MOMENTS OF VECTOR-VALUED FUNCTIONS AND MEASURES

MILOSLAV DUCHOŇ — CAMILLE DEBIÈVE

ABSTRACT. There are investigated conditions under which the elements of a normed vector space are the moments of a vector-valued measure, and of a Bochner integrable function, respectively, both with values in a Banach space.

Introduction

If g is a real-valued function of bounded variation on the unit interval I of the real line, the numbers

$$a_k = \int_0^1 t^k dg(t) \quad (k = 0, 1, \dots)$$

are called the moments of g , g being, e.g., distribution of mass of particle, body, or electric charge. Hausdorff [H] has shown that for a sequence a_k of real numbers to be moment sequence of some non-decreasing g (the case of particular interest in some cases), it is necessary and sufficient that a_k be totally monotone. Recall that a sequence is called totally monotone if all its difference sequences are non-negative.

We shall search conditions under which, given a sequence a_n , $n = 0, 1, 2, \dots$ of complex numbers,

- (a) there exists a measure on $[a, b]$ such that a_n are the moments of the measure, respectively,
- (b) there exists a function in $L_p([a, b])$, $1 \leq p \leq \infty$, such that a_n are the moments of this function.

2000 Mathematics Subject Classification: 44A60, 28B99.

Keywords: Banach space, vector valued measure, Bochner integral, moments of measures and functions.

Supported by the Slovak grant agency VEGA, grant number 2/4137/07.

Then, in Section 3, a_n are taken to be elements of a normed vector space and conditions are given under which a_n are the moments of (a) a vector-valued measure, and (b) of a Bochner integrable function, respectively.

1. Preliminaries

The Hausdorff moment problem [H, Lo, W] is the following: given a prescribed set of real numbers $\{a_n\}_0^\infty$, find a bounded non-decreasing function $u(t)$ on the closed interval $[0, 1]$ such that its moments are equal to the prescribed values, that is,

$$\int_{[0,1]} t^n du(t) = a_n, \quad n = 0, 1, 2, \dots$$

The integral is a Riemann-Stieltjes integral. Equivalently, find a nonnegative measure μ on borelian subsets in $[0, 1]$ with

$$\int_{[0,1]} t^n d\mu(t) = a_n, \quad n = 0, 1, 2, \dots$$

We shall need the operator ∇^k ($k = 0, 1, 2, \dots$) defined by

$$\begin{aligned} \nabla^0 a_n &= a_n, & \nabla^1 a_n &= a_n - a_{n+1}, \\ \nabla^k a_n &= a_n - \binom{k}{1} a_{n+1} + \binom{k}{2} a_{n+2} - \dots + (-1)^k a_{n+k}, & n &= 1, 2, \dots \end{aligned}$$

for any sequence of real or complex numbers $\{a_n\}_0^\infty$.

Remark that we confine ourselves to interval $[0, 1]$ for simplicity, this is, however, no limitation of generality, for any bounded interval $[a, b]$.

DEFINITION. For each $f \in L_1([0, 1])$, the elements

$$a_n = \int_0^1 t^n f(t) dt, \quad n = 1, 2, \dots$$

are called the moments of f . For a measure μ on $[0, 1]$, the elements

$$a_n = \int_0^1 t^n \mu(dt), \quad n = 0, 1, 2, \dots$$

are called the moments of μ .

For a given sequence $a = \{a_n\}_0^\infty$, put

$$l_{k,m} = \binom{k}{m} \nabla^{k-m} a_m, \quad (k, m = 0, 1, 2, \dots).$$

Let us mention that this definition is slightly different from the definition of $\lambda_{k,m}$ given in [W], but we obtain similar results.

In order to remain in some reasonable frame in the following we shall suppose satisfied the

Condition A. There is a finite positive constant L such that

$$\sum_{k=0}^N |l_{N,k}| < L, \quad N = 0, 1, 2, \dots$$

Following [W, p. 107] for the sequence $\{a_n\}$, define an operator $L_N(t)$ by

$$L_N(t) = (1 + N) l_{N, [Nt]}, \quad N = 1, 2, \dots$$

$[Nt]$ means the largest integer contained in Nt .

