AN INTEGRAL FOR A BANACH VALUED FUNCTION

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ABSTRACT. Using partitions of the unity ((PU) -partition), a new definition of an integral is given for a function $f: [a, b] \rightarrow X$, where X is a Banach space, and it is proved that this integral is equivalent to the Bochner integral.

Introduction

Using the idea of an integral given by F e a u v e a u in [5], an integral ($(PU)^*$ -integral) for a Banach valued function is defined on an interval $[a, b]$, as a limit of Riemann-type sums, using partitions of the unity ((PU) -partition) rather than the usual partitions of the interval. Using some properties of the (PU) -integral of real valued functions defined on a compact Hausdorff space (see [10]), the measurability of a $(PU)^*$ -integrable function and the equivalence of this integral with the Bochner integral are proved.

We observe that, while the (PU) -integral is equivalent to the Lebesgue integral and hence to the McShane integral, the $(PU)^*$ -integral is equivalent to the Bochner integral but not to the McShane one.

Preliminaries

In this paper, X denotes a Banach space with a norm $\| \cdot \|_X$ and X^* its dual space, $[a, b]$ is a real interval with $a < b$, $\mathcal{M}$ is a σ -algebra of subsets of $[a, b]$ to which the open sets belong, μ is a non-atomic, finite, complete Radon measure on $\mathcal{M}$.

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DEFINITION 1 ([10]). A partition of the unity $((PU)$ -partition) of $[a, b]$ is a finite collection $P = \{(\theta_i, x_i)\}_{i=1}^p$, where $x_i \in [a, b]$ and θ_i are non negative, μ -measurable and μ -integrable real functions on $[a, b]$ such that $\sum_{i=1}^p \theta_i(x) = 1$ a.e. in $[a, b]$.

DEFINITION 2 ([9]). A gauge δ on $[a, b]$ is a map from $[a, b]$ to $(0, +\infty)$.

DEFINITION 3 ([10]). If δ is a gauge on $[a, b]$, a (PU) -partition $P = \{(\theta_i, x_i)\}_{i=1}^p$ is δ -fine if $S_{\theta_i} \subset (x_i - \delta(x_i), x_i + \delta(x_i))$ ($i = 1, 2, \dots, p$), where $S_{\theta_i} = \{x \in [a, b] \text{ such that } \theta_i(x) \neq 0\}$.

Now, we will give the following definition:

DEFINITION 4. A function $f: [a, b] \rightarrow X$ is $(PU)^*$ -integrable on $[a, b]$ if for every given $\epsilon > 0$ there is a gauge δ_ϵ such that $\sum_{i=1}^p \|(f(x_i) - f(x'_i)) \int_a^b \theta_i, d\mu\| < \epsilon$ for each couple $P = \{(\theta_i, x_i)\}_i$ and $P' = \{(\theta_i, x'_i)\}_i$ of δ_ϵ -fine (PU) -partitions.

We say that δ_ϵ is ϵ -adapted to f .

Main results

1. The $(PU)^*$ -integral

PROPOSITION 1.1. *Let $f: [a, b] \rightarrow X$ be a $(PU)^*$ -integrable function and, for every $\epsilon > 0$, δ_ϵ be a gauge ϵ -adapted to f . For every (PU) -partition $P_\epsilon = \{(\theta_{i,\epsilon}, x_{i,\epsilon})\}$ δ_ϵ -fine, put $\sigma(f, P_\epsilon) = \sum_i f(x_{i,\epsilon}) \int_a^b \theta_{i,\epsilon} d\mu$. If ϵ converges to 0, then the function $\epsilon \rightarrow \sigma(f, P_\epsilon)$ converges; this limit does not depend on the chosen family $\{P_\epsilon, \epsilon > 0\}$ and it is called the $(PU)^*$ -integral of f . Denote this limit by $(PU)^* \int_a^b f$.*

PROOF. The proof is similar to that in [5, Theorem 2.1]. First, consider an increasing family of gauges (δ_ϵ) with $\epsilon > 0$, so $\delta_\alpha \leq \delta_\gamma$ for $0 < \alpha < \gamma$.

