ON THE FAMILY OF $[\lambda, \varrho]$ -CONTINUOUS FUNCTIONS

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ABSTRACT. In this paper, we introduce the class of $[\lambda, \varrho]$ -continuous functions. First, we investigate some relations between the classes of $[\lambda_1, \varrho_1]$ and $[\lambda_2, \varrho_2]$ -continuous functions for different pairs $[\lambda_1, \varrho_1]$, $[\lambda_2, \varrho_2]$. Next, we give an equivalent condition for a function in order that it will belong to the discussed class.

1. Preliminaries

Before stating and proving our theorems, we will collect some of the notions, definitions and notations which appear frequently in the sequel. We apply standard symbols and notations. By $\mathbb{R}$ we denote the set of real numbers, by $\mathbb{N}$ we denote the set of positive integers. The symbol $|\cdot|$ stands for the Lebesgue measure on the real line and also for the absolute value. Let f be a real-valued function defined on an open interval.

Let E be a measurable subset of $\mathbb{R}$ and let $x \in \mathbb{R}$. The numbers

$$\underline{d}^+(E, x) = \liminf_{t \rightarrow 0^+} \frac{|E \cap [x, x+t]|}{t}$$

and

$$\overline{d}^+(E, x) = \limsup_{t \rightarrow 0^+} \frac{|E \cap [x, x+t]|}{t}$$

are called the right-hand lower density of E at x and right-hand upper density of E at x . The left-hand lower and upper densities of E at x are defined analogously. If

$$\underline{d}^+(E, x) = \overline{d}^+(E, x) \quad \text{and} \quad \underline{d}^-(E, x) = \overline{d}^-(E, x),$$

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then we call these numbers the right-hand density and left-hand density of E at x , respectively. The numbers

$$\overline{d}(E, x) = \limsup_{\substack{t \rightarrow 0^+ \\ k \rightarrow 0^+ \\ t+k \neq 0}} \frac{|E \cap [x-t, x+k]|}{k+t}$$

and

$$\underline{d}(E, x) = \liminf_{\substack{t \rightarrow 0^+ \\ k \rightarrow 0^+ \\ t+k \neq 0}} \frac{|E \cap [x-t, x+k]|}{k+t}$$

are called the upper and lower density of E at x , respectively. If $\overline{d}(E, x) = \underline{d}(E, x)$, we call this number the density of E at x and denote it by $d(E, x)$.

Let us observe that

$$\overline{d}(E, x) = \max \left\{ \overline{d}^+(E, x), \overline{d}^-(E, x) \right\}$$

and

$$\underline{d}(E, x) = \min \left\{ \underline{d}^+(E, x), \underline{d}^-(E, x) \right\}.$$

Moreover, it is clear that

$$\overline{d}^+(E, x) = 1 - \underline{d}^+(\mathbb{R} \setminus E, x) \quad \text{and} \quad \underline{d}^+(E, x) = 1 - \overline{d}^+(\mathbb{R} \setminus E, x).$$

Similarly,

$$\overline{d}^-(E, x) = 1 - \underline{d}^-(\mathbb{R} \setminus E, x) \quad \text{and} \quad \underline{d}^-(E, x) = 1 - \overline{d}^-(\mathbb{R} \setminus E, x).$$

A. M. Bruckner, R. J. O'Malley and B. S. Thomson [2] proposed the notion of system of paths and developed a framework within which a number of generalized derivatives can be expressed. We use this idea for studying some notion of generalized continuity.

DEFINITION 1.1. Let E be a measurable subset of $\mathbb{R}$, $x \in \mathbb{R}$ and $0 < \lambda \leq \varrho < 1$. We say that the point x is a point of $[\lambda, \varrho]$ -type density of E when $\underline{d}(E, x) > \lambda$ and $\overline{d}(E, x) > \varrho$.

DEFINITION 1.2. A real-valued function f defined on an open interval I is called $[\lambda, \varrho]$ -continuous at x provided there is a measurable set $E \subset I$ such that the point x is a point of $[\lambda, \varrho]$ -type density of E , $x \in E$ and $f|_E$ is continuous at x . If f is $[\lambda, \varrho]$ -continuous at every point of I , we simply say that f is $[\lambda, \varrho]$ -continuous.

