UNIQUENESS INTERVALS AND TWO-POINT BOUNDARY VALUE PROBLEMS

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ABSTRACT. Consider a linear n th order differential equation with continuous coefficients and continuous forcing term. The maximal uniqueness interval for a classical 2-point boundary value problem will be calculated by an algorithm that uses an auxiliary linear system of differential equations, called a Mikusinski system. This system always has higher order than n . The algorithm leads to a graphical representation of the uniqueness profile and to a new method for solving 2-point boundary value problems. The ideas are applied to construct a graphic for the conjugate function associated with the n th order linear homogeneous differential equation. Details are given about how to solve classical 2-point boundary value problems, using auxiliary Mikusinski systems and Green's function.

1. Introduction

Let L be an n th order linear differential operator with continuous coefficients given by

$$Ly \equiv y^{(n)}(t) + \sum_{i=0}^{n-1} p_i(t)y^{(i)}(t).$$

Consider the two-point boundary value problem

$$\begin{aligned} Ly &= f(t), \\ y(a) &= y'(a) = \cdots = y^{(p-1)}(a) = 0, \\ y(b) &= y'(b) = \cdots = y^{(q-1)}(b) = 0, \end{aligned}$$

where $p + q = n$ and $f(t)$ is continuous.

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FIGURE 1. *Uniqueness profile.* Plot of $W(t)$ for the fourth order equation. The boundary conditions are $y(0.9) = 0$, $y'(0.9) = 0$, $y(b) = 0$, $y'(b) = 0$.

The solution $y(t)$ is unique if and only if the homogeneous problem has only the solution $y(t) \equiv 0$. Let $y_j(t, t_0)$ be the unique solution of $Ly = 0$ with initial conditions $y_j^{(i)}(t_0, t_0) = \delta_{ij}$ and define $W(t)$ to be the Wronskian determinant of the functions $y_j(t, a)$ for $j = q, \dots, n-1$. Then a nontrivial solution of $Ly = 0$ exists satisfying the boundary conditions at $t = a$ and $t = b$ if and only if $W(b) = 0$. The interval $a \leq t \leq b_1$ is called a *uniqueness interval* provided $W(b) \neq 0$ for $a < b < b_1$, but $W(b_1) = 0$.

A *uniqueness profile* is a plot of the function $W(t)$ on some subinterval of $a \leq t < \infty$. The first crossing of $W(t)$ with the t -axis defines the maximal interval of uniqueness of the 2-point boundary value problem. Shown in Figure 1 is a uniqueness profile for the boundary value problem

$$\begin{aligned} y^{iv} &= p(t)y'', \\ y(a) &= y'(a) = 0, \\ y(b) &= y'(b) = 0. \end{aligned}$$

The piecewise continuous function $p(t)$ equals -1 on $\frac{3\pi}{2} < t < \infty$ and zero elsewhere, except for a small linear connection interval.

2. Mikusinski system

The presentation will concentrate on the case of a fourth order differential equation

$$y^{iv} = p_0(x)y + p_1(x)y' + p_2(x)y'' + p_3(x)y'''$$

and the boundary conditions

$$y(a) = y'(a) = 0, \quad y(b) = y'(b) = 0,$$

with specializations $p_0 = p_1 = p_3 = 0$ and $p_2 = p(t)$.

Let $w(n_1, n_2)$ denote the Wronskian of $y_0^{(n_1)}(t), y_1^{(n_2)}(t)$. Define state variables by the equations

$$T_1 = w(0, 1),$$

$$T_2 = w(0, 2),$$

$$T_3 = w(0, 3),$$

$$T_4 = w(1, 2),$$

$$T_5 = w(1, 3),$$

$$T_6 = w(2, 3).$$

Then determinant differentiation rules and use of the fourth order differential equation imply the Mikusinski system

$$T_1' = T_2,$$

$$T_2' = T_3 + T_4,$$

$$T_3' = T_5 + p(x)T_2,$$

$$T_4' = T_5,$$

$$T_5' = T_6 + p(x)T_4,$$

$$T_6' = 0.$$

Because of the known initial conditions for $y_j^{(i)}$, the initial data for the state variables is

$$T_1(a) = 0, \quad T_2(a) = 0, \quad T_3(a) = 0, \quad T_4(a) = 0, \quad T_5(a) = 0, \quad T_6(a) = 1.$$

In summary, the uniqueness profile for the fourth order boundary value problem on $a \leq t < \infty$ is the plot of the unique solution of component T_1 of the sixth order Mikusinski system $T' = MT$ with initial condition $T(a)$ equal to the last column of the identity matrix.

