ON A NON-HOMOGENEOUS DIFFERENCE EQUATION FROM PROBABILITY THEORY

JEAN-LUC GUILBAULT — MARIO LEFEBVRE

ABSTRACT. The so-called gambler's ruin problem in probability theory is considered for a Markov chain having transition probabilities depending on the current state. This problem leads to a non-homogeneous difference equation with non-constant coefficients for the expected duration of the game. This mathematical expectation is computed explicitly.

1. Introduction

We consider a Markov chain $\{X_n, n = 0, 1, \dots\}$ having state space $S := \{0, \dots, N\}$ and for which the (time-homogeneous) transition probabilities

$$p_{i,j} := P[X_{n+1} = j \mid X_n = i]$$

are given by

$$p_{i,i+1} = \frac{1}{4}(1 - ci), \quad p_{i,i-1} = \frac{1}{4}(1 + ci) \quad \text{and} \quad p_{i,i} = \frac{1}{2} \quad (1)$$

for $i = 1, \dots, N - 1$. The states 0 and N are assumed to be absorbing, so that $p_{0,j} = \delta_{0,j}$ and $p_{N,j} = \delta_{N,j}$. Moreover, because $p_{i,j} \in [0, 1]$ for all $i, j \in S$ and the state N must be accessible from $N - 1$, the positive constant c must satisfy the condition

$$c < \frac{1}{N - 1}. \quad (2)$$

Using the results in Cox and Miller (1965, p. 213), we can state that the Markov chain defined above may be regarded as a discrete version of the Ornstein-Uhlenbeck process. Various discrete versions of the Ornstein-Uhlenbeck process have been considered by, in particular, Renshaw (1997), Anishchenko *et al.* (2002, p. 53), Sprott (2003, p. 234), Larralde (2004 a,b), Kontoyiannis and Meyn (2005), and Milstein *et al.* (2007).

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Assume that $X_0 = i \in \{1, \dots, N-1\}$, and let $T := \inf\{n > 0 : X_n = 0 \text{ or } N\}$. We want to determine the value of

$$D_i := E[T \mid X_0 = i]. \quad (3)$$

The quantity D_i is thus the expected value of the time (that is, the number of transitions) needed for the Markov chain to hit either 0 or N , starting from i . An application of this type of problem is the following: we can imagine that a *gambler*, having initially i units of money, bets 1 unit at each play of a game for which he has a probability $p_{i,i+1}$ (respectively $p_{i,i-1}$, $p_{i,i}$) of winning (resp. losing, drawing). If $X_T = 0$, we say that the gambler is *ruined*, hence the classical problem called the *gambler's ruin problem*. Here, D_i is the *expected duration of the game*.

Conditioning on the outcome of the first play, we can write that D_i satisfies the non-homogeneous difference equation with non-constant coefficients

$$D_i = \frac{1-ci}{4}D_{i+1} + \frac{1+ci}{4}D_{i-1} + \frac{1}{2}D_i + 1, \quad (4)$$

that is,

$$\frac{[1-c(i+1)]}{2}D_{i+2} - D_{i+1} + \frac{[1-c(i+1)]}{2}D_i = -2. \quad (5)$$

The boundary conditions are

$$D_0 = D_N = 0. \quad (6)$$

When $\{X_n, n = 0, 1, \dots\}$ is a *random walk*, that is, when $p_{i,i+1} = p = 1-p_{i,i-1}$ for $i = 1, \dots, N-1$, where $p \in (0, 1)$, equation (4) becomes

$$D_i = pD_{i+1} + (1-p)D_{i-1} + 1.$$

It is not difficult to find the solution to this difference equation that satisfies $D_0 = D_N = 0$ (see for instance Feller, 1968, p. 348). However, it turns out that solving (5), (6) is much more difficult. In Section 2, we will compute D_i explicitly, and we will end this work with a few remarks in Section 3.

2. Expected duration of the game

In Lefebvre and Guilbault (2008), the authors calculated the first hitting place probability

$$p_i := P[X_T = N \mid X_0 = i]. \quad (7)$$

This probability is the solution of

$$p_i = \frac{(1-ci)}{2}p_{i+1} + \frac{(1+ci)}{2}p_{i-1}, \quad (8)$$

for $i = 1, \dots, N - 1$, that satisfies the boundary conditions $p_0 = 0$ and $p_N = 1$. To obtain an explicit expression for p_i , the authors first set $x = i - 1$ and let $y(x) = p_{i-1}$, so that equation (8) could be rewritten as

$$\left[\frac{1 - c(x+1)}{2} \right] y(x+2) - y(x+1) + \left[\frac{1 + c(x+1)}{2} \right] y(x) = 0 \quad (9)$$

for $x = 0, \dots, N - 2$. The boundary conditions were $y(0) = 0$ and $y(N) = 1$. Notice that (9) is a *hypergeometric difference equation* (see Batchelder, 1967, p. 68).