We will denote the class of $[\lambda, \varrho]$ -continuous functions by $\mathcal{C}_{[\lambda, \varrho]}$.

Straightforward from the definition, it follows:

LEMMA 1.1. If $0 < \lambda_1 \leq \lambda_2 < 1$ and $0 < \varrho_1 \leq \varrho_2 < 1$, then $\mathcal{C}_{[\lambda_2, \varrho_2]} \subset \mathcal{C}_{[\lambda_1, \varrho_1]}$.

2. Relationship between $\mathcal{C}_{[\lambda, \varrho]}$ for different numbers λ and ϱ

In the preface we investigate some relationship among the classes of $[\lambda_1, \varrho_1]$ and $[\lambda_2, \varrho_2]$ -continuous functions for different pairs $[\lambda_1, \varrho_1]$, $[\lambda_2, \varrho_2]$. But first, we review two standard properties of the density of a set at a point and the proof of a useful lemma.

The proof of the following lemma is straightforward and follows exactly from the properties of upper and lower limits of the sum of two positive functions.

LEMMA 2.1. *Let E and F be measurable sets and let $x \in \mathbb{R}$. Then*

1. $\bar{d}^+(E \cup F, x) \leq \bar{d}^+(E, x) + \bar{d}^+(F, x)$.
2. $\underline{d}^+(E \cup F, x) \leq \underline{d}^+(E, x) + \bar{d}^+(F, x)$.

It is easily seen that the analogous properties as in Lemma 2.1 are true for the left-hand density.

LEMMA 2.2. *For every real number α such that $0 \leq \alpha \leq 1$ there exists a sequence of intervals $\{I_n = [a_n, b_n]: n \geq 1\}$ such that $0 < \dots < b_{n+1} < a_n < \dots$, $\underline{d}^+(\bigcup_{n=1}^{\infty} I_n, 0) = \alpha$ and $\bar{d}^+(\bigcup_{n=1}^{\infty} I_n, 0) = 1$.*

Proof. Fix any $0 \leq \alpha \leq 1$. Let $\{x_n\}_{n \geq 1}$ be a decreasing sequence converging to 0 such that

$$\lim_{n \rightarrow \infty} \frac{x_n - x_{n+1}}{x_n} = 1 \quad (2.1)$$

(for example $x_n = \frac{1}{n!}$). Hence, $\lim_{n \rightarrow \infty} \frac{x_{n+1}}{x_n} = 0$. Let

$$I_n = [x_{n+1}, x_{n+1} + \alpha(x_n - x_{n+1})].$$

Therefore, $|I_n| = \alpha|x_{n+1}, x_n|$. By (2.1), $\underline{d}^+(\bigcup_{n=1}^{\infty} I_n, 0) = \alpha$.

Let

$$y_n = x_{n+1} + \alpha(x_n - x_{n+1}).$$

We have $\bar{d}^+(\bigcup_{n=1}^{\infty} I_n, 0) = 1$, because

$$\lim_{n \rightarrow \infty} \frac{\left| \bigcup_{k=1}^{\infty} I_k \cap [0, y_n] \right|}{y_n} \geq \lim_{n \rightarrow \infty} \frac{|I_n|}{y_n} = \lim_{n \rightarrow \infty} \frac{y_n - x_{n+1}}{y_n} \geq 1 - \lim_{n \rightarrow \infty} \frac{1}{\alpha} \cdot \frac{x_{n+1}}{x_n} = 1.$$

□

EXAMPLE 2.1. Let ϱ_1, ϱ_2 be arbitrary real numbers such that $0 < \varrho_2 < \varrho_1 < 1$. Let $\{I_n = [a_n, b_n]: 0 < \dots < b_{n+1} < a_n < \dots, n \geq 1\}$ be a sequence of intervals such that $\bar{d}^+(\bigcup_{n=1}^{\infty} I_n, 0) = \frac{\varrho_1 + \varrho_2}{2}$. Let $\{K_n = [c_n, d_n]: n \geq 1\}$ be a sequence of intervals such that $I_n \subset \text{Int } K_n$ for $n \in \mathbb{N}$ and

$$\bar{d}^+\left(\bigcup_{n=1}^{\infty} (K_n \setminus I_n), 0\right) = 0.$$

Let a function $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0 & \text{if } x \in (-\infty, 0] \cup [b_1, \infty) \cup \bigcup_{n=1}^{\infty} I_n, \\ 1 & \text{if } x \in \bigcup_{n=1}^{\infty} [d_{n+1}, c_n], \\ \text{linear on the intervals } [c_n, a_n] \text{ and } [b_{n+1}, d_{n+1}], & n = 1, 2, \dots \end{cases}$$