Remarks. The Mikusinski system $T' = MT$ can be defined for any set of 2-point boundary conditions. The initial data $T(a)$ is always the last column of the identity matrix. The algorithm for computer algebra generation of M appears in Gustafson—Ridenhour [2].

Jan Mikusiński (1913–1987) is credited with the idea of using Wronskians to discover properties of solutions of linear differential equations, hence the naming convention for the system $T' = MT$. It is not claimed that this system is first identified here, because in truth we (Ridenhour—Gustafson) discovered it during computations done in 1988.

What is claimed for state variable T_1 is also true for all other state variables. In the case of constant coefficients p_j , each state variable is a solution of a linear constant-coefficient differential equation $\mathbf{M}_0 y = 0$, where the characteristic equation of $\mathbf{M}_0$ equals the characteristic equation $\det(M - \lambda I) = 0$ of the Mikusinski system $T' = MT$. To prove the claim, multiply the Cayley-Hamilton matrix identity by T and use $T^{(r)} = MT^{(r-1)}$, $1 \leq r \leq n$. Equation $\mathbf{M}_0 y = 0$ can be selected to have even lower order, by using the minimal polynomial matrix identity, instead of the characteristic polynomial matrix identity.

3. Conjugate functions and uniqueness

Following P. Hartman [3, 4] and T. Sherman [7], the first conjugate point of $Ly = 0$ to $t = a$ is the least number $b > a$ such that on $a \leq t \leq b$ there is a nontrivial solution with n zeros counting multiplicities. It is known that either $b = \infty$ or else $b = \eta_1(a)$ is a continuous function of the point a .

A more precise description of η_1 is possible, in terms of the nontrivial solutions of 2-point boundary value problems. Define $b(a, k)$ to be the first crossing greater than a of the uniqueness profile for boundary conditions $y(a) = \dots = y^{(k-1)}(a) = 0$, $y(b) = \dots = y^{(n-k-1)}(b) = 0$, for $k = 1, \dots, n-1$. If there is no first crossing, then define $b(a, k) = \infty$. Then (see Sherman [6])

$$\eta_1(a) = \min\{b(a, k) : 0 \leq k \leq n-1\}.$$

The minimization implies some continuity properties of η_1 , but also there is generally a corner in the graph of η_1 , a point of non-differentiability, each time the minimum switches from $b(a, i)$ to $b(a, j)$ with $i \neq j$. This typical behavior is illustrated in Figure 2, which is a plot of $\eta_1(a)$ from $a = 0$ to $a = 9$ for the sixth order equation

$$y^{(6)} = p(t)y^{(4)}.$$

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FIGURE 2. A typical conjugate function plot. The sixth order equation is used to illustrate the piecewise continuous character of η_1 . The piecewise sections are realized by $(4, 2)$, $(3, 3)$, $(2, 4)$ and $(5, 1)$ boundary value problems, respectively.

For illustration purposes, coefficient $p(t)$ is piecewise linear, with $p(t) = -1$ on $\frac{3\pi}{2} < t < 3\pi$ and zero elsewhere, except on two small linear connection intervals.

Remark. The Figure 2 shows that the conjugate function for a constant-coefficient differential equation is not the typical case. For such equations, $\eta_1(0) = b(0, k)$ for some k . Translation-invariance of constant equations then implies that $\eta_1(a) = b(a, k)$. Accordingly, there are no corners in the graph of η_1 . Furthermore, continuity of η_1 then implies continuity of $b(\cdot, k)$, which is not true for non-constant coefficients.

