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ON THE RATIONAL RECURSIVE SEQUENCE $x_{n+1} = \frac{ax_{n-1}}{b+cx_nx_{n-1}}$

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ABSTRACT. In this paper we consider the difference equation

$$x_{n+1} = \frac{ax_{n-1}}{b + cx_nx_{n-1}}, \qquad n = 0, 1, \dots \tag{E}$$

with positive parameters and nonnegative initial conditions. We use the explicit formula for the solutions of equation (E) in investigating their behavior.

1. Introduction

In this paper we consider the following rational difference equation

$$x_{n+1} = \frac{ax_{n-1}}{b + cx_nx_{n-1}}, \qquad n = 0, 1, \dots \tag{E}$$

where a, b, c are positive real numbers and the initial conditions x_{-1}, x_0 are nonnegative real numbers such that x_{-1} or x_0 or both are positive real numbers. Equation (E) in the case of negative b was considered in [1].

The purpose of this paper is to use the explicit formula for solutions of equation (E) in investigating their behavior. We will show that when $a < b$, the zero equilibrium is a global attractor for all positive solutions of equation (E) and that all positive solutions of equation (E) are bounded.

There has been a lot of work concerning the asymptotic behavior of solutions of rational difference equations. Second order rational difference equations were investigated, for example in [1–11]. This paper is motivated by the short notes [2] and [9], where the authors studied the rational difference equation

$$x_{n+1} = \frac{x_{n-1}}{1 + x_nx_{n-1}}, \qquad n = 0, 1, \dots$$

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2. Main results

Let $p = \frac{b}{a}$, $q = \frac{c}{a}$. Then equation (E) can be rewritten as

$$x_{n+1} = \frac{x_{n-1}}{p + q x_n x_{n-1}}, \qquad n = 0, 1, \dots \tag{E1}$$

The change of variables $x_n = \frac{1}{\sqrt{q}} y_n$ reduces the above equation to

$$y_{n+1} = \frac{y_{n-1}}{p + y_n y_{n-1}}, \qquad n = 0, 1, \dots \tag{E2}$$

where $p \in R_+$ and the initial conditions y_{-1}, y_0 are nonnegative real numbers such that y_{-1} or y_0 or both are positive real numbers. Hereafter, we focus our attention on equation (E2) instead of equation (E). Note, that the solution $\{y_n\}$ with $y_{-1} = 0$ or $y_0 = 0$ of equation (E2) is oscillatory. In fact, in this case we have

$$\{y_n\} = \left\{0, y_0, 0, \frac{y_0}{p}, 0, \frac{y_0}{p^2}, \dots\right\}$$

or

$$\{y_n\} = \left\{y_{-1}, 0, \frac{y_{-1}}{p}, 0, \frac{y_{-1}}{p^2}, 0, \dots\right\}.$$

Obviously, if $p = 1$, these solutions are 2-periodic.

Thus, let us assume that y_{-1} and y_0 are positive. Then it is clear that $y_n > 0$ for all $n \geq -1$. In the sequel, we will only consider positive solutions of equation (E2).

The equilibria of equation (E2) are the solutions of the equation

$$\bar{y} = \frac{\bar{y}}{p + \bar{y}^2}.$$

Hence, $\bar{y} = 0$ is always an equilibrium point of equation (E2). Clearly, when $p \geq 1$ it is a unique equilibrium point. The local asymptotic behavior of the zero equilibrium of equation (E2) is characterized by the following result.

Theorem A ([11]). *The following statements are true.*

(i) *If $p > 1$, then $\bar{y} = 0$ is locally asymptotically stable.*

(ii) *If $p < 1$, then $\bar{y} = 0$ is a repeller.*

Applying Theorem 2.1 obtained by Cinar in [3] (with $a = b = \frac{1}{p}$) to equation (E2) we get the explicit formula for every solution $\{y_n\}$ with positive initial conditions y_{-1}, y_0. We can write it in the following form

$$y_n = \begin{cases} y_{-1} \dfrac{\prod_{i=0}^{\frac{n+1}{2}-1} \left[p^{2i} + y_0 y_{-1} \sum_{k=0}^{2i-1} p^k \right]}{\prod_{i=0}^{\frac{n+1}{2}-1} \left[p^{2i+1} + y_0 y_{-1} \sum_{k=0}^{2i} p^k \right]} & \text{for odd} \quad n, \\ y_0 \dfrac{\prod_{i=0}^{\frac{n}{2}-1} \left[p^{2i+1} + y_0 y_{-1} \sum_{k=0}^{2i} p^k \right]}{\prod_{i=0}^{\frac{n}{2}-1} \left[p^{2i+2} + y_0 y_{-1} \sum_{k=0}^{2i+1} p^k \right]} & \text{for even} \quad n. \end{cases} \tag{1}$$

We will use the explicit formula for solutions of equation (E2) in investigating their asymptotic behavior. We will consider the cases, when $p \geq 1$ and $p \in (0, 1)$.