The function f is continuous at every point except 0. If

$$E_1 = \bigcup_{n=1}^{\infty} I_n \cup (-\infty, 0],$$

then the function f restricted to E_1 is constant, so in particular, it is continuous at zero. Moreover,

$$\underline{d}^+(E_1, 0) = \frac{\varrho_1 + \varrho_2}{2} > \varrho_2 \quad \text{and} \quad \overline{d}^+(E_1, 0) = \frac{\varrho_1 + \varrho_2}{2} > \varrho_2.$$

Hence,

$$f \in \mathcal{C}_{[\varrho_2, \varrho_2]}.$$

But, if

$$E_2 = \{x: f(x) < 1\},$$

then

$$\begin{aligned} \overline{d}^+(E_2, 0) &\leq \overline{d}^+\left(\bigcup_{n=1}^{\infty} K_n, 0\right) \\ &= \overline{d}^+\left(\bigcup_{n=1}^{\infty} I_n \cup \left(\bigcup_{n=1}^{\infty} (K_n \setminus I_n)\right), 0\right) \\ &\leq \overline{d}^+\left(\bigcup_{n=1}^{\infty} I_n, 0\right) + \overline{d}^+\left(\bigcup_{n=1}^{\infty} (K_n \setminus I_n), 0\right) \\ &= \frac{\varrho_1 + \varrho_2}{2} + 0 < \varrho_1. \end{aligned}$$

If E is a measurable set such that $0 \in E$ and $f|_E$ is continuous at 0, then there exists $\delta > 0$ for which $E \cap (-\delta, \delta) \subset E_2$. Therefore, $\overline{d}^+(E, 0) \leq \overline{d}^+(E_2, 0) < \varrho_1$. Hence, $f \notin \mathcal{C}_{[\lambda_1, \varrho_1]}$ for every λ_1 such that $0 < \lambda_1 \leq \varrho_1$.

COROLLARY 2.1. *Let $0 < \lambda_1 \leq \varrho_1 < 1$ and $0 < \lambda_2 \leq \varrho_2 < 1$. If $\mathcal{C}_{[\lambda_2, \varrho_2]} \subset \mathcal{C}_{[\lambda_1, \varrho_1]}$, then $\varrho_1 \leq \varrho_2$.*

Proof. Assume that $\mathcal{C}_{[\lambda_2, \varrho_2]} \subset \mathcal{C}_{[\lambda_1, \varrho_1]}$ and suppose that $\varrho_1 > \varrho_2$. Example 2.1 shows that there exists a function $f: \mathbb{R} \rightarrow \mathbb{R}$ such that $f \in \mathcal{C}_{[\varrho_2, \varrho_2]} \setminus \mathcal{C}_{[\lambda_1, \varrho_1]}$ for every real numbers $\varrho_1, \varrho_2, \lambda_1$ such that $0 < \varrho_2 < \varrho_1 < 1$ and $0 < \lambda_1 \leq \varrho_1$. Since $\mathcal{C}_{[\varrho_2, \varrho_2]} \subset \mathcal{C}_{[\lambda_2, \varrho_2]}$, we have $f \in \mathcal{C}_{[\lambda_2, \varrho_2]} \setminus \mathcal{C}_{[\lambda_1, \varrho_1]}$. Hence, $\mathcal{C}_{[\lambda_2, \varrho_2]} \not\subset \mathcal{C}_{[\lambda_1, \varrho_1]}$, a contradiction $\square$

EXAMPLE 2.2. Let λ_1, λ_2 be arbitrary real numbers such that $0 < \lambda_2 < \lambda_1 < 1$. By Lemma 2.2, there exists a sequence of intervals

$$\{I_n = [a_n, b_n]: 0 < \dots < b_{n+1} < a_n < \dots, n \geq 1\}$$

such that

$$\underline{d}^+\left(\bigcup_{n=1}^{\infty} I_n, 0\right) = \frac{\lambda_1 + \lambda_2}{2} \quad \text{and} \quad \overline{d}^+\left(\bigcup_{n=1}^{\infty} I_n, 0\right) = 1.$$

Let $\{K_n = [c_n, d_n]: n \geq 1\}$ be a sequence of intervals such that $I_n \subset \text{Int } K_n$ for positive integers n and $\overline{d}^+(\bigcup_{n=1}^{\infty} (K_n \setminus I_n), 0) = 0$.