We have [W, p. 110] for $p > 1$,

$$\int_0^1 |L_N(t)|^p dt = \frac{(N+1)^p}{N} \sum_{m=0}^{N-1} |l_{N,m}|^p < 2L, \quad N = 0, 1, 2, \dots$$

If f is in $L_p([0, 1])$ and a_n are its moments, then [W, p. 111]

$$\lim_{N \rightarrow \infty} \int_0^1 t^n L_N(t) dt = a_n, \quad n = 0, 1, \dots \quad (1)$$

Hence for every continuous, and more generally, for every bounded measurable function g there exists

$$\lim_{N \rightarrow \infty} \int_0^1 g(t) L_N(t) dt = A.$$

If a_n are moments of f , then for all $t \in [0, 1]$, we have

$$\int_0^1 |L_N(s)| ds \leq \int_0^1 K_N(t, s) |f(s)| ds, \quad (2)$$

where

$$K_N(t, s) = (1 + N) \binom{N}{[Nt]} s^{[Nt]} (1 - s)^{N - [Nt]}, \quad (N = 1, 2, \dots; 0 \leq s, t \leq 1).$$

It is easy to prove the following lemma.

LEMMA 1. *For every $t \in [0, 1]$*

$$\int_0^1 |K_N(t, s)| \, ds \leq \frac{N+1}{N} \sum_{m=0}^{N-1} \int_0^1 \binom{N}{m} s^m (1-s)^{N-m} \, ds \leq M < \infty,$$

for $N = 1, 2, \dots$

2. Scalar-valued moments

At least the part of the following theorem is already known see [W, pp. 100–114], where the L_p case ($1 < p \leq \infty$) and (iii)–(iii)’ are explicitly given). Recall that L_p , ($1 \leq p \leq \infty$), denotes the space of all measurable functions f such that $|f|^p$ is integrable on $[0, 1]$.

THEOREM 2. *Given a sequence $a_n, n = 0, 1, 2, \dots$, of complex numbers, there exists*

- (i) *a function $f \in L_1$ such that a_n are the moments of f ;*
- (ii) *a function $f \in L_p, 1 < p \leq \infty$, such that a_n are the moments of f ;*
- (iii) *a complex, regular Borel measure μ such that a_n are the moments of μ ;*
- (iv) *a nonnegative regular Borel measure μ such that a_n are the moments of μ ;*
- (v) *a continuous function f such that a_n are the moments of f
if and only if the functions $L_k(t)\{a_n\}$*
- (i)’ *converge in the L_1 -norm;*
- (ii)’ *are bounded in the L_p -norm;*
- (iii)’ *are bounded in the L_1 -norm;*
- (iv)’ *are nonnegative;*
- (v)’ *converge uniformly.*

For completeness, we give a little different and more compact proof of all parts using the following lemmas.

LEMMA 3. *For each N , define the linear map $T_N: L_p \rightarrow L_p$ (for all $p, 1 \leq p \leq \infty$) by*

$$(T_N g)(t) = \int_0^1 K_N(t, s) g(s) \, ds, \quad g \in L_p.$$

Then each T_N is continuous for all $p, 1 \leq p \leq \infty$, and there exists a constant $D < \infty$ such that $\|T_N\| \leq D$ for all N .

Proof. When $p = 1$, we have, by Lemma 1, for each $g \in L_1$,

$$\begin{aligned} \sup_N \|T_N g\| &= \sup_N \int_0^1 \left| \int_0^1 K_N(t, s) g(s) \, ds \right| dt \\ &\leq \sup_N \int_0^1 \left(|g(s)| \int_0^1 |K_N(t, s)| \, dt \right) ds \leq M \|g\|. \end{aligned}$$

Therefore $\sup \|T_N\| = D_1 < \infty$. When $p = \infty$, for each N ,

$$\begin{aligned} \|T_N\| &= \sup_{\|g\|=1} \operatorname{ess\,sup}_t \left| \int_0^1 K_N(t, s) g(s) \, ds \right| \\ &\leq \sup_{\|g\|=1} \operatorname{ess\,sup}_t \int_0^1 |K_N(t, s)| |g(s)| \, ds = D_2 < \infty. \end{aligned}$$