Theorem 1. *Assume that $p \geq 1$. Then every positive solution $\{y_n\}$ of equation* (E2) *converges to zero.*

Proof. Let $\{y_n\}$ be a solution of equation (E2) satisfying the initial conditions $y_{-1} > 0$ and $y_0 > 0$. It is enough to prove that the subsequences $\{y_{2n}\}$ and $\{y_{2n-1}\}$ converge to zero as $n \to \infty$. From (1) we have

$$\begin{aligned} y_{2n} &= y_0 \frac{\prod_{i=0}^{n-1} \left[p^{2i+1} + y_0 y_{-1} \sum_{k=0}^{2i} p^k \right]}{\prod_{i=0}^{n-1} \left[p^{2i+2} + y_0 y_{-1} \sum_{k=0}^{2i+1} p^k \right]} \\ &= y_0 \exp \left[\sum_{i=0}^{n-1} \ln \frac{p^{2i+1} + y_0 y_{-1} \sum_{k=0}^{2i} p^k}{p^{2i+2} + y_0 y_{-1} \sum_{k=0}^{2i+1} p^k} \right] \\ &= y_0 \exp \left[\sum_{i=0}^{n-1} \ln \left(1 - \frac{p^{2i+2} - p^{2i+1} + p^{2i+1} y_0 y_{-1}}{p^{2i+2} + y_0 y_{-1} \sum_{k=0}^{2i+1} p^k} \right) \right] \\ &\leq y_0 \exp \left[- \sum_{i=0}^{n-1} \frac{p^{2i+2} - p^{2i+1} + p^{2i+1} y_0 y_{-1}}{p^{2i+2} + y_0 y_{-1} \sum_{k=0}^{2i+1} p^k} \right] \\ &= y_0 \exp \left[- \sum_{i=0}^{n-1} \frac{p^{2i+1} (p - 1 + y_0 y_{-1})}{p^{2i+2} + y_0 y_{-1} \sum_{k=0}^{2i+1} p^k} \right]. \end{aligned}$$

So, we have

$$y_{2n} \le y_0 \exp\left[-(p-1+y_0y_{-1})\sum_{i=0}^{n-1}\frac{p^{2i+1}}{p^{2i+2}+y_0y_{-1}\sum\limits_{k=0}^{2i+1}p^k}\right]. \tag{2}$$

Since $p \ge 1$, $p-1+y_0y_{-1} > 0$ and from the above inequality we obtain

$$\begin{aligned} y_{2n} &\le y_0 \exp\left[-(p-1+y_0y_{-1})\sum_{i=0}^{n-1}\frac{p^{2i+1}}{p^{2i+2}+y_0y_{-1}p^{2i+1}\sum\limits_{k=0}^{2i+1}1}\right] \\ &= y_0 \exp\left[-(p-1+y_0y_{-1})\sum_{i=0}^{n-1}\frac{1}{p+y_0y_{-1}(2i+2)}\right]. \end{aligned}$$

Because $\sum_{i=0}^{n-1}\frac{1}{p+y_0y_{-1}(2i+2)} \to \infty$ as $n \to \infty$, so $y_{2n} \to 0$ as $n \to \infty$.
Similarly, we obtain

$$y_{2n-1} \le y_{-1} \exp\left[-(p-1+y_0y_{-1})\sum_{i=0}^{n-1}\frac{1}{p+y_0y_{-1}(2i+1)}\right] \to 0, \quad \text{as} \quad n \to \infty.$$

This completes the proof. □

Theorem 1 extends Theorem 1 of S t e v i ć [9].

Theorem 1 and Theorem A imply the following result.

