arbitrary decreasing sequence convergent to zero. Then there exists $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$ there exists $k_n \in \mathbb{N}$ such that $d_n \leq c_{k_n}$. Hence $M_{d_n} \supset M_{c_{k_n}}$, so

$$p^+(M_{d_n}, 0) \leq p^+(M_{c_{k_n}}, 0).$$

Consequently, $\lim_{n \rightarrow \infty} p^+(M_{d_n}, 0) = 0$.

Let $f \in C_0$. Due to the simplification, we use the intervals of the form $[0, x] \times [0, y]$, where $y \leq x$, $x > 0$, $y > 0$.

Let $f, g \in C_0$ and $g(x) \leq f(x)$ for $x \in (0, 1]$. If $A \subset [0, \infty) \times [0, \infty)$ is a measurable set fulfilling the condition

$$\lim_{\substack{(x,y) \rightarrow 0 \\ g(x) \leq y \leq x}} \frac{m_2(A \cap ([0, x] \times [0, y]))}{x \cdot y} = 0 \quad (21)$$

then, obviously,

$$\lim_{\substack{(x,y) \rightarrow 0 \\ f(x) \leq y \leq x}} \frac{m_2(A \cap ([0, x] \times [0, y]))}{x \cdot y} = 0. \quad (22)$$

The next theorem gives some sufficient condition for which the implication $(22) \implies (21)$ holds.

THEOREM 20. *Let $f, g \in C_0$, $g(x) \leq f(x)$ for $x \in (0, 1]$. If the pair (f, g) fulfills the condition $(*)$, then the conditions (21) and (22) are equivalent.*

PROOF. Let $A \subset [0, \infty) \times [0, \infty)$ be a measurable set. Let $f, g \in C_0$, $g(x) \leq f(x)$ for $x \in (0, 1]$ and the pair (f, g) fulfills $(*)$. Suppose that (22) is fulfilled. We will prove that (21) also holds.

Let $\epsilon > 0$ and let $\{c_n\}_{n \in \mathbb{N}}$ be an arbitrary decreasing sequence tending to zero. From $(*)$ there exists $n_0 \in \mathbb{N}$ such that

$$p^+(M_{c_{n_0}}, 0) < \frac{\epsilon}{3}.$$

Hence, there exists $\delta_1 > 0$ such that if $h \in (0, \delta_1)$ and $(a, b) \subset (0, h) \cap M'_{c_{n_0}}$, then $(b - a)/b < \epsilon/2$.

We may assume that $\delta_1 \in M_{c_{n_0}}$. From (22) it follows that there exists $\delta_2 > 0$ such that if $\rho((x, y), (0, 0)) < \delta_2$ and $f(x) \leq y \leq x$. Then

$$\frac{m_2(A \cap ([0, x] \times [0, y]))}{x \cdot y} < \frac{\epsilon c_{n_0}}{2}. \quad (23)$$

Since $f(x) \leq x$, so, for $x \in (0, \delta_2/\sqrt{2})$, we have $\rho((x, f(x)), (0, 0)) < \delta_2$. Consequently,

$$\frac{m_2(A \cap ([0, x] \times [0, f(x)]))}{x \cdot f(x)} < \frac{\epsilon c_{n_0}}{2}. \quad (24)$$

We may assume that $\delta_2/\sqrt{2} \in M_{c_{n_0}}$.

Put $\delta = \min(\delta_1, \delta_2/\sqrt{2})$. We will prove that if $x \in (0, \delta)$ and $g(x) \leq y \leq x$, then

$$\frac{m_2(A \cap ([0, x] \times [0, y]))}{x \cdot y} < \epsilon, \quad (25)$$

i.e., the condition (21) holds. There are two possibilities:

1° $x \in (0, \delta) \cap M_{c_{n_0}}$. Then, if $f(x) \leq y \leq x$, the inequality (25) follows from (23) ($c_{n_0} < 1$). Suppose that $g(x) \leq y < f(x)$. We have $x \in M_{c_{n_0}}$, so $g(x) \geq c_{n_0} \cdot f(x)$ and $c_{n_0}f(x) \leq y < f(x)$. Consequently, from (24) we obtain

$$\frac{m_2(A \cap ([0, x] \times [0, y]))}{x \cdot y} \leq \frac{m_2(A \cap ([0, x] \times [0, f(x)]))}{x \cdot c_{n_0}f(x)} < \frac{\epsilon}{2}. \quad (26)$$

