THE STRUCTURE OF THE FRÉCHET SPACE s REGARDING THE SERIES $\sum f_n(x_n)$

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This paper is dedicated to the legacy of the late professor Tibor Šalát, to whom the second author is indebted greatly.

ABSTRACT. We investigate the subsets of the Fréchet space s of all sequences of real numbers equipped with the Fréchet metric ρ from the Baire category point of view. In particular, we concentrate on the “convergence” sets of the series $\sum f_n(x_n)$ that is, sets of sequences $x = (x_n)$ for which the series converges, or has a sum (perhaps infinite), or oscillates. Provided all f_n are continuous real functions, sufficient conditions are given for the “convergence” sets to be of the first Baire category or residual in s .

1. Introduction

Nowadays, there is an amount of papers exploring the properties of the Fréchet space s of sequences of real numbers equipped with the metric ρ ,

$$\rho(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{|x_n - y_n|}{1 + |x_n - y_n|}, \quad x = (x_n), \quad y = (y_n) \in s,$$

especially regarding the convergence of series composed of the elements of the space s .

For example, in [LŠ] it was shown that the set of all elements $x = (x_n)$ in s for which the series $\sum a_n x_n$ converges (for a fixed $a = (a_n)$ with $a_n \neq 0$ for infinitely many n) is of first Baire category in s . Perhaps even stronger statements can be found in [Ne] with generalizations in [EŠ1] and [EŠ2]. For more results see for example [GL], [La], [LG], etc.

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Our approach, however, will be a more general one. Given a sequence $(f_n)_{n=1}^\infty$ of real functions, $f_n: \mathbb{R} \rightarrow \mathbb{R}$, we make the following notation:

$$\begin{aligned} A &= A(f_1, f_2, \dots) := \left\{ x \in s; \exists \sum_{n=1}^\infty f_n(x_n) \text{ finite or infinite} \right\}, \\ A_0 &= A_0(f_1, f_2, \dots) := \left\{ x \in s; \exists \sum_{n=1}^\infty f_n(x_n) \text{ finite} \right\}, \\ O &= O(f_1, f_2, \dots) := \left\{ x \in s; \sum_{n=1}^\infty f_n(x_n) \text{ oscillates} \right\}, \\ O^* &= O^*(f_1, f_2, \dots) := \left\{ x \in s; \sum_{n=1}^\infty f_n(x_n) \text{ oscillates from } -\infty \text{ to } +\infty \right\}. \end{aligned}$$

Obviously,

$$A_0 \subseteq A \quad \text{and} \quad O^* \subseteq O,$$

and we have a decomposition of the Fréchet space s :

$$s = A \cup O.$$

It is easy to see that in general case one of the sets A , O in the foregoing union may be empty: for example, if all functions f_n are non-negative (or non-positive), then $O = \emptyset$ and $s = A$. It is also easy to see that with a suitable choice of f_n we can get $s = A^+$ or $s = A^-$ (see the definition below). Moreover, if, for example $|f_n(t)| \leq n^{-2}$ for each $t \in \mathbb{R}$, $n \in \mathbb{N}$, then $s = A_0$.

On the other hand, if $f_n(t) = (-1)^n$, for each $t \in \mathbb{R}$, $n \in \mathbb{N}$, then $A = \emptyset$ (and $s = O \setminus O^*$). Similarly, if $f_n(t) = n \cdot (-1)^n$, then $O^* = s$.

The reference results mentioned earlier can be viewed as special cases of our situation if we put $f_n(t) = a_n \cdot t$, $t \in \mathbb{R}$, $n \in \mathbb{N}$, or if we put $f_n \equiv f$, $n \in \mathbb{N}$.

2. Results

Consider two more sets:

$$\begin{aligned} A^+ &= A^+(f_1, f_2, \dots) := \left\{ x \in s; \sum_{n=1}^\infty f_n(x_n) = +\infty \right\}, \\ A^- &= A^-(f_1, f_2, \dots) := \left\{ x \in s; \sum_{n=1}^\infty f_n(x_n) = -\infty \right\}. \end{aligned}$$

Then, of course, $A = A_0 \cup A^+ \cup A^-$.

PROPOSITION 2.1. *If any of the sets A_0 , A^+ , A^- , O , O^* is nonempty, then the set is dense in the space s .*

Proof. The result follows from the properties of the Fréchet metric ρ and from the fact that the convergence character of a series remains the same if only a finite number of its terms has been changed. $\square$

Evidently, more specific properties of these sets can be shown if the functions f_n are continuous.