Next, they transformed (9) into its *normal form*:

$$(x - a)y(x+2) + ay(x+1) - xy(x) = 0, \quad (10)$$

where $a := \frac{2}{c}$ and $x \in \{1 + \frac{1}{c}, \dots, N - 1 + \frac{1}{c}\}$. They found (see Batchelder, 1967, Chapter III) that a fundamental system of solutions of equation (10) is $y_1(x) \equiv 1 \in \mathbb{R}$ and

$$y_2(x) = (-1)^x \frac{\Gamma(x)}{\Gamma(x-a)} F\left(-a, 1, x-a, \frac{1}{2}\right), \quad (11)$$

in which $F(\cdot, \cdot, \cdot, \cdot)$ is a hypergeometric function (see Abramowitz and Stegun, 1965, p. 556).

Finally, they showed that the general solution of (10) can be expressed as

$$y(x) = c_1 + c_2(-1)^{[x]} \frac{\Gamma(x)}{\Gamma(x-a)} F\left(-a, 1, x-a, \frac{1}{2}\right), \quad (12)$$

where $[\]$ denotes the integer part, and c_1 and c_2 are arbitrary constants.

Remarks.

- i) The integer part of x was used because the function $y(x)$ must be real-valued in the application considered. Moreover, the constants c_1 and c_2 are also assumed to be real.
- ii) When the constant c is of the form $c = 1/(N+j)$ or $c = 2/[2(N+j)-1]$, where $j \in \{0, 1, \dots\}$, equation (10) is *completely reducible* (see Batchelder, 1967, p. 123–124). The case when $c = 2/[2(N+j)-1]$ does not cause any problem, but when $c = 1/(N+j)$ the function $y_2(x)$ can be expressed as

$$y_2(x) = (-1)^{a+1} \frac{a!}{2^a} \quad \text{for } x = 1, 2, \dots, a+1.$$

Therefore, $y_1(x) \equiv 1$ and $y_2(x)$ are not linearly independent. However, $y_2(x)$ being a continuous function of the parameter c , the authors could take the limit as c tends to $1/(N+j)$ to obtain the solution for this value of c .

Now, with $x = i - 1$ as above, and $z(x) := D_{i-1}$, the normal form of equation (5) is

$$(x - a)z(x + 2) + az(x + 1) - xz(x) = 2a. \quad (13)$$

We already know the general solution of the corresponding homogeneous equation. Hence, we must find a particular solution of (13) to obtain its general solution. To do so, we will make use of a method that is similar to the method of variation of parameters for ordinary differential equations. We assume that the constant c is different from $1/(N + n)$. As mentioned above, the solution for this value of c is obtained as a limit.

If $z_1(x)$ and $z_2(x)$ form a fundamental system of solutions for the associated homogeneous equation, and if $D(x)$ is the Casorati determinant, then a particular solution of (13) is (see Batchelder, 1967, p. 13)

$$z_p(x) = c_1(x)z_1(x) + c_2(x)z_2(x),$$

where

$$\Delta c_n(x) = \frac{M_{2n}(x)}{D(x+1)} \frac{2a}{(x-a)} \quad \text{for } n = 1, 2$$

and $M_{2n}(x)$ is the cofactor of the second element in the n th column of

$$D(x+1) = \begin{vmatrix} 1 & z_2(x+1) \\ 1 & z_2(x+2) \end{vmatrix}.$$

Hence,

$$\Delta c_1(x) = -\frac{z_2(x+1)}{D(x+1)} \frac{2a}{(x-a)} \quad (14)$$

and

$$\Delta c_2(x) = \frac{1}{D(x+1)} \frac{2a}{(x-a)}. \quad (15)$$

Remark. In the sequel, we assume that $\frac{\Gamma(z)}{\Gamma(w)} = \lim_{\epsilon \rightarrow 0} \frac{\Gamma(\epsilon+z)}{\Gamma(\epsilon+w)}$.

