EXISTENCE OF POSITIVE SOLUTIONS OF DELAY DIFFERENTIAL EQUATIONS

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ABSTRACT. The article is concerned with the existence of positive solutions of the delay differential equations. The asymptotic properties of solutions are also treated.

1. Introduction

This article is concerned with the existence and asymptotic properties of solutions of the delay differential equation of the form

$$\dot{x}(t) + p(t)x(t) + q(t)x(\tau(t)) = 0, \quad t \geq t_0. \quad (1)$$

With respect to (1) throughout we shall assume the following:

- i) $\tau, q \in C([t_0, \infty), [0, \infty))$, $q(t) \not\equiv 0$, $p \in C([t_0, \infty), R)$,
- ii) τ is increasing, $\tau(t) < t$ and $\lim_{t \rightarrow \infty} \tau(t) = \infty$.

A solution of equation (1) is called oscillatory if it has arbitrarily large zeros and otherwise it is nonoscillatory.

The problem of the existence of nonoscillatory solutions of delay differential equations received less attention as oscillation and asymptotic behavior problem. It is due mainly to the technical difficulties arising in analysis of the existence problem. We refer the reader to [1]–[10] and the references cited therein. However, mostly in the papers the investigation of the existence of nonoscillatory solutions which are bounded by the positive functions, is absent. To know such properties of solutions is important in applications.

In this paper we are interested in the existence of solutions of equation (1) which are bounded by positive functions.

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The following fixed point theorem will be used to prove the main results in the next section.

LEMMA 1.1 (Schauder's Fixed Point Theorem [4], [5]). *Let Ω be a closed, convex and nonempty subset of a Banach space X . Let $S: \Omega \rightarrow \Omega$ be a continuous mapping such that $S\Omega$ is a relatively compact subset of X . Then S has at least one fixed point in Ω . That is, there exists an $x \in \Omega$ such that $Sx = x$.*

2. Existence of positive solutions

In this section we shall investigate the existence of positive solutions for equation (1). The main result is in the following theorem.

THEOREM 2.1. *Suppose that for $t \geq t_0$*

$$1 < k_1 \leq k_2, \quad p(t) + k_1 q(t) \geq 0$$

and

$$\ln k_1 \leq \int_{\tau(t)}^t [p(s) + k_1 q(s)] ds \quad \text{and} \quad \int_{\tau(t)}^t [p(s) + k_2 q(s)] ds \leq \ln k_2. \quad (2)$$

Then equation (1) has a solution which is bounded by positive functions.

Proof. We choose $T \geq t_0 + \tau(T)$ and set

$$\begin{aligned} u(t) &= \exp \left(- \int_T^t [p(s) + k_2 q(s)] ds \right), \\ v(t) &= \exp \left(- \int_T^t [p(s) + k_1 q(s)] ds \right), \quad t \geq T. \end{aligned}$$

Let $C([t_0, \infty), R)$ be the set of all continuous functions with the norm

$$\|x\| = \sup_{t \geq t_0} |x(t)| < \infty.$$

Then $C([t_0, \infty), R)$ is a Banach space. We define a close, bounded and convex subset Ω of $C([t_0, \infty), R)$ as follows:

$$\begin{aligned} \Omega = \{ & u(t) \leq x(t) \leq v(t), & t \geq T, \\ & x(\tau(t)) \leq k_2 x(t), & t \geq T, \\ & x(\tau(t)) \geq k_1 x(t), & t \geq T, \\ & x(t) = 1, & \tau(T) \leq t \leq T \}. \end{aligned}$$

Define the map $S: \Omega \rightarrow C([t_0, \infty), R)$ as follows

$$(Sx)(t) = \begin{cases} \exp\left(-\int_T^t \left[p(s) + q(s) \frac{x(\tau(s))}{x(s)}\right] ds\right), & t \geq T, \\ 1, & \tau(T) \leq t \leq T. \end{cases}$$

We shall show that for any $x \in \Omega$ we have $Sx \in \Omega$. For every $x \in \Omega$ and $t \geq T$ we get

$$(Sx)(t) \leq \exp\left(-\int_T^t [p(s) + k_1 q(s)] ds\right) = v(t).$$

Furthermore for $t \geq T$ we have

$$(Sx)(t) \geq \exp\left(-\int_T^t [p(s) + k_2 q(s)] ds\right) = u(t).$$

For $t \in [\tau(T), T]$ we obtain $(Sx)(t) = 1$, that is $(Sx)(t) \in \Omega$. Further for every $x \in \Omega$ and $\tau(t) \geq T$ we get

$$\begin{aligned} (Sx)(\tau(t)) &= \exp\left(-\int_T^{\tau(t)} \left[p(s) + q(s) \frac{x(\tau(s))}{x(s)}\right] ds\right) \\ &= (Sx)(t) \exp\left(-\int_{\tau(t)}^t \left[p(s) + q(s) \frac{x(\tau(s))}{x(s)}\right] ds\right). \end{aligned} \tag{3}$$

With regard to (2) and (3) we have

$$(Sx)(\tau(t)) \leq (Sx)(t) \exp\left(-\int_{\tau(t)}^t [p(s) + k_2 q(s)] ds\right) \leq k_2 (Sx)(t), \quad \tau(t) \geq T,$$

$$(Sx)(\tau(t)) \geq (Sx)(t) \exp\left(-\int_{\tau(t)}^t [p(s) + k_1 q(s)] ds\right) \geq k_1 (Sx)(t), \quad \tau(t) \geq T.$$

For $\tau(T) \leq \tau(t) \leq T$ we obtain $(Sx)(\tau(t)) = 1$, that is $(Sx)(\tau(t)) \in \Omega$. Thus we have proved that $Sx \in \Omega$ for any $x \in \Omega$.