THEOREM 2.1. *Let $f_n: \mathbb{R} \rightarrow \mathbb{R}$, $n \in \mathbb{N}$, be continuous real functions.*

- (i) *If $A^+(A^-)$ is nonempty, then the set $A_0 \cup A^-$ ($A_0 \cup A^+$) is of the first Baire category in s . In particular, A_0 is of first Baire category.*
- (ii) *If both A^+ and A^- are nonempty, then the set A is of the first Baire category in s .*

Proof. For each $k \in \mathbb{N}$ and $x \in s$ put $s_k(x) := \sum_{n=1}^k f_n(x_n)$ and define the function $F_k: s \rightarrow \mathbb{R}$ by

$$F_k(x) = \frac{s_k(x)}{1 + |s_k(x)|}.$$

Then each F_k is continuous (this follows from the continuity of the functions f_n and from the fact that the convergence in s is equivalent to the convergence along the coordinates). Thus, the restriction $F_k \upharpoonright_A$ is continuous on A and for each $x \in A$ there is the limit

$$F(x) = \lim_{k \rightarrow \infty} F_k(x)$$

such that $F: A \rightarrow [-1, 1]$ where

$$F^{-1}(1) = A^+, \quad F^{-1}(-1) = A^- \quad \text{and} \quad F^{-1}((-1, 1)) = A_0.$$

Moreover, the function F belongs to the first Baire class of functions on A .

Suppose now that $A^+ \neq \emptyset$ (the case for A^- is symmetric). Thus A^+ is dense in A . Since $A^+ = F^{-1}(1)$, it follows that any point $x \in A_0 \cup A^-$ is a point of discontinuity of F , or $x \in D(F)$, since F assumes the value 1 arbitrarily close to x . Therefore, $A_0 \cup A^- \subseteq D(F)$.

It is known (see [Si, p. 185]) that $D(f)$ is a first Baire category set whenever f is a first Baire class function. Hence, $A_0 \cup A^-$ is of the first Baire category in A and in s .

If both sets A^+ and A^- are nonempty (and hence dense), then both $F^{-1}(1)$ and $F^{-1}(-1)$ are dense in A ; that is, every point of A is a discontinuity point of F , so $A = D(F)$ is of first Baire category in A and in s . $\square$

Note 2.1. Consider for a second the situation where $f_n \equiv f$ for each $n \in \mathbb{N}$. Then $A(f_1, f_2, \dots) = A(f)$, etc. The assumption (i) of the previous theorem is satisfied provided f has at least one positive (or negative) value that is, if $\mathbb{R} \neq Z_f := \{t \in \mathbb{R}; f(t) = 0\}$. The assumption (ii) holds if f has a positive and a negative value. Hence, we obtain the following two corollaries:

COROLLARY 2.1. *Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function.*

- (i) *If $f \not\equiv 0$ (or $Z_f \neq \mathbb{R}$), then $A_0(f)$ is of the first Baire category in s .*

- (ii) If there are $t_1, t_2 \in \mathbb{R}$ with $f(t_1) < 0 < f(t_2)$, then $A(f)$ is of the first Baire category in s .

COROLLARY 2.2. *Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be continuous. Then either $A_0(f) = s$ or $A_0(f)$ is of first Baire category in s .*

PROOF. If $f \equiv 0$, then $Z_f = \mathbb{R}$ and $A_0(f) = s$. Otherwise, $Z_f \neq \mathbb{R}$ and the preceding corollary implies the result. $\square$

The latter statement resembles the well-known 0–1 *theorem* from the *classical measure and probability theory*. Now, let us turn our attention to the “oscillation” sets O and O^* instead of the “convergence” sets A and A_0 .

THEOREM 2.2. *Let $f_n: \mathbb{R} \rightarrow \mathbb{R}$, $n \in \mathbb{N}$, be continuous functions. Then the set $O^*(f_1, f_2, \dots)$ is of G_δ -type.*

PROOF. Recall that $s_k(x)$ denotes the k -th partial sum of our series $\sum f_n(x_n)$ and for $t_1, t_2 \in \mathbb{R}$, $t_1 < t_2$, consider the set $H(t_1, t_2)$ of the form:

$$H(t_1, t_2) = \left\{ x \in s; \exists I_1 \subseteq \mathbb{N} \text{ infinite } \forall k \in I_1 : s_k(x) < t_1, \right. \\ \left. \exists I_2 \subseteq \mathbb{N} \text{ infinite } \forall k \in I_2 : s_k(x) > t_2 \right\}.$$

It follows from the definition of the set $O^*(f_1, f_2, \dots)$ that

$$O^*(f_1, f_2, \dots) = \bigcap_{t=1}^{\infty} H(-t, t).$$

Now, for each $k \in \mathbb{N}$ and $t_1, t_2 \in \mathbb{R}$, $t_1 < t_2$, define the sets

$$H_k^-(t_1) = \{x \in s; s_k(x) < t_1\}, \\ H_k^+(t_2) = \{x \in s; s_k(x) > t_2\}.$$