Putting $D = \max(D_1, D_2)$, we have that, for each N , T_N is a linear mapping of each L_p space into itself ($1 \leq p \leq \infty$) and T_N is continuous for $p = 1$ and $p = \infty$ with norm at most D . The result now follows from the Riesz convexity theorem [E, p. 526]. $\square$

LEMMA 4. *If f is in $L_p([0, 1])$, $1 \leq p < \infty$, then $T_N f$ converges to f in the L_p -norm.*

Proof. Let $\epsilon > 0$ be given. Then there exists $\psi \in C([0, 1])$ such that $\|f - \psi\|_p < \epsilon$. So, for all sufficiently large N ,

$$\begin{aligned} \|T_N f - f\| &\leq \|T_N(f - \psi)\| + \|T_N \psi - \psi\| + \|\psi - f\| \\ &\leq \|T_N\| \|f - \psi\| + \epsilon + \epsilon \\ &< D\epsilon + 2\epsilon. \end{aligned} \quad \square$$

LEMMA 5. *If f is in $L_\infty([0, 1])$, then $T_N f$ is a bounded subset of $L_\infty([0, 1])$ and so weak-star compact and (some subsequence of) $T_N f$ converges to f in the weak-star topology of L_∞ .*

Proof. For all f in $L_\infty([0, 1])$ the sequence $(T_N f)$ is bounded in $L_\infty([0, 1])$, and so weak-star compact. $\square$

Proof of Theorem 2. Suppose that f is in L_1 and a_n are the moments of f . Then $L_N = T_N f$ and so (i)' follows from Lemma 4. Also if f is in L_p , (ii)' follows from Lemma 4 ($1 < p < \infty$) and Lemma 5 ($p = \infty$).

If the a_n are the moments of a complex measure μ , $a_n = \int_0^1 s^n \mu(ds)$ then, by Lemma 1

$$\begin{aligned} \|L_N\|_1 &= \int_0^1 |L_N(t)| dt = \frac{N+1}{N} \sum_{m=0}^{N-1} |\lambda_{N,m}| \\ &\leq \frac{N+1}{N} \sum_{m=0}^N \binom{N}{m} \int_0^1 s^m (1-s)^{N-m} |\mu|(ds) \\ &\leq M |\mu|([0, 1]), \end{aligned}$$

which shows that (iii)' follows from (iii).

We now show the converse implications. First, observe that, if n is fixed and $N > n$, then

$$\int_0^1 t^n L_N(t) dt \rightarrow a_n, \quad (n = 0, 1, 2, \dots), \quad N \rightarrow \infty. \quad (3)$$

If the L_N converge in the L_1 -norm, they converge to an integrable function f . So

$$\begin{aligned} \left| \int_0^1 (f(t) - L_N(t)) t^n dt \right| &\leq \int_0^1 |f(t) - L_N(t)| |t^n| dt \\ &\leq \sup_t |t^n| \|f - L_N\|_1. \end{aligned}$$

Therefore,

$$\lim_N \left| \int_0^1 f(t) t^n dt - \int_0^1 L_N(t) t^n dt \right| = 0.$$

Hence, by (3),

$$\int_0^1 f(t) t^n dt = a_n.$$

Suppose that L_N are bounded in L_p -norm, $1 < p \leq \infty$. We may as well assume that $\|L_N\|_p \leq 1$, $N = 1, 2, \dots$. Then the L_N lie in the unit ball of the conjugate space of L_q (where $p^{-1} + q^{-1} = 1$). Since this unit ball is weak-star compact, there is a function f in L_p with $\|f\|_p \leq 1$ such that every weak-star neighbourhood of f contains L_N for infinitely many values of N . In other words,

given any g in L_q , the numbers

$$\int_0^1 L_N(t)g(t) dt$$

are near $\int_0^1 f(t)g(t) dt$ for infinitely many values of N . But each t^n is in L_q and so, by (3)