For $0 < \alpha < \gamma$, consider two (PU) -partitions $P_\alpha = (\theta_{i,\alpha}, x_{i,\alpha})_{i=1}^n$ and $P_\gamma = (\theta'_{j,\gamma}, x'_{j,\gamma})_{j=1}^p$ δ_α -fine and δ_γ -fine respectively, and, since $\delta_\alpha \leq \delta_\gamma$, P_α is also δ_γ -fine.

Using the same construction as in [10, Proposition 1.5], it is possible to construct two partitions $P'_\gamma = (h_{ij,\gamma}, x_i)_{i,j}$ and $P''_\gamma = (h_{ij,\gamma}, x'_j)_{i,j}$ δ_γ -fine with $h_{ij} = (\theta_i \cdot \theta'_j)$ $i = 1, 2, \dots, n$ $j = 1, \dots, p$ and such that

$$\|\sigma(f, P_\alpha) - \sigma(f, P_\gamma)\| = \|\sigma(f, P'_\gamma) - \sigma(f, P''_\gamma)\| \leq \gamma.$$

Since the family $\{\sigma(f, P_\epsilon)\}_\epsilon$ satisfies the Cauchy-property, it converges.

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Now, let $(\delta_\epsilon)_{\epsilon>0}$ be a general family of gauges adapted to f and $(\epsilon_n)_{n \in \mathbb{N}}$ be a decreasing sequence of positive numbers converging to 0. The sequence $(\sigma(f, P_{\epsilon_n}))_{n \in \mathbb{N}}$ is convergent.

In fact, define a new family of gauges $(\bar{\delta}_\epsilon)_{\epsilon>0}$ as follows:

$$\bar{\delta}_\epsilon = \delta_{\epsilon_0} \text{ for every } \epsilon \geq \epsilon_0,$$

$$\bar{\delta}_\epsilon = \min\{\delta_{\epsilon_0}, \delta_{\epsilon_1}, \dots, \delta_{\epsilon_n}\} \text{ for } \epsilon \in [\epsilon_n, \epsilon_{n-1}[\text{ for every } n > 0.$$

The family $(\bar{\delta}_\epsilon)_{\epsilon>0}$ is increasing and adapted to f . So, the family $\{\sigma(f, P_{\epsilon_n})\}_n$ converges, and the function $\epsilon \rightarrow \sigma(f, P_\epsilon)$ is convergent for $\epsilon \rightarrow 0$.

Now, we will prove that the limit is independent from the choice of the family $\{P_\epsilon\}$.

Let $(\delta_\epsilon)_{\epsilon>0}$ and $(\delta'_\epsilon)_{\epsilon>0}$ be two families of gauges adapted to f , and $\mathcal{P} = \{P_\epsilon = (\theta_{i,\epsilon}, x_{i,\epsilon})_i\}_\epsilon$, $\mathcal{P}' = \{P'_\epsilon = (\theta'_{i,\epsilon}, x'_{i,\epsilon})_i\}_\epsilon$ be two families of δ_ϵ and δ'_ϵ -fine (PU)-partitions, respectively, for varying $\epsilon > 0$. For every $\epsilon > 0$, define a family of gauges $\{\delta''_\epsilon\}$ and a family of δ''_ϵ -fine (PU)-partitions $\mathcal{P}'' = \{P''_\epsilon = (x''_{i,\epsilon}, \theta''_{i,\epsilon})_i\}_\epsilon$ as follows:

$$\delta''_\epsilon = \delta_\epsilon \text{ and } P''_\epsilon = P_\epsilon \text{ if } \epsilon \text{ is a positive rational,}$$

$$\delta''_\epsilon = \delta'_\epsilon \text{ and } P''_\epsilon = P'_\epsilon \text{ if } \epsilon \text{ is a positive irrational.}$$

The function $\epsilon \rightarrow \sigma(f, P''_\epsilon)$ is convergent for $\epsilon \rightarrow 0$, and its limit is equal to the limit of $\{\sigma(f, P'_\epsilon)\}$ and to the limit of $\{\sigma(f, P_\epsilon)\}$. $\square$

PROPOSITION 1.2. *If f, g are two (PU)*-integrable functions on $[a, b]$, the following classical results for an integral are verified:*