We now show that S is continuous. Let $x_i = x_i(t) \in \Omega$ be such that $x_i(t) \rightarrow x(t)$ as $i \rightarrow \infty$. Because Ω is closed, $x = x(t) \in \Omega$. For $t \geq T$ we have

$$\begin{aligned} |(Sx_i)(t) - (Sx)(t)| &= \left| \exp \left(- \int_T^t \left[p(s) + q(s) \frac{x_i(\tau(s))}{x_i(s)} \right] ds \right) \right. \\ &\quad \left. - \exp \left(- \int_T^t \left[p(s) + q(s) \frac{x(\tau(s))}{x(s)} \right] ds \right) \right|. \end{aligned}$$

So we conclude that

$$\lim_{i \rightarrow \infty} \|(Sx_i)(t) - (Sx)(t)\| = 0.$$

This means that S is continuous.

The family of functions $\{Sx : x \in \Omega\}$ is uniformly bounded on $[\tau(T), \infty)$. It follows from the definition of Ω . This family is also equicontinuous on $[\tau(T), \infty)$. Then by Arzela–Ascoli theorem the $S\Omega$ is relatively compact subset of

$$C([t_0, \infty), R).$$

By Lemma 1.1 there is an $x_0 \in \Omega$ such that $Sx_0 = x_0$. We see that $x_0(t)$ is a positive solution of the equation (1). The proof is complete. $\square$

COROLLARY 2.1. *Suppose that $k > 1$, $p(t) + kq(t) \geq 0$ and*

$$\int_{\tau(t)}^t [p(s) + kq(s)] ds = \ln k, \quad t \geq t_0.$$

Then equation (1) has a solution

$$x(t) = \exp \left(- \int_T^t [p(s) + kq(s)] ds \right), \quad t \geq T.$$

COROLLARY 2.2. *Suppose $1 < k_1 < k_2$, $0 \leq \alpha < k_1$, $p(t) + \alpha q(t) = 0$ and*

$$\frac{\ln k_1}{k_1 - \alpha} \leq \int_{\tau(t)}^t q(s) ds \leq \frac{\ln k_2}{k_2 - \alpha}, \quad t \geq t_0.$$

Then equation (1) has a solution which is bounded by positive functions.

3. Asymptotic properties

In this section some asymptotic properties of positive solutions of (1) are treated.

THEOREM 3.1. *Suppose that $0 < \alpha < 1$, $p(t) + \alpha q(t) \geq 0$, $t \geq t_0$ and*

$$\liminf_{t \rightarrow \infty} \int_{\tau(t)}^t q(s) \, ds > \max_{k > 1} \frac{\ln k}{k - \alpha}. \quad (4)$$

Then equation (1) has not a positive solution $x(t)$ with the property

$$x(\tau(t)) \geq kx(t), \quad t \geq T \geq t_0, \quad k \in (1, \infty).$$

Proof. Assume to the contrary that (1) has a positive solution $x(t)$ with the property $x(\tau(t)) \geq kx(t)$, $t \geq T \geq t_0$ for some $k \in (1, \infty)$. Then

$$\begin{aligned} \dot{x}(t) &\leq q(t) [\alpha x(t) - x(\tau(t))], \\ \frac{\dot{x}(t)}{x(t)} &\leq q(t) \left[\alpha - \frac{x(\tau(t))}{x(t)} \right], \\ \frac{\dot{x}(t)}{x(t)} &\leq (\alpha - k)q(t), \quad t \geq T. \end{aligned} \quad (5)$$

Integrating the last inequality we get

$$\begin{aligned} \ln \frac{x(\tau(t))}{x(t)} &\geq (k - \alpha) \int_{\tau(t)}^t q(s) \, ds, \\ \frac{x(\tau(t))}{x(t)} &\geq \exp \left((k - \alpha) \int_{\tau(t)}^t q(s) \, ds \right), \quad t \geq T. \end{aligned}$$

With regard to condition (4) there exists a constant $c > 0$ such that

$$\int_{\tau(t)}^t q(s) \, ds \geq c > \max_{k > 1} \frac{\ln k}{k - \alpha}, \quad t \geq t_1 \geq T. \quad (6)$$