Defining the function f in the same way as in the previous example, we obtain that $f \in \mathcal{C}_{[\lambda_2, \varrho_2]}$ for any $\lambda_2 \leq \varrho_2 < 1$. But, if $E_2 = \{x: f(x) < 1\}$, then

$$\begin{aligned} \underline{d}^+(E_2, 0) &\leq \underline{d}^+\left(\bigcup_{n=1}^{\infty} K_n, 0\right) \\ &= \underline{d}^+\left(\bigcup_{n=1}^{\infty} I_n \cup \left(\bigcup_{n=1}^{\infty} (K_n \setminus I_n)\right), 0\right) \\ &\leq \underline{d}^+\left(\bigcup_{n=1}^{\infty} I_n, 0\right) + \overline{d}^+\left(\bigcup_{n=1}^{\infty} (K_n \setminus I_n), 0\right) \\ &= \frac{\lambda_1 + \lambda_2}{2} + 0 < \lambda_1. \end{aligned}$$

If E is a measurable set such that $0 \in E$ and $f|_E$ is continuous at 0, then there exists $\delta > 0$ for which $E \cap (-\delta, \delta) \subset E_2$. Therefore, $\underline{d}^+(E, 0) \leq \underline{d}^+(E_2, 0) < \lambda_1$. Hence, $f \notin \mathcal{C}_{[\lambda_1, \lambda_1]}$.

COROLLARY 2.2. *Let $0 < \lambda_1 \leq \varrho_1 < 1$ and $0 < \lambda_2 \leq \varrho_2 < 1$. If $\mathcal{C}_{[\lambda_2, \varrho_2]} \subset \mathcal{C}_{[\lambda_1, \varrho_1]}$, then $\lambda_1 \leq \lambda_2$.*

Proof. Assume that $\mathcal{C}_{[\lambda_2, \varrho_2]} \subset \mathcal{C}_{[\lambda_1, \varrho_1]}$ and suppose that $\lambda_1 > \lambda_2$. Example 2.2 shows that there exists a function $f: \mathbb{R} \rightarrow \mathbb{R}$ such that $f \in \mathcal{C}_{[\lambda_2, \varrho_2]} \setminus \mathcal{C}_{[\lambda_1, \lambda_1]}$ for every real numbers $\lambda_1, \lambda_2, \varrho_2$ such that $0 < \lambda_2 < \lambda_1 < 1$ and $0 < \lambda_2 \leq \varrho_2$. Since $\mathcal{C}_{[\lambda_1, \lambda_1]} \subset \mathcal{C}_{[\lambda_1, \varrho_1]}$, we have $f \in \mathcal{C}_{[\lambda_2, \varrho_2]} \setminus \mathcal{C}_{[\lambda_1, \varrho_1]}$. Hence, $\mathcal{C}_{[\lambda_2, \varrho_2]} \not\subset \mathcal{C}_{[\lambda_1, \varrho_1]}$, a contradiction. $\square$

Combining Corollaries 2.1, 2.2 and Lemma 1.1, we obtain:

LEMMA 2.3. $\mathcal{C}_{[\lambda_2, \varrho_2]} \subset \mathcal{C}_{[\lambda_1, \varrho_1]} \Leftrightarrow 0 < \lambda_1 \leq \lambda_2 < 1 \wedge 0 < \varrho_1 \leq \varrho_2 < 1$.

We will prove an equivalent condition for a function in order that it will belong to $\mathcal{C}_{[\lambda, \varrho]}$.

LEMMA 2.4. *Let $0 < \lambda < 1$. If E is a measurable set and if $\underline{d}^+(E, x) \geq \lambda$ then for each positive integer n there exists $\varepsilon_n > 0$ such that*

$$\frac{|E \cap [x + \frac{a}{2n}, x + b]|}{b} > \lambda - \frac{1}{n}$$

for each numbers $0 < a < b < \varepsilon_n$.