- $\alpha)$ $f + g$ and kf , for k real number, are (PU)*-integrable,
- $\beta)$ the operator $f \rightarrow (PU)^* \int_a^b f$ is linear,
- $\gamma)$ f is (PU)*-integrable in every subinterval $[c, d]$,
- $\delta)$ if $c \in (a, b)$, then f is (PU)*-integrable in $[a, c]$ and $[c, b]$ and satisfies the Chasles relation

$$(PU)^* \int_a^b f = (PU)^* \int_a^c f + (PU)^* \int_c^b f.$$

PROPOSITION 1.3. *If f is (PU)*-integrable, then for $\epsilon > 0$ and for every δ_ϵ -fine partition $\{(\chi_{I_i}, c_i)\}$, where χ_{I_i} is the characteristic function of the interval I_i , we have*

$$\sum_i \left\| f(c_i) \int_a^b \chi_{I_i} d\mu - (PU)^* \int_{x_{i-1}}^{x_i} f \right\| \leq \epsilon,$$

where $(I_i = [x_{i-1}, x_i], c_i)_i$ is a δ_ϵ -fine partition of $[a, b]$.

Proof. The proof follows the idea of the classical Saks-Henstock lemma, (see [5, p. 922]). $\square$

PROPOSITION 1.4. *If f is $(PU)^*$ -integrable on $[a, b]$, then the function $F: t \rightarrow (PU)^* \int_a^t f$ with $t \in [a, b]$ is absolutely continuous.*

Proof. The proof is analogous to that in [5, Theorem 3.3], but (PU) -partitions $\{(\chi_{I_i})\}$ are used instead of the usual partitions. $\square$

PROPOSITION 1.5. *Let $f: [a, b] \rightarrow X$ be a continuous and differentiable function with the derivative f' $(PU)^*$ -integrable on $[a, b]$. Then*

$$(PU)^* \int_a^b f' = f(b) - f(a).$$

Proof. The proof is analogous to that in [5, Theorem 3.4]. $\square$

PROPOSITION 1.6. *If f is $(PU)^*$ -integrable, then the function $F(t) = (PU)^* \int_a^t f$ is differentiable a.e. on $[a, b]$ and $F'(t) = f(t)$ a.e. $t \in [a, b]$.*

Proof. The proof is the same as for the real valued functions (see [7, p. 145]). $\square$

2. Integrability of the norm

PROPOSITION 2.1. *If f is $(PU)^*$ -integrable, then the function $\|f\|: [a, b] \rightarrow [0, +\infty)$ is Lebesgue-integrable and we have the relation*

$$\left\| (PU)^* \int_a^b f \right\| \leq \int_a^b \|f\| d\mu.$$

Proof. By [10], a real function is (PU) -integrable on $[a, b]$ if there is a real number λ such that for every $\epsilon > 0$, there exists a gauge δ on $[a, b]$ with the property that for every δ -fine (PU) -partition $\{(\theta_i, c_i)\}_i$ of the interval $[a, b]$, the following inequality

$$\left| \sum_i f(c_i) \int_a^b \theta_i d\mu - \lambda \right| \leq \epsilon$$

holds.

The $(PU)^*$ -integrability of $\|f\|$ is an immediate consequence of the definition of $(PU)^*$ -integrability of f and the (PU) -integrability of $\|f\|$ (see [10, Prop. 1.5 and 3.3]). Moreover, observe that a gauge δ_ϵ , ϵ -adapted to f , is also ϵ -adapted to $\|f\|$, so the last relation is a consequence of the inequality

$$\left\| \sum_i f(c_i) \int_a^b \theta_i d\mu \right\| \leq \sum_i \|f(c_i)\| \int_a^b \theta_i d\mu$$

for every δ -fine (PU) -partition. $\square$

3. Measurability of a $(PU)^*$ -integrable function

PROPOSITION 3.1. *If f is a.e. a null function then it is $(PU)^*$ -integrable.*

Proof. If f is a.e. a null function, the real function $\|f\|$ is null a.e. and, by the completeness of the measure, it is μ -measurable and μ -integrable and, by [10, Proposition 2.2], it is (PU) -integrable on $[a, b]$. So, for $\epsilon > 0$, there exists a gauge δ_ϵ such that

$$\sum_i \|f(c_i)\| \int_a^b \theta_i d\mu \leq \epsilon$$

for each δ -fine (PU) -partition $\{(\theta_i, c_i)\}_i$.