4. Green's function and uniqueness profiles

The Green's function $G(t, s)$ for the boundary value problem

$$Ly = f(t),$$

$$\begin{aligned} y(a) &= y'(a) = \cdots = y^{(p-1)}(a) = 0, \\ y(b) &= y'(b) = \cdots = y^{(q-1)}(b) = 0, \end{aligned}$$

where $p + q = n$, can be defined as follows. Let $u_0(t, s), \dots, u_{n-1}(t, s)$ be the fundamental set of solutions to $Ly = 0$ defined by $u_i^{(j)}(s, s) = \delta_{ij}$ (standard basis). Then

$$G(t, s) = \frac{\det B(t, s)}{\det W},$$

where the matrices B and W are given by:

$$W = \begin{bmatrix} u_q(b, a) & \cdots & u_{n-1}(b, a) \\ u'_q(b, a) & \cdots & u'_{n-1}(b, a) \\ \vdots & & \vdots \\ u_q^{(p-1)}(b, a) & \cdots & u_{n-1}^{(p-1)}(b, a) \end{bmatrix},$$

$$B(t, s) = \begin{bmatrix} K(t, s) & u_q(t, a) & \cdots & u_{n-1}(t, a) \\ u'_{n-1}(b, s) & u'_q(b, a) & \cdots & u'_{n-1}(b, a) \\ \vdots & \vdots & & \vdots \\ u_{n-1}^{(p-1)}(b, s) & u_q^{(p-1)}(b, a) & \cdots & u_{n-1}^{(p-1)}(b, a) \end{bmatrix},$$

where $K(t, s) = 0$ for $t < s$, and $K(t, s) = u_{n-1}(t, s)$ for $t \geq s$. Verified in Gustafson–Ridenhour [2] is the formula

$$G(t, s) = K(t, s) + H(t, s),$$

where $H(t, s)$ is defined by

$$H(t, s) = \sum_{i=0}^{p-1} v_i(s) u_{q+i}(t, a)$$

and

$$v_i(s) = \sum_{j=0}^{q-1} (-1)^{1+i} z_{ij} c_j(s) - c_{q+i}(s).$$

The symbols z_{ij} are constants, initially defined as quotients of determinants, but ultimately computed as quotients of components of the Mikusinski system evaluated at $t = b$. The functions c_j are a standard solution basis for the formal adjoint system.

Function v_i can be computed from its initial values by a numeric initial value problem solver, such as RK4 or LSODE. The initial values of v_i are expressed entirely in terms of the constants z_{ij} , which are values of components of the Mikusinski system.

The end result is that $G(t, s)$ can be computed using initial value solvers applied to the system and its adjoint system, provided some one-time initial value solver computations z_{ij} are done.

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The solution of the boundary value problem for $Ly = f$ can be expressed as

$$y(t) = \int_a^b G(t, s) f(s) \, ds.$$

Because $G(t, s)$ can be computed for any pair (t, s) , then quadrature rules can be applied to reduce the computational effort to a small number of points in $a \leq t \leq b$, for example, Gaussian quadrature 40-point rules.

REFERENCES

- [1] GUSTAFSON, G. B.—RIDENHOUR, J. R.: *Lower order branching and conjugate function discontinuity*, J. Differential Equations **27** (1978), 167–179.
- [2] GUSTAFSON, G. B.—RIDENHOUR, J. R.: *Uniqueness intervals for multipoint boundary value problems* (preprint 2008).
- [3] HARTMAN, P.: *Unrestricted n -parameter families*, Rend. Circ. Mat. Palermo (2) **7** (1958), 123–142.
- [4] HARTMAN, P.: *Ordinary Differential Equations*, John Wiley and Sons, Inc., New York, 1964.
- [5] MIKUSIŃSKI, J.: *Sur l'équation $x^{(n)} + A(t)x = 0$* , Ann. Polon. Math. **1** (1955), 207–221.
- [6] SHERMAN, T. L.: *Properties of solutions of N th order linear differential equations*, Pacific J. Math. **15** (1965), 1045–1060.
- [7] SHERMAN, T. L.: *Conjugate points and simple zeros for ordinary linear differential equations*, Trans. Amer. Math. Soc. **146** (1969), 397–411.

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