$$\int_0^1 f(t)t^n dt = a_n.$$

Suppose that

$$\int_0^1 |L_N(t)| dt \leq \alpha$$

for all N . Define, for each N , the scalar-valued map Φ_N on $C([0, 1])$ by

$$\Phi_N(\psi) = \int_0^1 \psi(t)L_N(t) dt, \quad \psi \in C([0, 1]).$$

Then, for all N , $\|\Phi_N\| \leq \alpha$. That is, all the Φ_N are in the (weak-star compact) ball of radius α of the dual space of $C([0, 1])$. Hence there exists a regular Borel measure μ such that, for all $\epsilon > 0$ and all $\psi \in C([0, 1])$, there exists N such that

$$\left| \int_0^1 \psi(t)L_N(t) dt - \int_0^1 \psi(t)\mu(dt) \right| < \epsilon. \quad (4)$$

Since each t^n is in $C([0, 1])$, we have, by (3),

$$\int_0^1 t^n \mu(dt) = a_n. \quad \square$$

Remark. By using $\lim_{N \rightarrow \infty} \|L_N - f\|_1 = 0$, in conjunction with results in general integration theory referring to criteria for weak compactness for subsets of L_1 (see [DS, pp. 294–295] and [E, pp. 274–276]) the following characterization of L_1 moments can be established.

Recalling an operator $L_N(t)$ defined by

$$L_N(t) = (N+1)l_{N,[Nt]} \quad (k = 1, 2, \dots; 0 \leq t \leq 1)$$

the following four conditions are equivalent:

- (a) a subsequence $(L_{N_k})_{k=1}^\infty$ exists such that the set functions $E \rightarrow \int_E L_{N_k}(t) dt$ ($k = 1, 2, \dots$) are uniformly (or equi-) absolutely continuous, that is, to each $e > 0$ there corresponds $d > 0$ such that

$$\sup_k \left| \int_E L_{N_k}(t) dt \right| \leq e$$

for all measurable sets E satisfying $m(E) < d$;

- (b) some subsequence $(L_{N_k})_{k=1}^\infty$ converges weakly in L^1 ;
 (c) some subsequence $(L_{N_k})_{k=1}^\infty$ converges in L^1 ;
 (d) $(a_n)_{n \in \mathbb{Z}} \in Mo(\mathbb{Z})$, i.e., $a = Mo f$, $f \in L^1$, Mo denotes the moment.

If any one of these conditions holds, and if $f = \lim_k L_{N_k}$ weakly or strongly in L^1 , then $a = Mo f$, $L_N = L_N f$, and so $\lim_{N \rightarrow \infty} \|L_N - f\|_1 = 0$.

Remark. The uniform absolute continuity of the set functions can be expressed as uniform absolute continuity of the point functions

$$F_k(x) = \int_0^x L_{N_k}(t) dt$$

the condition being precisely that to each $e > 0$ there corresponds a number $d = d(e) > 0$ such that

$$\sup_k \sum_{j=1}^r |F_k(b_j) - F_k(a_j)| \leq e$$

for any finite sequence $((a_j, b_j))_{j=1}^r$ of disjoint open intervals (a_j, b_j) the sum of whose lengths does not exceed d . In this connection see, for example, [HS, Theorem (19.53) and its proof].

3. Vector-valued moments

Let X be a Banach space. Note that at some places, e.g., in Condition A the absolute value will be replaced by norm in X .

THEOREM 6. *Given a sequence $a_n, n = 0, 1, 2, \dots$ of elements of X , there exists a regular measure $\mu: \mathcal{B}([0, 1]) \rightarrow X$ of finite total variation such that a_n are the moments of μ if and only if there exists a constant $0 < H < \infty$ such that*

$$\int_0^1 \|L_N(t)\| dt \leq H, \quad N = 1, 2, \dots$$

Proof. Suppose that such a measure exists. Then, for each N , by (1)

$$\begin{aligned} \int_0^1 \|L_N\| dt &= \int_0^1 \left\| \int_0^1 K_N(t, s) \mu(ds) \right\| dt \\ &\leq \int_0^1 \int_0^1 |K_N(t, s)| dt |\mu|(ds) \\ &\leq M |\mu|([0, 1]). \end{aligned}$$