Proof. Let n be an arbitrary positive integer. Since $\underline{d}^+(E, x) \geq \lambda$, it follows that there exists $\varepsilon_n > 0$ such that

$$\frac{|E \cap [x, x + c]|}{c} > \lambda - \frac{1}{2n}$$

for each $c \in (0, \varepsilon_n)$. If $0 < a < b < \varepsilon_n$, then

$$\begin{aligned} \left| E \cap \left[x + \frac{a}{2n}, x + b \right] \right| &= |E \cap [x, x + b]| - \left| E \cap \left[x, x + \frac{a}{2n} \right] \right| \\ &\geq b \left(\lambda - \frac{1}{2n} \right) - \frac{a}{2n} \\ &> b\lambda - \frac{b}{n}. \end{aligned}$$

Thus

$$\frac{|E \cap [x + \frac{a}{2n}, x + b]|}{b} > \lambda - \frac{1}{n}.$$

□

LEMMA 2.5. *Let $0 < \varrho < 1$. If E is a measurable set, $\overline{d}^+(E, x) \geq \varrho$, and n is a positive integer, then there exists a sequence $\{\alpha_m\}_{m \geq 1}$ of positive numbers such that $\lim_{m \rightarrow \infty} \alpha_m = 0$, $\alpha_{m+1} < \alpha_m$ and*

$$\frac{|E \cap [x + \frac{\alpha_m}{2n}, x + \alpha_m]|}{\alpha_m} > \varrho - \frac{1}{n}$$

for every $m \geq 1$.

Proof. Let n be an arbitrary natural number. Since $\overline{d}^+(E, x) \geq \varrho$, it follows that there exists a decreasing sequence $\{\beta_m\}_{m \geq 1}$ of positive numbers, $\lim_{m \rightarrow \infty} \beta_m = 0$, such that

$$\lim_{m \rightarrow \infty} \frac{|E \cap [x, x + \beta_m]|}{\beta_m} = \overline{d}^+(E, x) \geq \varrho.$$

Therefore, there exists m_0 such that

$$\frac{|E \cap [x, x + \beta_m]|}{\beta_m} > \varrho - \frac{1}{2n}$$

for every $m \geq m_0$.

Hence,

$$\begin{aligned} \frac{|E \cap [x + \frac{\beta_m}{2n}, x + \beta_m]|}{\beta_m} &\geq \frac{|E \cap [x, x + \beta_m]|}{\beta_m} - \frac{|E \cap [x, x + \frac{\beta_m}{2n}]|}{\beta_m} \\ &> \left(\varrho - \frac{1}{2n}\right) - \frac{1}{2n} \\ &= \varrho - \frac{1}{n} \end{aligned}$$

for every $m \geq m_0$. So, if $\alpha_m = \beta_{m+m_0}$ for $m \geq 1$, then the sequence $\{\alpha_m\}_{m \geq 1}$ satisfies all required properties. $\square$

In the same way, we can prove the analogous properties for left-hand lower and upper density at x .

LEMMA 2.6. *Let $0 < \lambda \leq \varrho < 1$. Let $\{E_n : n \in \mathbb{N}\}$ be a descending family of measurable sets such that $x \in \bigcap_{n=1}^{\infty} E_n$, $\underline{d}(E_n, x) \geq \lambda$ and $\overline{d}(E_n, x) \geq \varrho$ for $n \geq 1$. Then there exists a measurable set E such that $\underline{d}(E, x) \geq \lambda$, $\overline{d}(E, x) \geq \varrho$, $x \in E$ and for every positive integer n there exists $\delta_n > 0$ such that*

$$E \cap [x - \delta_n, x + \delta_n] \subset E_n.$$

Proof. By assumption, $\underline{d}^+(E_n, x) \geq \lambda$ and $\underline{d}^-(E_n, x) \geq \lambda$ for $n \geq 1$. Furthermore, $\overline{d}^+(E_n, x) \geq \varrho$ or $\overline{d}^-(E_n, x) \geq \varrho$ for $n \geq 1$. There exists an infinite sequence $\{E_{n_k} : k \in \mathbb{N}\}$ such that $\overline{d}^+(E_{n_k}, x) \geq \varrho$ or $\overline{d}^-(E_{n_k}, x) \geq \varrho$ for $k \geq 1$.