Corollary 1. *Assume $p > 1$. Then the unique equilibrium $\bar{y} = 0$ of equation* (E2) *is globally asymptotically stable.*

Note, that equation (E2) is a special case of equation (3.3) in [11], but our result relating the global attractivity of the zero equilibrium is stronger.

For $p \in (0,1)$ we have the following result about the subsequences of the even terms $\{y_{2n}\}_{n=0}^{\infty}$ and the odd terms $\{y_n\}_{n=-1}^{\infty}$ of every positive solution $\{y_n\}$ of equation (E2).

Theorem 2. *Assume that $p \in (0,1)$. Let $\{y_n\}$ be a solution of equation* (E2) *with positive initial conditions y_{-1}, y_0. Then the following statements are true:*

(i) *If $y_0y_{-1} < 1-p$, then the subsequences $\{y_{2n}\}$ and $\{y_{2n-1}\}$ are both increasing and bounded.*

(ii) *If $y_0y_{-1} > 1-p$, then the subsequences $\{y_{2n}\}$ and $\{y_{2n-1}\}$ are both decreasing and bounded.*

(iii) *If $y_0y_{-1} = 1-p$, then the subsequences $\{y_{2n}\}$ and $\{y_{2n-1}\}$ are both constant sequences.*

Proof.

(i) Let $\{y_n\}$ be a positive solution of equation (E2). From (1) for the subsequence $\{y_{2n}\}$ we have

$$y_{2n} = y_0 \frac{\prod\limits_{i=0}^{n-1}\left[p^{2i+1} + y_0y_{-1}\sum\limits_{k=0}^{2i} p^k\right]}{\prod\limits_{i=0}^{n-1}\left[p^{2i+2} + y_0y_{-1}\sum\limits_{k=0}^{2i+1} p^k\right]},$$

and so for $n \geq 0$

$$\frac{y_{2n+2}}{y_{2n}} = \frac{\prod\limits_{i=0}^{n}\left[p^{2i+1} + y_0y_{-1}\sum\limits_{k=0}^{2i} p^k\right]\prod\limits_{i=0}^{n-1}\left[p^{2i+2} + y_0y_{-1}\sum\limits_{k=0}^{2i+1} p^k\right]}{\prod\limits_{i=0}^{n}\left[p^{2i+2} + y_0y_{-1}\sum\limits_{k=0}^{2i+1} p^k\right]\prod\limits_{i=0}^{n-1}\left[p^{2i+1} + y_0y_{-1}\sum\limits_{k=0}^{2i} p^k\right]}$$

$$= \frac{p^{2n+1} + y_0y_{-1}\sum\limits_{k=0}^{2n} p^k}{p^{2n+2} + y_0y_{-1}\sum\limits_{k=0}^{2n+1} p^k}. \tag{3}$$

Since $y_0y_{-1} < 1 - p$, we have

$$y_0y_{-1}p^{2n+1} < p^{2n+1} - p^{2n+2}.$$

Hence

$$y_0y_{-1}\left(\sum_{k=0}^{2n+1} p^k - \sum_{k=0}^{2n} p^k\right) < p^{2n+1} - p^{2n+2},$$

and therefore

$$p^{2n+1} + y_0y_{-1}\sum_{k=0}^{2n} p^k > p^{2n+2} + y_0y_{-1}\sum_{k=0}^{2n+1} p^k.$$

From the above inequality and (3) it follows that the subsequence $\{y_{2n}\}$ is increasing. Similarly we obtain that the subsequence $\{y_{2n-1}\}$ is increasing. Now, we will show that the solution $\{y_n\}$ is bounded.
From (2) we have

$$y_{2n} \leq y_0 \exp\left[-(p-1+y_0y_{-1})\sum_{i=0}^{n-1}\frac{p^{2i+1}}{p^{2i+2} + y_0y_{-1}\sum\limits_{k=0}^{2i+1} p^k}\right]$$

$$= y_0 \exp\left[(1-p-y_0y_{-1})(1-p)\sum_{i=0}^{n-1}\frac{p^{2i+1}}{p^{2i+2}(1-p) + y_0y_{-1}(1-p^{2i+2})}\right].$$

Since for $p \in (0,1)$ the series

$$\sum_{i=0}^{n-1} \frac{p^{2i+1}}{p^{2i+2}(1-p) + y_0 y_{-1}(1-p^{2i+2})}$$

is convergent and we get the boundedness of $\{y_{2n}\}$. Similarly we obtain the boundedness of the subsequence $\{y_{2n-1}\}$.