2° $x \in (0, \delta) \cap M'_{c_{n_0}}$. Obviously, $M'_{c_{n_0}}$ is an open set, so there exists a component interval (a, b) of $M'_{c_{n_0}}$ such that $x \in (a, b)$. Then $(a, x) \subset (0, \delta_1) \cap M'_{c_{n_0}}$ and, consequently, $(x - a)/x < \epsilon/2$. Clearly, $a \in M_{c_{n_0}}$, so $g(a) \geq c_{n_0} \cdot f(a)$. Simultaneously for $t \in (a, x)$ we have $g(t) < c_{n_0} \cdot f(t)$, so, from the continuity of f and g , we obtain $g(a) = c_{n_0} \cdot f(a)$.

Now, let $g(x) \leq y \leq x$. Obviously, $g(a) \leq y$. If $y \leq a$ then $g(a) \leq y \leq a$ and $a \in (0, \delta) \cap M_{c_{n_0}}$, so, from 1°, we obtain

$$\frac{m_2(A \cap ([0, a] \times [0, y]))}{a \cdot y} < \frac{\epsilon}{2}. \quad (27)$$

Consequently, using (27), we have

$$\begin{aligned} \frac{m_2(A \cap ([0, x] \times [0, y]))}{x \cdot y} &\leq \frac{m_2(A \cap ([0, a] \times [0, y]))}{x \cdot y} + \frac{(x - a)y}{x \cdot y} \\ &\leq \frac{m_2(A \cap ([0, a] \times [0, y]))}{a \cdot y} + \frac{x - a}{x} < \epsilon, \end{aligned} \quad (28)$$

because $(a, x) \subset (0, \delta_1) \cap M'_{c_{n_0}}$.

If $a < y$, then $a < y \leq x < \delta \leq \delta_2/\sqrt{2}$, so $\rho((y, y), (0, 0)) < \delta_2$, $f(y) \leq y$ and from (23) we obtain

$$\begin{aligned} \frac{m_2(A \cap ([0, x] \times [0, y]))}{x \cdot y} &\leq \frac{m_2(A \cap ([0, y] \times [0, y]))}{x \cdot y} + \frac{(x - y)y}{x \cdot y} \\ &\leq \frac{m_2(A \cap ([0, y] \times [0, y]))}{y^2} + \frac{x - a}{x} < \epsilon, \end{aligned} \quad (29)$$

because $(y, x) \subset (a, x) \subset (0, \delta_1) \cap M'_{c_{n_0}}$.

Finally, from (26), (28) and (29) we obtain (21). $\square$

COROLLARY 21. *Let $f, g \in C_0$, $g(x) \leq f(x)$ for $x \in (0, 1]$. If the pair (f, g) fulfills the condition $(*)$ then for arbitrary measurable set $A \subset \mathbb{R}^2$, we have $D_f(A) = D_g(A)$, so $f \sim g$ and $\mathcal{T}_f = \mathcal{T}_g$.*

Remark 22. There exist functions $f, g \in C_0$ such that $\mathcal{T}_f \setminus \mathcal{T}_g \neq \emptyset$ and $\mathcal{T}_g \setminus \mathcal{T}_f \neq \emptyset$.

Take $q \in (0, 1)$. Put

$$f_0(x) = \begin{cases} \frac{1}{1!} & \text{for } x \in [q^1, q^0], \\ \frac{1}{(2n+1)!} & \text{for } x \in [q^{4n+1}, q^{4n-2}], \quad n \geq 1, \\ \text{linear and continuous on } [q^{4n-2}, q^{4n-3}], & n \geq 1 \end{cases}$$

and

$$g_0(x) = \begin{cases} \frac{1}{(2n+2)!} & \text{for } x \in [q^{4n+3}, q^{4n}], \quad n \geq 0, \\ \text{linear and continuous on } [q^{4n}, q^{4n-1}], & n \geq 1. \end{cases}$$

It is not difficult to see that there exists $c > 0$ such that $c \cdot f_0(x) \leq x$ and $c \cdot g_0(x) \leq x$ for $x \in (0, 1]$. Put $f = c \cdot f_0$ and $g = c \cdot g_0$.