Then, for each couple of δ -fine (PU) -partitions $P = \{(\theta_i, c_i)\}_i$ and $P' = \{(\theta'_i, c'_i)\}_i$, we have

$$\sum_i \|f(c_i) - f(c'_i)\| \int_a^b \theta_i d\mu \leq \sum_i \|f(c_i)\| \int_a^b \theta_i d\mu + \sum_i \|f(c'_i)\| \int_a^b \theta'_i d\mu \leq 2\epsilon.$$

So, f is $(PU)^*$ -integrable and, by Proposition 2.1, $(PU)^* \int_a^b f = 0$. $\square$

PROPOSITION 3.2. *If f is $(PU)^*$ -integrable and if $g = f$ a.e. in $[a, b]$, then g is $(PU)^*$ -integrable and $(PU)^* \int_a^b f = (PU)^* \int_a^b g$.*

Proof. It is an immediate consequence of the previous proposition and of the linearity of the integral. $\square$

Let us recall the classical definition of the measurability in the strong sense.

DEFINITION 3.1. A function $f: [a, b] \rightarrow X$ is measurable if it is the limit of a sequence of simple measurable functions a.e. in $[a, b]$.

PROPOSITION 3.3. *If f is a $(PU)^*$ -integrable function, then f is measurable.*

Proof. We will use the “Petti’s measurability theorem” (see [4, p. 42]).

For $x^* \in X^*$, if f is $(PU)^*$ -integrable, the real function $(x^* \circ f)$ is (PU) -integrable; in fact, for $\epsilon > 0$ there is a gauge $\delta > 0$ such that

$$\sum_i \left\| (f(c_i) - f(c'_i)) \int_a^b \theta_i d\mu \right\| \leq \epsilon$$

for each couple of δ -fine partitions $P = \{(\theta_i, c_i)\}_i$, $P' = \{(\theta_i, c'_i)\}_i$.

Consider the relation:

$$\begin{aligned} & \left| \sum_i \left[x^*(f(c_i)) - x^*(f(c'_i)) \right] \int_a^b \theta_i d\mu \right| \\ &= \left| x^* \left[\sum_i (f(c_i) - f(c'_i)) \int_a^b \theta_i d\mu \right] \right| \\ &\leq \|x^*\|_{X^*} \left\| \sum_i (f(c_i) - f(c'_i)) \int_a^b \theta_i d\mu \right\| \\ &\leq \|x^*\|_{X^*} \sum_i \left\| (f(c_i) - f(c'_i)) \int_a^b \theta_i d\mu \right\| \\ &\leq \|x^*\|_{X^*} \epsilon. \end{aligned}$$

So, by [10, Proposition 1.5], $(x^* \circ f)$ is (PU) -integrable and, hence, μ -measurable.

To prove that f is μ -essentially separably valued, observe that the continuity of the function F on $[a, b]$ implies the compactness of $F([a, b])$. Thus $F([a, b])$ is separable.

If $\mathcal{V}(F([a, b]))$ is a closed linear space spanned by $F([a, b])$, then $\mathcal{V}(F([a, b]))$ is separable and contains the set $\mathcal{H} = \{f(t) : F'(t) = f(t), t \in [a, b]\}$.

So, $\mathcal{H}$ is separable and f is measurable. $\square$

4. Convergence theorems

DEFINITION 4.1. A sequence f_n of $(PU)^*$ -integrable functions is uniformly $(PU)^*$ -integrable on $[a, b]$ if for each $\epsilon > 0$ there is δ_ϵ such that

$$\sum_i \left\| (f_n(c_i) - f_n(c'_i)) \int_a^b \theta_i d\mu \right\| \leq \epsilon$$

for each n and for each couple of δ -fine (PU) -partitions $P = \{(\theta_i, c_i)\}_i$ and $P' = \{(\theta_i, c'_i)\}_i$.