Without loss of generality, we may assume that $\overline{d}^+(E_n, x) \geq \varrho$ for every $n \geq 1$.

Let $n \in \mathbb{N}$. By Lemma 2.4, there exists $\varepsilon_n > 0$ such that

$$\frac{|E_n \cap [x + \frac{a}{2n}, x + b]|}{b} > \lambda - \frac{1}{n} \quad (2.2)$$

for each numbers $0 < a < b < \varepsilon_n$. By Lemma 2.5, there exists a decreasing convergent to zero sequence $\{\alpha_m^n\}_{m \geq 1}$ such that

$$\frac{|E_n \cap [x + \frac{\alpha_m^n}{2n}, x + \alpha_m^n]|}{\alpha_m^n} > \varrho - \frac{1}{n} \quad (2.3)$$

for every $m \geq 1$. We can see that there exists a convergent to zero sequence $\{\alpha_{m_n}^n\}_{m \geq 1}$ such that

$$\frac{|E_n \cap [x + \frac{\alpha_{m_n}^n}{2n}, x + \alpha_{m_n}^n]|}{\alpha_{m_n}^n} > \varrho - \frac{1}{n} \quad (2.4)$$

for each $m \geq 1$.

We inductively construct a decreasing sequence of positive numbers $\{a_n\}_{n \geq 1}$ such that

$$a_1 < \varepsilon_1 \quad \text{and} \quad a_{n+1} < \min \left\{ \varepsilon_{n+1}, \frac{a_n}{2n}, \frac{\alpha_{m_n}^n}{2n} \right\} \quad \text{for } n \geq 1.$$

Let

$$F = \bigcup_{n=1}^{\infty} (E_n \cap [x + a_{n+2}, x + a_{n+1}]).$$

We show that $\underline{d}^+(F, x) \geq \lambda$. Let $y \in [x, x + a_2]$, $y = x + c$. There exists $n \geq 2$ such that $c \in [a_{n+1}, a_n]$. By the definition of a set F , monotonicity of $\{E_n\}_{n \in \mathbb{N}}$ and the inequality $\frac{a_{n+1}}{2n} > a_{n+2}$, we have

$$F \cap [x + a_{n+2}, x + c] \supset E_n \cap \left[x + \frac{a_{n+1}}{2n}, x + c \right].$$

From (2.2), we obtain

$$\begin{aligned} |F \cap [x, x + c]| &> \left| F \cap \left[x + \frac{a_{n+1}}{2n}, x + c \right] \right| \\ &\geq \left| E_n \cap \left[x + \frac{a_{n+1}}{2n}, x + c \right] \right| \\ &> c \left(\lambda - \frac{1}{n} \right). \end{aligned}$$

Hence,

$$\frac{|F \cap [x, x + c]|}{c} > \lambda - \frac{1}{n}.$$

Thus

$$\liminf_{y \rightarrow x^+} \frac{|F \cap [x, y]|}{y - x} \geq \lambda,$$

since $n \rightarrow \infty$ as $y \rightarrow x$. It follows that $\underline{d}^+(F, x) \geq \lambda$.

Now, we prove that $\overline{d}^+(F, x) \geq \varrho$. Since $\{E_n : n \geq 1\}$ is a descending family of measurable sets and since

$$\frac{\alpha_{m_n}^n}{2n} \geq a_{n+1} > a_{n+2},$$

we have

$$F \cap [x, x + \alpha_{m_n}^n] \supset E_n \cap [x + a_{n+2}, x + \alpha_{m_n}^n] \supset E_n \cap \left[x + \frac{\alpha_{m_n}^n}{2n}, x + \alpha_{m_n}^n \right].$$

By (2.4),

$$\frac{|F \cap [x + \frac{\alpha_{m_n}^n}{2n}, x + \alpha_{m_n}^n]|}{\alpha_{m_n}^n} > \varrho - \frac{1}{n}$$

for $n \geq 1$.