LEMMA 2.1. *When $c - a \neq 0$ and $z \neq 0$,*

$$F(a, 1, c+1, z) = \frac{\Gamma(c+1)}{\Gamma(c)} \frac{\Gamma(c-a)}{\Gamma(c-a+1)} \frac{1}{z} \{1 + (z-1)F(a, 1, c, z)\}.$$

Proof. The result follows directly from the following relation (see Abramowitz and Stegun, 1965, p. 558):

$$c(1-z)F(a, b, c, z) - cF(a, b-1, c, z) + (c-a)zF(a, b, c+1, z) = 0,$$

and from the fact that $F(a, 0, c, z) \equiv 1$. □

Next, for equation (13) the Casorati determinant is

$$\begin{aligned} D(x) = & (-1)^{[x]+1} \frac{\Gamma(x+1)}{\Gamma(x+1-a)} F\left(-a, 1, x+1-a, \frac{1}{2}\right) \\ & - (-1)^{[x]} \frac{\Gamma(x)}{\Gamma(x-a)} F\left(-a, 1, x-a, \frac{1}{2}\right) \end{aligned} \quad (16)$$

and, making use of Lemma 2.1, we may write that

$$D(x) = 2(-1)^{[x]+1} \frac{\Gamma(x)}{\Gamma(x-a)}.$$

Hence, the determinant is never equal to zero if $c \neq 1/(N+n)$, with $n \in \mathbb{N}$. Furthermore, from equation (15),

$$\Delta c_2(x) = (-1)^{[x]} a \frac{\Gamma(x-a)}{\Gamma(x+1)}.$$

Remark. When $\frac{1}{c} = N+n-\frac{1}{2}$, x is not an integer and the Casorati determinant is never equal to zero for all admissible values of x , as should be. In this case,

$$\Delta c_2(x) = (-1)^{[x]} a \prod_{j=0}^a (x-j)^{-1}.$$

However, if $\frac{1}{c} = N+n$, $D(x)$ is a polynomial of degree $a = 2(N+n)$, which is zero over the set $S_1 = \{1, 2, \dots, a\}$, but non-zero over $\mathbb{N} \setminus S_1$, that is, to the right of the singularity at $x = a$.

Now, an inverse difference $\Delta^{-1}f(x)$ of the function $f(x)$ is known every time either of the following formal series: $-\sum_{n=0}^{\infty} f(x+n)$, $\sum_{n=1}^{\infty} f(x-n)$ converges (see Batchelder, 1967, p. 6). However, these series usually diverge. Nevertheless, they enable us to demonstrate the following proposition.

PROPOSITION 2.1. *Let*

$$f(x) = \eta^x \frac{\Gamma(x+\alpha)}{\Gamma(x+\beta)}.$$

Then, everywhere the limit $\lim_{\epsilon \rightarrow 0} \frac{\Gamma(x+\epsilon+\alpha)}{\Gamma(x+\epsilon+\beta)}$ exists, we have

$$\Delta^{-1}f(x) = -\eta^x \frac{\Gamma(x+\alpha)}{\Gamma(x+\beta)} F(x+\alpha, 1, x+\beta, \eta)$$

or, equivalently,

$$\Delta^{-1}f(x) = -\frac{\eta^x}{(1-\eta)} \frac{\Gamma(x+\alpha)}{\Gamma(x+\beta)} F\left[\beta-\alpha, 1, x+\beta, \frac{\eta}{(\eta-1)}\right], \quad (17)$$

provided that $\eta < 1$ or that $\beta-\alpha > 1$ if $\eta = 1$. In the case of the second equation, η must be strictly smaller than 1.

P r o o f. Making use of the formal series, we may write that

$$\Delta^{-1}f(x) = -\eta^x \frac{\Gamma(x+\alpha)}{\Gamma(x+\beta)} F(x+\alpha, 1, x+\beta, \eta).$$

We can show (see E r d é l y i *et al.*, 1953, p. 68) that this function is entire when $|\eta| < 1$. The series remains convergent when $\eta = 1$ if $\beta - \alpha > 1$. Moreover, by analytic continuation, this formula is valid for any $\eta < 1$ and can then be written as in (17) (see A b r a m o w i t z and S t e g u n, 1965, p. 558), which is an entire function for any $\eta < 1$. $\square$

Remark. When $\eta < 0$, it is better to take the integer part of x and write $\eta^{[x]}$ instead of η^x , in order to obtain only real values.