Conversely, suppose that $\int_0^1 \|L_N\| dt \leq H$ for all N . If we define

$$\Phi_N(\psi) = \int_0^1 \psi(t) L_N(t) dt, \quad \psi \in C([0, 1]),$$

then $\|\Phi_N\| \leq H$ for all N . Since, for each n , $\lim_N \Phi_N(t^n) = a_n$, we have that $\lim_N \Phi_N(\psi)$ exists for all ψ which are linear combinations of the t^n and so, as $\|\Phi_N\| \leq H$ for all N , we conclude that $\lim_N \Phi_N(\psi)$ exists for all $\psi \in C([0, 1])$. Denote this limit by Φ . To obtain our required measure, we use the following lemma (see [D, § 19, pp. 380, 383]). For each subset A of $[0, 1]$, let $C([0, 1], A)$ denote the space of continuous functions on $[0, 1]$ vanishing outside A . If $F: C([0, 1]) \rightarrow X$ is a linear mapping, define for each A ,

$$|||F_A||| = \sup \sum \|F(\psi_i)\|,$$

where supremum is over all finite families ψ_i in $C([0, 1])$ with $\sum |\psi_i(t)| \leq \chi_A(t)$ for all $t \in [0, 1]$.

LEMMA. *If $F: C([0, 1]) \rightarrow X$ is a linear mapping, then there exists a regular measure $\mu: \mathcal{B}([0, 1]) \rightarrow X$ with finite variation such that*

$$F(\psi) = \int_0^1 \psi(t) \mu(dt), \quad \psi \in C([0, 1]),$$

if and only if $|||F_A||| < \infty$ for all $A \in \mathcal{B}([0, 1])$.

Let A be in $\mathcal{B}([0, 1])$ and let $\{\psi_i; i = 1, 2, \dots, n\}$ be a finite family of functions in $C([0, 1], A)$ with

$$\sum_{i=1}^n |\psi_i(t)| \leq \chi_A(t), \quad t \in [0, 1].$$

Then for each N ,

$$\sum_{i=1}^n |\psi_i(t)| \leq \chi_A(t), \quad t \in [0, 1].$$

So for each N ,

$$\begin{aligned} \sum_{i=1}^n \|\Phi_N(\psi_i)\| &= \sum_{i=1}^n \left\| \int_0^1 \psi_i(t) L_N(t) dt \right\| \\ &\leq \sum_{i=1}^n \int_0^1 |\psi_i(t)| \|L_N(t)\| dt \\ &\leq \int_0^1 \chi_A(t) \|L_N(t)\| dt \leq H. \end{aligned}$$

Hence $\sum_1^n \|\Phi(\psi_i)\| \leq H$ and so $\|\Phi_A\| \leq H$. Therefore, there exists a regular measure $\mu: \mathcal{B}([0, 1]) \rightarrow X$ with finite variation such that

$$\Phi(\psi) = \int_0^1 \psi(t) \mu(dt), \quad \psi \in C([0, 1]).$$

But each t^n is in $C([0, 1])$ and so,

$$a_n = \int_0^1 t^n \mu(dt).$$

and thus theorem is proved. $\square$

Recall a definition of a Bochner integrable function. If $f: [a, b] \rightarrow X$ is simple, i.e., $f(s) = \sum_{i=1}^n \chi_{E_i}(s) x_i$, where χ_E denotes the indicator function of the set $E \subset [a, b]$, $x_i \in X$, then for any $E \in \mathcal{B}([a, b])$

$$\int_E f d\lambda = \sum_{i=1}^n \lambda(E \cap E_i) x_i,$$

where λ is a probability measure on $[a, b]$. Such functions are λ -measurable. Any function $f: [a, b] \rightarrow X$ which is the λ -almost everywhere limit of a sequence of simple functions is (called) λ -measurable.

A λ -measurable function $f: [a, b] \rightarrow X$ is called Bochner integrable if there exists a sequence of simple functions (f_n) such that

$$\lim_{n \rightarrow \infty} \int_{[a, b]} \|f_n(s) - f(s)\| d\lambda(s) = 0.$$

In this case $\int_E f d\lambda$ is defined for each measurable set E in $[a, b]$ by

$$\int_E f d\lambda = \lim_n \int_E f_n d\lambda.$$

A λ -measurable function $f: [a, b] \rightarrow X$ is Bochner integrable if and only if $\int_{[a, b]} \|f\| d\lambda < \infty$.