(ii) The proof is similar to the proof of (i) and will be omitted.

(iii) If $y_0 y_{-1} = 1 - p$ then from (E2) we get

$$y_{n+1} = \frac{y_{n-1}}{p + y_n y_{n-1}} = y_{n-1}.$$

Hence

$$\{y_{2n}\} = \{y_0, y_0, y_0, \ldots\} \quad \text{and} \quad \{y_{2n-1}\} = \{y_{-1}, y_{-1}, y_{-1}, \ldots\}$$

and, the solution

$$\{y_n\} = \{y_{-1}, y_0, y_{-1}, y_0, \ldots y_{-1}, y_0\}$$

is 2-periodic.

This completes the proof. □

From Theorem 1 and Theorem 2 we get the following corollary.

Corollary 2. *Every positive solution of equation* (E) *is bounded.*

We say the sequences $\{a_n\}$, $\{b_n\}$ are equivalent (Cauchy equivalent) if $\lim_{n\to\infty}(a_n - b_n) = 0$. If there exists a k-periodic sequence $\{c_n\}$ equivalent to $\{a_n\}$, we say that $\{a_n\}$ is asymptotically k-periodic sequence.

The next corollary follows from Theorem 2 and from the expression of equation (E2).

Corollary 3. *Assume that $p \in (0,1)$. Then every positive solution of equation* (E2) *is asymptotically 2-periodic sequence $\{r, s, r, s, r, s, \ldots\}$, where $rs = 1 - p$.*

Moreover, it is clear from formula (1) that, for a fixed p, numbers r and s depend only on the initial data y_{-1}, y_0.

3. Numerical results

Example 1. Let $y_{-1} = 2$, $y_0 = 5$ be the initial conditions of equation (E2) with $p = 2$. Then, by Theorem 1, the solution converges to zero.

The Table 1 sets forth the values of y_n for selected small n's. Note that Theorem 5.2 from [11] in this case can not be applied.

TABLE 1. The values of y_n for selected small n's.

n	y_n	n	y_n
1	0.1666666666	2	1.764705882
3	0.07264957264	4	0.8291991495
5	0.03526265806	6	0.4086255174
7	0.01750521080	8	0.2035846305
9	0.008737036910	10	0.1017018654
29	$8.527211545 * 10^{-6}$	30	$9.928883343 * 10^{-5}$
699	$1.218311909 * 10^{-106}$	700	$1.418573558 * 10^{-105}$

EXAMPLE 2. Let $y_{-1} = 500$, $y_0 = 100$ be the initial conditions of equation (E2) with $p = \frac{4}{5}$. Then $y_0y_{-1} = 50000$, $1 - p = \frac{1}{5}$. So condition $y_0y_{-1} > 1 - p$ holds and by Theorem 2, the subsequences $\{y_{2n}\}$ and $\{y_{2n-1}\}$ are both decreasing.

The Table 2 sets forth the values of y_n for selected small n's.

TABLE 2. The values of y_n for selected small n's.

n	y_n	n	y_n
1	0.009999840002	2	55.55604937
3	0.007376952648	4	45.92037709
5	0.006478100367	6	41.84177433
7	0.006048334655	8	39.73302154
9	0.005813925263	10	38.53815313
29	0.005460160780	30	36.67435849
49	0.005456447604	50	36.65440759
99	0.005456404175	100	36.65417618

EXAMPLE 3. Let $y_{-1} = 0.001$, $y_0 = 2$ be the initial conditions of equation (E2) with $p = 0.9$. Then $y_0y_{-1} = 0.002$, $1-p = 0.1$. So condition $y_0y_{-1} < 1-p$ holds and by Theorem 2, the subsequences $\{y_{2n}\}$ and $\{y_{2n-1}\}$ are both increasing.

The Table 3 sets forth the values of y_n for selected small n's.

TABLE 3. The values of y_n for selected small n's.

n	y_n	n	y_n
1	0.00110864745	2	2.216760874
3	0.001228475933	4	2.455637323
5	0.001360413317	6	2.718395588
7	0.001505384658	8	3.006767998
9	0.00166427951	10	3.322380518
25	0.003484539186	26	6.888785559
49	0.006449046007	50	12.37982939
99	0.007251009859	100	13.77325644
299	0.007255975296	300	13.78174482

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