Next, from the previous proposition, if $c \neq 1/(N+n)$, with $n \in \mathbb{N}$, we have

$$c_2(x) = \frac{a}{2} (-1)^{[x]+1} \frac{\Gamma(x-a)}{\Gamma(x+1)} F\left(a+1, 1, x+1, \frac{1}{2}\right).$$

Finally [see equation (11)],

$$c_2(x)z_2(x) = -\frac{a}{2x} F\left(a+1, 1, x+1, \frac{1}{2}\right) F\left(-a, 1, x-a, \frac{1}{2}\right). \quad (18)$$

Remarks.

- i) In the preceding equation, the second hypergeometric function is absolutely convergent when the state space of the Markov chain is $\{0, 1, \dots, N\}$, because $x-1 > N-1 > 0$.
- ii) In the particular case when $a = 2(N+n)-1$, since $x-a$ is not a relative integer, we may write (see A b r a m o w i t z and S t e g u n, 1965, p. 561)

$$c_2(x)z_2(x) = -\frac{a}{2x} F\left(a+1, 1, x+1, \frac{1}{2}\right) \sum_{n=0}^a \frac{(-a)_n}{2^n (x-a)_n}.$$

The term $\Delta c_1(x)$ can be written as follows [see equation (14)]:

$$\Delta c_1(x) = aF\left(-a, 1, x+1-a, \frac{1}{2}\right) \frac{\Gamma(x-a)}{\Gamma(x-a+1)}. \quad (19)$$

In order to obtain an explicit expression for $c_1(x)$, we will use the *Fundamental Theorem* on summations.

PROPOSITION 2.2. *If $\Delta F(x) = f(x)$, and if $a, b (\geq a)$ are not integers, then*

$$\sum_{x=a}^b f(x) = \Delta^{-1}f(x)|_a^{b+1} = F(b+1) - F(a).$$

Now, since

$$\Delta \sum_{t=a}^{x-1} f(t) = \Delta [F(x) - F(a)] = f(x),$$

we have

$$\Delta^{-1} f(x) = \sum_{t=a}^{x-1} f(t)$$

(with the convention that $\sum_{t=a}^{a-1} f(t) = 0$). Using this principle, equation (14) and the linearity of the inverse difference operator Δ^{-1} , we deduce that

$$\begin{aligned} c_1(x) &= a\Delta^{-1} \left\{ F \left(-a, 1, x+1-a, \frac{1}{2} \right) \frac{\Gamma(x-a)}{\Gamma(x-a+1)} \right\} \\ &= a \sum_{t=1+\frac{1}{c}}^{x-1} F \left(-a, 1, t+1-a, \frac{1}{2} \right) \frac{\Gamma(t-a)}{\Gamma(t-a+1)}, \end{aligned}$$

where the summation $\sum_{t=1+\frac{1}{c}}^{x-1} f(t)$ is over $x-1-\frac{1}{c}$ elements. Furthermore, because $z_1(x) \equiv 1$, we have $c_1(x)z_1(x) = c_1(x)$, which is indeed a difference.

From the preceding results, we may state that a particular solution of equation (13) is given by

$$\begin{aligned} z_p(x) &= -\frac{a}{2x} F \left(a+1, 1, x+1, \frac{1}{2} \right) F \left(-a, 1, x-a, \frac{1}{2} \right) \\ &\quad + a \sum_{t=1+\frac{1}{c}}^{x-1} F \left(-a, 1, t+1-a, \frac{1}{2} \right) \frac{\Gamma(t-a)}{\Gamma(t-a+1)}. \end{aligned} \quad (20)$$

This solution, evaluated at $x = 1 + \frac{a}{2}$, becomes (see Abramowitz and Stegun, 1965, p. 557)

$$z_p \left[1 + \left(\frac{a}{2} \right) \right] = \frac{a\pi}{2} \left\{ \frac{\Gamma \left(1 + \frac{a}{2} \right)}{\Gamma \left(\frac{1}{2} + \frac{a}{2} \right)} - \frac{1}{\sqrt{\pi}} \right\} \frac{\Gamma \left(\frac{-a}{2} \right)}{\Gamma \left(\frac{1}{2} - \frac{a}{2} \right)}.$$

It follows that $z_p \left[1 + \left(\frac{a}{2} \right) \right] = 0$ when $c = 2/[2(N+n)-1]$, with $n \in \mathbb{N}$.