These sets are open (see the proof of Theorem 2.1), so for each $t \in \mathbb{N}$, the set

$$H(-t, t) = \left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} H_k^-(-t) \right) \cap \left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} H_k^+(t) \right)$$

is of G_δ -type. It follows that also $O^*(f_1, f_2, \dots)$ is of G_δ -type. $\square$

THEOREM 2.3. *Let $f_n: \mathbb{R} \rightarrow \mathbb{R}$, $n \in \mathbb{N}$, be continuous functions. If the set $O^*(f_1, f_2, \dots)$ is nonempty, then it is residual in s . In that case, $O(f_1, f_2, \dots)$ is also residual in s .*

PROOF. If $O^*(f_1, f_2, \dots)$ is nonempty, then it is dense in s (see Proposition 2.1) and, by the preceding theorem, it is of G_δ -type. According to Kuratowski [Ku, p. 49] a dense G_δ set is residual (i.e., is a complement of a first category set). The second statement obviously follows from $O^* \subseteq O$. $\square$

COROLLARY 2.3. *Let $f_n: \mathbb{R} \rightarrow \mathbb{R}$, $n \in \mathbb{N}$, be continuous functions infinitely many of which are surjective (i.e., for infinitely many $n \in \mathbb{N}$ it is $f_n[\mathbb{R}] = \mathbb{R}$). Then the sets $O^*(f_1, f_2, \dots)$ and $O(f_1, f_2, \dots)$ are residual in s .*

Proof. Let $\{n_k\}_{k=1}^\infty$ be a sequence of natural numbers with $n_k < n_{k+1}$ and $f_{n_k}[\mathbb{R}] = \mathbb{R}$. Define the element $d = (d_i)$ of s in the following way: choose $d_1, \dots, d_{n_1-1} \in \mathbb{R}$ arbitrarily. By the surjectivity of f_{n_1} , we can find d_{n_1} in $\mathbb{R}$ such that

$$f_{n_1}(d_{n_1}) < -(f_1(d_1) + \dots + f_{n_1-1}(d_{n_1-1})) - 1.$$

Choose again $d_{n_1+1}, \dots, d_{n_2-1} \in \mathbb{R}$ arbitrarily and by the surjectivity of f_{n_2} , find d_{n_2} in $\mathbb{R}$ such that

$$f_{n_2}(d_{n_2}) > -(f_1(d_1) + \dots + f_{n_2-1}(d_{n_2-1})) + 1.$$

Alternating in this way, we construct the sequence $d = (d_i) \in s$ by means of induction such that the partial sums of $\sum f_n(d_n)$ for each $k \in \mathbb{N}$ satisfy

$$\begin{aligned} s_{n_{2k}}(d) &> k, \\ s_{n_{2k-1}}(d) &< -k. \end{aligned}$$

For each $k \in \mathbb{N}$ the element d belongs to the set $H(-k, k)$ defined in the proof of Theorem 2.2, that is,

$$d \in \bigcap \{H(-k, k); k \in \mathbb{N}\} = O^*$$

and hence O^* is nonempty. The rest follows from the preceding theorem. $\square$

Given a sequence $(d_n)_{n=1}^\infty$, we denote the set of all its accumulation points by $(d_n)'_n$. The following theorem improves a similar result from [Ne].

THEOREM 2.4. *Let $f_n: \mathbb{R} \rightarrow \mathbb{R}$, $n \in \mathbb{N}$, be continuous functions. Let M denote the set of those $x \in s$ such that each number in $\mathbb{R}^* = [-\infty, \infty]$, the extended reals, is an accumulation point of the sequence $(s_k(x))_{k=1}^\infty$, that is,*

$$M = \left\{ x \in s; (s_k(x))'_k = \mathbb{R}^* \right\}.$$

If the set M is nonempty, then it is residual in s .

Proof. Let $t \in \mathbb{R}$, $m, k \in \mathbb{N}$. Denote by $M(t, m, k)$ the set

$$M(t, m, k) = \left\{ x \in s; |s_k(x) - t| < \frac{1}{m} \right\}.$$

Because of the continuity of the k -th partial sum $s_k: s \rightarrow \mathbb{R}$, each set $M(t, m, k)$ is open in s . Therefore, the set

$$M(t) = \bigcap_{m=1}^\infty \bigcap_{n=1}^\infty \bigcup_{k=n}^\infty M(t, m, k)$$

is of G_δ type. It is also easy to see that if $\mathbb{Q}$ denotes the set of all rationals, then

$$M = \bigcap_{t \in \mathbb{Q}} M(t)$$

and hence, M is also a G_δ set. In view of our previous remarks, it suffices to show that M is dense in s , for a dense G_δ set is residual.