The following theorem concerns moments of Bochner integrable function.

THEOREM 7. *Given a sequence $a_n, n = 0, 1, 2, \dots$, of elements of X , there exists an X -valued Bochner integrable function f on $[0, 1]$ such that a_n are the moments of f if and only if*

$$\lim_{N, J \rightarrow \infty} \int_0^1 \|L_N(t) - L_J(t)\| dt = 0.$$

Proof. Suppose that f is Bochner integrable and a_n are the moments of f . Let $\{A_i\}_1^n$ be a finite family in $\mathcal{B}([0, 1])$ and $\{\beta_i^n\}$ a finite family of vectors in X and define $g: [0, 1] \rightarrow X$ by

$$g(t) = \sum_{i=1}^n \beta_i \chi_{A_i}(t).$$

Then

$$\begin{aligned} \int_0^1 \left\| \int_0^1 K_N(t, s) g(s) ds - g(t) \right\| dt &= \int_0^1 \left\| \sum_{i=1}^n \beta_i \left(\int_0^1 K_N(t, s) \chi_{A_i}(s) ds - \chi_{A_i}(t) \right) \right\| dt \\ &\leq \sum_{i=1}^n \left(\|\beta_i\| \int_0^1 \left| \int_0^1 K_N(t, s) \chi_{A_i}(s) ds - \chi_{A_i}(t) \right| dt \right) \end{aligned}$$

which, by Lemma 4, tends to 0 as $N \rightarrow \infty$. Therefore, since the set of all such g is dense in the space of all Bochner integrable functions and as

$$L_N(t) = \int_0^1 K_N(t, s) f(s) ds,$$

we have

$$\lim_N \int_0^1 \|L_N(t) - f(t)\| dt = 0.$$

Conversely, suppose that the sequence L_N is Cauchy in the norm of the space of all Bochner integrable functions. Since this space is Banach, L_N converges in the Bochner space norm to a Bochner integrable function f .

So, for each n ,

$$\begin{aligned} \left\| \int_0^1 (f(t) - L_N(t)) t^n dt \right\| &\leq \int_0^1 \|f(t) - L_N(t)\| |t^n| dt \\ &\leq \sup_t |t^n| \|L_N - f\|_B \rightarrow 0, \quad N \rightarrow \infty. \end{aligned}$$

Thus, by (3),

$$a_n = \int_0^1 t^n f(t) dt.$$

So the theorem is proved. □

REFERENCES

- [BDS] BARTLE, R.—DUNFORD, N.—SCHWARTZ, J. C.: *Weak compactness and vector measures*, Canad. J. Math. **7** (1955), 289–305.
- [D] DINCULEANU, N.: *Vector Measures*. Pergamon Press, New York, 1967.
- [DS] DUNFORD, N.—SCHWARTZ, J. T.: *Linear Operators. Part I*, in: Pure Appl. Math. (N.Y.), Vol. 6, Wiley-Intersci., New York, 1958.
- [E] EDWARDS, R. E.: *Functional Analysis: Theory and Applications*. Holt Rinehart and Winston, Inc., New York, 1965.
- [H] HAUSDORFF, F.: *Summations Methoden und Momentenfolgen II*, Math. Z. **9** (1921), 280–299.
- [HS] HEWITT, E.—STROMBERG, K.: *Real and Abstract Analysis*. Springer-Verlag, Berlin, 1965.
- [Lo] LORENTZ, G. G.: *Bernstein Polynomials*. Toronto University Press, Toronto, 1953.
- [W] WIDDER, D. V.: *The Laplace Transform*. Princeton University Press, Princeton, N. J., 1946.

Received December 17, 2008

Miloslav Duchoň
Mathematical Institute
Slovak Academy of Sciences
Štefánikova 49
SK-814-73 Bratislava
SLOVAKIA
E-mail: duchon@mat.savba.sk

Camille Debiève
Université Catholique de Louvain
Département de Mathématiques
2, Chemin du Cyclotron
B-1348 Louvain-la-Neuve
BELGIUM
E-mail: camille.debieve@uclouvain.be