Hence,

$$\begin{aligned}\overline{d}^+(F, x) &= \limsup_{h \rightarrow 0^+} \frac{|F \cap [x, x+h]|}{h} \\ &\geq \limsup_{n \rightarrow \infty} \frac{|F \cap [x, x + \alpha_{m_n}^n]|}{\alpha_{m_n}^n} \\ &\geq \limsup_{n \rightarrow \infty} \left(\varrho - \frac{1}{n} \right) = \varrho.\end{aligned}$$

From the definition of the set F , we conclude that for every positive integer n there exists $\delta_n = a_{n+1} > 0$ such that $F \cap [x, x + \delta_n] \subset E_n$.

Similarly, we can define a measurable set $G \subset (-\infty, x)$ for which $\underline{d}^-(G, x) \geq \lambda$ and there exists $\delta_n > 0$ such that $G \cap [x - \delta_n, x] \subset E_n$ for every positive integer n .

Thus, we get that the set $E = F \cup G \cup \{x\}$ establishes the lemma. $\square$

We use Lemma 2.6 to prove the next theorem.

THEOREM 2.1. *Let $0 < \lambda \leq \varrho < 1$ and let I be an open interval. Let $f: I \rightarrow \mathbb{R}$ be a measurable function. Then f is $[\lambda, \varrho]$ -continuous at x if and only if*

$$\lim_{\varepsilon \rightarrow 0^+} \underline{d}\left(\{y \in I: |f(x) - f(y)| < \varepsilon\}, x\right) > \lambda$$

and

$$\lim_{\varepsilon \rightarrow 0^+} \overline{d}\left(\{y \in I: |f(x) - f(y)| < \varepsilon\}, x\right) > \varrho.$$

Proof. First, suppose that the function f is $[\lambda, \varrho]$ -continuous at x . Let E be a measurable set, $x \in E$, $\underline{d}(E, x) > \lambda$, $\overline{d}(E, x) > \varrho$ and let $f|_E$ be continuous at x . Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that $E \cap (x - \delta, x + \delta) \subset \{y \in I: |f(x) - f(y)| < \varepsilon\}$. Hence,

$$\underline{d}\left(\{y \in I: |f(x) - f(y)| < \varepsilon\}, x\right) \geq \underline{d}(E, x)$$

and

$$\overline{d}\left(\{y \in I: |f(x) - f(y)| < \varepsilon\}, x\right) \geq \overline{d}(E, x).$$

Therefore,

$$\lim_{\varepsilon \rightarrow 0^+} \underline{d}\left(\{y \in I: |f(x) - f(y)| < \varepsilon\}, x\right) \geq \underline{d}(E, x)$$

and

$$\lim_{\varepsilon \rightarrow 0^+} \overline{d}\left(\{y \in I: |f(x) - f(y)| < \varepsilon\}, x\right) \geq \overline{d}(E, x).$$

Since $f|E$ is $[\lambda, \varrho]$ -continuous at x , we have

$$\lim_{\varepsilon \rightarrow 0^+} \underline{d}\left(\{y \in I: |f(x) - f(y)| < \varepsilon\}, x\right) > \lambda$$

and

$$\lim_{\varepsilon \rightarrow 0^+} \overline{d}\left(\{y \in I: |f(x) - f(y)| < \varepsilon\}, x\right) > \varrho.$$

Conversely, suppose that

$$\lambda_1 = \lim_{\varepsilon \rightarrow 0^+} \underline{d}\left(\{y \in I: |f(x) - f(y)| < \varepsilon\}, x\right) > \lambda$$

and

$$\varrho_1 = \lim_{\varepsilon \rightarrow 0^+} \overline{d}\left(\{y \in I: |f(x) - f(y)| < \varepsilon\}, x\right) > \varrho.$$

Applying Lemma 2.6 for $E_n = \{y \in (a, b): |f(x) - f(y)| < \frac{1}{n}\}$, we can construct a measurable set E such that $\underline{d}(E, x) \geq \lambda_1 > \lambda$, $\overline{d}(E, x) \geq \varrho_1 > \varrho$ and for every positive integer n there exists $\delta_n > 0$ such that $E \cap [x - \delta_n, x + \delta_n] \subset E_n$. From the latter condition, we conclude that $f|E$ is continuous at x . Hence, f is $[\lambda, \varrho]$ -continuous at x , and the proof is complete. $\square$

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