So, let $d = (d_i) \in M$ and let $z = (z_i) \in s$ and $\varepsilon > 0$ be arbitrary. It suffices to show that M contains a point from the open ρ -ball $B_\rho(z, \varepsilon)$ with the centre z and radius ε .

Choose $n \in \mathbb{N}$ such that $2^{-n} < \varepsilon$ and define the element $u = (u_i) \in s$ in the following way

$$u_i = \begin{cases} z_i, & i = 1, 2, \dots, n, \\ d_i, & i \geq n+1. \end{cases}$$

Then $\rho(u, z) < \sum_{i=n+1}^{\infty} 2^{-i} = 2^{-n} < \varepsilon$, so $u \in B_\rho(z, \varepsilon)$. Moreover, since u differs from $d \in M$ only in a finite number of terms, we have $u \in M$. Therefore, M is dense, G_δ and hence residual. $\square$

THEOREM 2.5. *Let $f_n: \mathbb{R} \rightarrow \mathbb{R}$, $n \in \mathbb{N}$, be continuous functions. If the set $s \setminus A_0(f_1, f_2, \dots)$ is nonempty, then it is residual in s , i.e., the set $A_0(f_1, f_2, \dots)$ is then of first Baire category in s .*

Proof. For $m, k, p \in \mathbb{N}$ put

$$C(m, k, p) = \left\{ x \in s; |f_{k+1}(x_{k+1}) + \dots + f_{k+p}(x_{k+p})| \leq \frac{1}{m} \right\}.$$

By the Cauchy-Bolzano criterion for the convergence of series we have

$$A_0(f_1, f_2, \dots) = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{\substack{p \in \mathbb{N} \\ k \geq n}} C(m, k, p).$$

For our convenience put

$$C(m) = \bigcup_{n=1}^{\infty} \bigcap_{\substack{p \in \mathbb{N} \\ k \geq n}} C(m, k, p),$$

so,

$$A_0(f_1, f_2, \dots) = \bigcap_{m \in \mathbb{N}} C(m).$$

Because of the continuity of all f_n functions, each set $C(m, k, p)$ is closed, i.e., each set $C(m)$ is of F_σ -type.

Next, $s \setminus A_0 = \bigcup_{m \in \mathbb{N}} (s \setminus C(m)) = \bigcup_{m \in \mathbb{N}} D(m)$ if we put $D(m) = s \setminus C(m)$. If $s \setminus A_0$ is nonempty, then there is $d = (d_j) \in s$ and $m_0 \in \mathbb{N}$ such that $d \in D(m_0)$. We want to show that $D(m_0)$ is dense in s .

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Let $z = (z_j) \in s$ and $\varepsilon > 0$ be arbitrary fixed. It suffices to show that $D(m_0) \cap B_\rho(z, \varepsilon) \neq \emptyset$. Since $d \in D(m_0)$, for each $n \in \mathbb{N}$ there is $k \in \mathbb{N}$, $k \geq n$, and $p \in \mathbb{N}$ such that

$$|f_{k+1}(d_{k+1}) + \cdots + f_{k+p}(d_{k+p})| > \frac{1}{m_0}. \quad (1)$$

Now, choose $l \in \mathbb{N}$ with $2^{-l} < \varepsilon$ and define the element $u = (u_j) \in s$ by

$$u_j = \begin{cases} z_j, & j = 1, 2, \dots, l, \\ d_j, & j \geq l+1. \end{cases}$$

Then, $\rho(u, z) < \sum_{j=l+1}^{\infty} 2^{-j} = 2^{-l} < \varepsilon$, that is, $u \in B_\rho(z, \varepsilon)$. Moreover, if $j \geq l+1$, then $u_j = d_j$ and it follows from (1) that if $n \geq l+1$, then there is $k \geq n$ and $p \in \mathbb{N}$ with

$$|f_{k+1}(u_{k+1}) + \cdots + f_{k+p}(u_{k+p})| > \frac{1}{m_0}$$

and consequently, $u \in D(m_0)$.

Therefore $D(m_0)$ is dense and G_δ , i.e., residual. Hence $s \setminus A_0 = \bigcup_{m \in \mathbb{N}} D(m)$ is residual and A_0 is of first Baire category in s . $\square$

The result of the preceding theorem can be reformulated to resemble Corollary 2.2 and the 0–1 *measure theorem* mentioned earlier. This corollary can also be deduced from Theorem 2.1.

COROLLARY 2.4. *Let $f_n: \mathbb{R} \rightarrow \mathbb{R}$, $n \in \mathbb{N}$, be continuous functions. Then either $A_0(f_1, f_2, \dots) = s$ or the set $A_0(f_1, f_2, \dots)$ is of first Baire category in s .*

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