The general solution of equation (13), subject to the boundary conditions

$$z \left[1 + \left(\frac{a}{2} \right) \right] = z \left[N+1 + \left(\frac{a}{2} \right) \right] = 0,$$

is now given in the following proposition.

PROPOSITION 2.3. *The average number of transitions D_i defined in (3) is*

$$D_i = z \left[i+1 + \left(\frac{1}{c} \right) \right],$$

where

$$z(x) = c_1 + c_2(-1)^{[x]} \frac{\Gamma(x)}{\Gamma(x-a)} F\left(-a, 1, x-a, \frac{1}{2}\right) + z_p(x),$$

$z_p(x)$ is given in (20),

$$c_2 = \frac{z_p\left[1 + \left(\frac{a}{2}\right)\right] - z_p\left[N + 1 + \left(\frac{a}{2}\right)\right]}{z_2\left[N + 1 + \left(\frac{a}{2}\right)\right] - z_2\left[1 + \left(\frac{a}{2}\right)\right]}$$

and

$$c_1 = -c_2 z_2\left[1 + \left(\frac{a}{2}\right)\right] - z_p\left[1 + \left(\frac{a}{2}\right)\right],$$

in which the function $z_2(x)$ is the same as the function $y_2(x)$ in (11), but with $(-1)^{[x]}$ instead of $(-1)^x$.

In the particular case when

$$c = \frac{2}{[2(N+n)-1]}, \quad \text{with } n \in \mathbb{N},$$

we have

$$c_1 = 0 \quad \text{and} \quad c_2 = \frac{-z_p\left[N + 1 + \left(\frac{a}{2}\right)\right]}{z_2\left[N + 1 + \left(\frac{a}{2}\right)\right]}.$$

3. Concluding remarks

We have obtained an explicit expression for the average number D_i of transitions needed for the Markov chain having the transition probabilities defined in (1) and starting at $X_0 = i$ to hit either 0 or N . To do so, we solved the appropriate difference equation, subject to the boundary conditions $D_0 = D_N = 0$. We saw that the problem of finding a particular solution of (4) is not trivial.

Instead of proceeding as above, we could have solved a system of linear equations. This is straightforward when N is small enough. However, in order to obtain an exact solution for any N , this technique is not convenient.

As in Lefebvre and Guilbault (2008), we could extend the problem solved in the current work by considering a Markov chain $\{X_n, n = 0, 1, \dots\}$ with state space $\{-M, \dots, -1, 0, 1, \dots, N\}$, where $M \in \{1, 2, \dots\}$. Moreover, the probabilities $p_{i,j}$ could be different when i is a negative state. That is, we could set

$$p_{i,i+1} = \frac{1-di}{4}, \quad p_{i,i-1} = \frac{1+di}{4} \quad \text{and} \quad p_{i,i} = \frac{1}{2} \quad \text{for } i \in \{-M+1, \dots, -1\},$$

in which the constant $d (> 0)$ must be such that $d < 1/(M-1)$ for the condition $0 \leq p_{i,j} \leq 1$ for all i, j to be respected and the state $-M$ to be accessible from $-M+1$. When the process hits the origin, we could set

$$p_{0,1} = p_0, \quad p_{0,-1} = \frac{1}{2} - p_0 \quad \text{and} \quad p_{0,0} = \frac{1}{2}, \quad \text{with} \quad p_0 \in (0, 1).$$

This type of Markov chain can have some interesting applications [see L e f e b v r e (2006)].

Finally, the fact that the Markov chain always has a $\frac{1}{2}$ probability of remaining in its current state does not influence the probability that it will hit 0 before N , for instance. However, it does of course increase the average number of transitions needed for the process to hit either 0 or N . We could also calculate the quantity D_i when

$$p_{i,i+1} = \frac{1-ci}{2} \quad \text{and} \quad p_{i,i-1} = \frac{1+ci}{2} \quad \text{for} \quad i \in \{1, \dots, N-1\}.$$

In this case, taking the limit as c decreases to zero, we should retrieve the formula (see F e l l e r, 1968, p. 349) $D_i = i(N-i)$.

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*Département de mathématiques
et de génie industriel
École Polytechnique
C.P. 6079, Succursale Centre-ville
Montréal, Québec H3C 3A7
CANADA
E-mail: jean-luc.guilbault@polymtl.ca
mlefebvre@polymtl.ca*