PROPOSITION 4.1. *Let f_n be a sequence of uniformly $(PU)^*$ -integrable functions defined on $[a, b]$, pointwise convergent to a function f . Then*

- 1) f is $(PU)^*$ -integrable on $[a, b]$,
- 2) $(PU)^* \int_a^b f = \lim_n (PU)^* \int_a^b f_n$,
- 3) $\lim_n \int_a^b \|f_n - f\| d\mu = 0$.

P r o o f.

1) Fix $\epsilon > 0$. Let δ_ϵ be a gauge ϵ -adapted to every f_n . So, for every couple $P = \{(\theta_i, c_i)\}_i$, $P' = \{(\theta_i, c'_i)\}_i$ of δ -fine (PU) -partitions and for every $n \in N$, we have

$$\sum_i \left\| (f_n(c_i) - f_n(c'_i)) \int_a^b \theta_i d\mu \right\| \leq \epsilon$$

and the limit for $n \rightarrow +\infty$ gives the condition of integrability for f .

2) If $P = \{(\theta_i, c_i)\}_i$ is a δ -fine partition, then $\sum_i f_n(c_i) \int_a^b \theta_i d\mu$ converges to $\sum_i f(c_i) \int_a^b \theta_i d\mu$. So, for $\epsilon > 0$ fixed, the following relations are verified for all $n > n_\epsilon$, with n_ϵ suitable:

$$\begin{aligned} & \left\| \sum_i f_n(c_i) \int_a^b \theta_i d\mu - (PU)^* \int_a^b f_n \right\| \leq \epsilon, \\ & \left\| \sum_i f(c_i) \int_a^b \theta_i d\mu - (PU)^* \int_a^b f \right\| \leq \epsilon, \\ & \left\| \sum_i f_n(c_i) \int_a^b \theta_i d\mu - \sum_i f(c_i) \int_a^b \theta_i d\mu \right\| \leq \epsilon. \end{aligned}$$

By these three relations, it follows

$$\left\| (PU)^* \int_a^b f_n - (PU)^* \int_a^b f \right\| \leq 3\epsilon.$$

3) The sequence $(f_n - f)_n$ is uniformly $(PU)^*$ -integrable, hence, the sequence $\|f_n - f\|$ is uniformly (PU) -integrable and converges to 0 (see [11, Proposition 1]). $\square$

Dominated convergence theorem. Let $f_n: [a, b] \rightarrow X$ be a sequence of $(PU)^*$ -integrable functions on $[a, b]$ converging to f , and let $g \geq 0$ be a μ -integrable real function defined on $[a, b]$ such that $\|f_n\| \leq g$ a.e. in $[a, b]$ for every n . Then f is $(PU)^*$ -integrable on $[a, b]$ and we have:

$$\lim_n \int_a^b \|f_n - f\| d\mu = 0, \quad \text{and} \quad \lim_n (PU)^* \int_a^b f_n = (PU)^* \int_a^b f.$$

Proof. The proof is similar to the proof for real valued functions. $\square$

5. Equivalence with the Bochner integral

By [4], a measurable function $f: [a, b] \rightarrow X$ is Bochner integrable if there exists such a sequence of simple measurable functions $f_n: [a, b] \rightarrow X$ that

$$\lim_n \int_a^b \|f_n - f\| d\mu = 0,$$

and the Bochner integral of f is

$$B \int_a^b f = \lim_n \int_a^b f_n.$$

We prove the following result.

PROPOSITION 5.1. *A function $f: [a, b] \rightarrow X$ is $(PU)^*$ -integrable if and only if it is Bochner-integrable. Moreover, the integrals coincide.*

Proof. Suppose that f is Bochner-integrable.

By the measurability of f , there exists a sequence $(s_n)_n$ of simple measurable functions such that $\lim_n \|s_n - f\| = 0$ a.e. in $[a, b]$ and it implies that there exists n_ϵ such that for all $n > n_\epsilon$ we have $\|s_n\| \leq \|f\| + 1$ a.e. in $[a, b]$, and according to the dominated convergence theorem, f is $(PU)^*$ -integrable. $\square$

The converse is a consequence of the Bochner's Characterization Theorem and the Propositions 2.1 and 3.3.

Finally, since f is the limit of simple functions and the integral of a simple function is the same for any type of integral, by the dominated convergent theorem, we have the equality of the integrals.

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