Then we obtain

$$\frac{x(\tau(t))}{x(t)} \geq \exp((k - \alpha)c) = c_1, \quad t \geq t_1,$$

where $c_1 > k$. Repeating this process by using (5) we get

$$\frac{x(\tau(t))}{x(t)} \geq \exp((c_i - \alpha)c) = c_{i+1}, \quad t \geq t_{i+1}, \quad (7)$$

where $c_{i+1} > c_i$, $i = 1, 2, \dots$, since $(c_i - \alpha)c > \ln c_i$. We see that $c_i \rightarrow \infty$ as $i \rightarrow \infty$. On the other hand for given $\epsilon > 0$ and sufficiently large i we have

$$c_{i+1} \leq c_i + \epsilon, \quad (c_i - \alpha)c = \ln c_{i+1} \leq \ln(c_i + \epsilon), \quad c \leq \frac{\ln(c_i + \epsilon)}{c_i - \alpha},$$

which is a contradiction with (6) for small $\epsilon > 0$.

According to (6) for arbitrary $t \geq t_1$ we can choose $t^* > t$ such that

$$\int_{\tau(t^*)}^t q(s) \, ds \geq \frac{c}{2} \quad \text{and} \quad \int_t^{t^*} q(s) \, ds \geq \frac{c}{2}.$$

Integrating the inequality

$$-\dot{x}(t) \geq q(t) [x(\tau(t)) - \alpha x(t)]$$

from $\tau(t^*)$ to t and then from t to t^* and using the nonincreasing character of $x(t)$ we obtain

$$\begin{aligned} x(\tau(t^*)) - x(t) &\geq \int_{\tau(t^*)}^t q(s) x(\tau(s)) \, ds - \alpha \int_{\tau(t^*)}^t q(s) x(s) \, ds \\ &\geq \frac{c}{2} x(\tau(t)) - \frac{c\alpha}{2} x(\tau(t^*)), \\ x(\tau(t^*)) &\geq \frac{c}{2 + c\alpha} x(\tau(t)), \quad t \geq t_1. \end{aligned}$$

Further we have

$$\begin{aligned} x(t) - x(t^*) &\geq \int_t^{t^*} q(s) x(\tau(s)) \, ds - \alpha \int_t^{t^*} q(s) x(s) \, ds \\ &\geq \frac{c}{2} x(\tau(t^*)) - \frac{c\alpha}{2} x(t), \\ x(t) &\geq \frac{c}{2 + c\alpha} x(\tau(t^*)), \quad t \geq t_1. \end{aligned}$$

It follows that

$$x(t) \geq \frac{c}{2 + c\alpha} x(\tau(t^*)) \quad \text{and} \quad x(\tau(t^*)) \geq \frac{c}{2 + c\alpha} x(\tau(t)), \quad t \geq t_1.$$

Hence we have

$$\frac{x(\tau(t))}{x(t)} \leq \left(\frac{2 + c\alpha}{c} \right)^2, \quad t \geq t_1.$$

This is contradiction with (7). The proof of Theorem 3.1 is complete. $\square$

EXAMPLE 1. Consider the delay differential equation

$$\dot{x}(t) + \frac{3}{10}x(t) + \frac{1}{6}x(t-1) = 0, \quad t \geq 1. \quad (8)$$

If we take $k_1 = 3/2$, $k_2 = 3$, then all conditions of Theorem 2.1 are satisfied and equation (8) has a solution which is bounded by functions

$$u(t) = \exp(-0.8(t-1)), \quad v(t) = \exp(-0.55(t-1)), \quad t \geq 1.$$

EXAMPLE 2. Consider the delay differential equation

$$\dot{x}(t) - (\ln 3)x(t) + \frac{\ln 3}{2}x(t-2) = 0, \quad t \geq 2. \quad (9)$$

If we take $k = 3$, then all conditions of Corollary 2.2 are satisfied and equation (9) has a solution

$$x(t) = \exp\left(-\frac{\ln 3}{2}(t-2)\right), \quad t \geq 2.$$

EXAMPLE 3. Consider the delay differential equation

$$\dot{x}(t) + x(t) + \frac{1}{t}x(0.5t) = 0, \quad t \geq 1. \quad (10)$$

Then we have

$$\liminf_{t \rightarrow \infty} \int_{0.5t}^t \frac{1}{s} ds = \ln 2 > \max_{k > 1} \frac{\ln k}{k - 0.5}.$$

Since all conditions of Theorem 3.1 are satisfied, then equation (10) has not a positive solution $x(t)$ with the property

$$x(0.5t) \geq kx(t), \quad t \geq 1, \quad k \in (1, \infty).$$

EXAMPLE 4. Consider the delay differential equation

$$\dot{x}(t) + \frac{2}{5t}x(t) + \frac{1}{5t}x\left(\frac{1}{2}t\right) = 0, \quad t \geq 1. \quad (11)$$

If we take $k_1 = 1.5$, $k_2 = 3$, then all conditions of Theorem 2.1 are satisfied and equation (11) has a positive solution.

Remark 1. Another original existence results for the equation

$$\dot{x}(t) + \sum_{i=1}^n p_i(t)x(t - \tau_i(t)) = 0, \quad t \geq t_0,$$

can be found in [6]. However, no result from [6] can be applied to the equation (11), due to the unbounded delay.

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