Observe that

$$\frac{f(x)}{g(x)} = 2n + 2 \text{ for } x \in [q^{4n+1}, q^{4n}], \quad n \geq 0$$

and

$$\frac{g(x)}{f(x)} = 2n + 3 \text{ for } x \in [q^{4n+3}, q^{4n+2}], \quad n \geq 0.$$

Let

$$A = \bigcup_{n=1}^{\infty} [q^{4n-1}, q^{4n-2}] \times [0, f(q^{4n-1})] \quad \text{and} \quad B = \bigcup_{n=0}^{\infty} [q^{4n+1}, q^{4n}] \times [0, g(q^{4n+1})].$$

Observe that for $h = q^{4n-2}$, $k = f(q^{4n-2})$, $n \geq 1$ we have

$$\frac{m_2(A \cap ([-h, h] \times [-k, k]))}{4hk} \geq \frac{1}{4} \cdot \frac{q^{4n-2} - q^{4n-1}}{q^{4n-2}} = \frac{1}{4}(1 - q) > 0,$$

so $(0, 0)$ is not a density point of $\mathbb{R}^2 \setminus A$ with respect to f and $\mathbb{R}^2 \setminus A \notin \mathcal{T}_f$. Simultaneously, for $h \in [q^{4n+3}, q^{4n-1}]$ and $g(h) \leq k \leq h$ we have

$$\frac{m_2(A \cap ([-h, h] \times [-k, k]))}{4hk} \leq \frac{1}{4} \cdot \frac{1}{2n+3}$$

because $A \cap ([-h, h] \times [-k, k]) \subset [0, h] \times [0, f(q^{4n+3})]$ and $\frac{g(h)}{f(q^{4n+3})} \geq 2n+3$ for $h \in [q^{4n+3}, q^{4n-1}]$. Hence, $(0, 0)$ is a density point of $\mathbb{R}^2 \setminus A$ with respect to g . Observe also that $\mathbb{R}^2 \setminus (A \cup \{(0, 0)\})$ is open in the natural topology on the plane, so $\mathbb{R}^2 \setminus A \subset D_g(\mathbb{R}^2 \setminus A)$ and $\mathbb{R}^2 \setminus A \in \mathcal{T}_g$. Similarly, one can prove that $\mathbb{R}^2 \setminus B \in \mathcal{T}_f \setminus \mathcal{T}_g$.

REFERENCES

- [LMZ] LUKEŠ, J.—MALÝ, J.—ZAJÍČEK, L.: *Fine Topology Methods in Real Analysis and Potential Theory*, Lecture Notes in Math., Vol. 1189, Springer-Verlag, Berlin, 1986.
- [O] OXToby, J. C.: *Measure and Category. A Survey of the Analogies Between Topological and Measure Spaces* (2nd ed.). Grad. Texts in Math., Vol. 2, Springer-Verlag, New York, 1980.
- [Os] OSTASZEWSKI, K.: *Continuity in the density topology*, Real Anal. Exchange **7** (1982), 259–270.
- [S] SAKS, S.: *Theory of the Integral*. Hafner Publ. Comp., New York, 1938.
- [Sch] SCHEINBERG, S.: *Topologies which generate a complete measure algebra*, Adv. Math. **7** (1971), 231–239.
- [T] TALL, F. D.: *The density topology*, Pacific J. Math. **62** (1976), 275–284.
- [W] WILCZYŃSKI, W.: *Density topologies*, in: Handbook of Measure Theory (E. Pap, ed.), North-Holland, Amsterdam, 2002, pp. 675–702.

Received December 9, 2008

Elżbieta Wagner-Bojakowska
Faculty of Mathematics and
Computer Science
Banacha 22
PL-90-238 Łódź
POLAND

The College of Computer Science
Chair of Mathematics and Physics
Rzgowska 17A
PL-93-008 Łódź
POLAND
E-mail: wagner@uni.lodz.pl

Władysław Wilczyński
Faculty of Mathematics and
Computer Science
Banacha 22
PL-90-238 Łódź
POLAND
E-mail: wwil@uni.lodz.pl