ψ -CONTINUOUS FUNCTIONS AND FUNCTIONS PRESERVING ψ -DENSITY POINTS

MAŁGORZATA FILIPCZAK — MAŁGORZATA TEREPETA

ABSTRACT. Let $\mathcal{T}_\psi$ be the ψ -density topology for a fixed function ψ . We will examine some new properties of the family of ψ -continuous functions (that means continuous functions $f: \mathbb{R} \rightarrow \mathbb{R}$ with ψ -density topology $\mathcal{T}_\psi$ in its domain and range). In the second part of the article we will discuss functions preserving ψ -density points.

We will use the following notations: $\mathbb{R}$ will denote the set of real numbers, $\mathcal{L}$ – the σ -algebra of Lebesgue measurable subsets of $\mathbb{R}$, m (m^*) – the Lebesgue measure (outer measure) on $\mathbb{R}$, A' – the complement of the set $A \subset \mathbb{R}$. Let $\mathcal{C}$ be a family of nondecreasing continuous functions $\psi: (0, \infty) \rightarrow (0, \infty)$ such that $\lim_{t \rightarrow 0^+} \psi(t) = 0$.

We say that $x \in \mathbb{R}$ is a ψ -density point of a measurable set $A \subset \mathbb{R}$ if and only if

$$\lim_{h \rightarrow 0^+} \frac{m(A' \cap [x - h, x + h])}{2h\psi(2h)} = 0.$$

A point $x \in \mathbb{R}$ is said to be a ψ -dispersion point of a measurable set A if it is a ψ -density point of the complement of A .

In the definition of a ψ -density point we use a symmetric interval of the length $2h$ and with the center in the point x . As shown in [TW-B, Theorem 0.1], we cannot replace such an interval with an arbitrary interval I including x . Despite this fact, we can prove the following property.

Property 1. The point x_0 is a ψ -density point of a measurable set S if and only if

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall I \ni x_0 \quad \left(m(I) < \delta \implies \frac{m(I \setminus S)}{m(I) \psi(2m(I))} < \varepsilon \right). \quad (1)$$

P r o o f. If the condition (1) holds then it is sufficient to put $I = [x_0 - h, x_0 + h]$ and we get that x_0 is a ψ -density point of the set S . Suppose that x_0 is a ψ -density point of a measurable set S . Hence for any $\varepsilon_1 > 0$ there exists a positive number δ_1 such that for all $h \in (0, \delta_1)$ we have

$$\frac{m([x_0 - h, x_0 + h] \setminus S)}{2h\psi(2h)} < \varepsilon_1.$$

For a fixed $\varepsilon > 0$ we match δ_1 to $\varepsilon_1 = \frac{1}{2}\varepsilon$. Let I be an interval containing x_0 and having the length less than $\frac{1}{2}\delta_1$. By h we denote the least positive number such that $I \subset [x_0 - h, x_0 + h]$. Then $h \leq m(I) \leq 2h < \delta_1$ and

$$\frac{m(I \setminus S)}{m(I) \psi(2m(I))} \leq \frac{m([x_0 - h, x_0 + h] \setminus S)}{h\psi(2h)} < 2\varepsilon_1 = \varepsilon.$$

□

For any $A \in \mathcal{L}$ we denote

$$\Phi_\psi(A) = \{x \in \mathbb{R} : x \text{ is a } \psi\text{-density point of } A\}.$$

From [TW-B, Theorem 1.4], we obtain that the family $\mathcal{T}_\psi = \{A \in \mathcal{L} : A \subset \Phi_\psi(A)\}$ is a topology which is stronger than the natural topology $\mathcal{T}_o$ and weaker than the density topology $\mathcal{T}_d$. Topology $\mathcal{T}_\psi$ is invariant under multiplication if and only if for any $\alpha > 0$.

$$\limsup_{x \rightarrow 0^+} \frac{\psi(\alpha x)}{\psi(x)} < \infty. \quad (2)$$

If the condition (2) is not fulfilled for a certain $\alpha > 1$, then there is a set $A \in \mathcal{T}_\psi$ such that $\frac{1}{\alpha}A$ is not a set from $\mathcal{T}_\psi$ (compare with [TW-B, Theorem 2.8]).

Fix a function $\psi \in \mathcal{C}$. We will examine some properties of continuous functions $f : (\mathbb{R}, \mathcal{T}_\psi) \rightarrow (\mathbb{R}, \mathcal{T}_\psi)$. We will call such functions ψ -continuous and the family of them will be denoted by $\mathcal{C}_{\psi\psi}$. Each ψ -continuous function is ψ -approximately continuous, hence it is $\mathcal{DB}_1$. In [FT1] we showed that

$$\mathcal{C}_{oo} \setminus \mathcal{C}_{\psi\psi} \neq \emptyset \quad \text{and} \quad \mathcal{C}_{\psi\psi} \setminus \mathcal{C}_{oo} \neq \emptyset,$$

where $\mathcal{C}_{oo}$ is the family of continuous functions.

The idea of ψ -continuity of a function is strictly connected with the notion of a function preserving ψ -density points.

DEFINITION 2. We will say that a homeomorphism h preserves ψ -density points if for any measurable set $S \subset \mathbb{R}$ and any $x_0 \in \Phi_\psi(S)$

$$\lim_{t \rightarrow 0^+} \frac{m^*\left(\left(h(S)\right)' \cap [h(x_0) - t, h(x_0) + t]\right)}{2t\psi(2t)} = 0.$$

PROPOSITION 3. *If h is a homeomorphism preserving ψ -density points, then h satisfies Lusin's condition (N).*

Proof. Let Z be a set of Lebesgue measure zero. There exists a G_δ -set $A \supset Z$ of measure zero. Then $h(A)$ is also a G_δ -set, so it is measurable. Suppose that $m(h(A)) > 0$. Hence $h(A)$ has density 1 at a certain point $y_0 \in h(A)$, so

$$\lim_{t \rightarrow 0^+} \frac{m(h(A) \cap [y_0 - t, y_0 + t])}{2t} = 1.$$

The point $h^{-1}(y_0)$ is from A and it is a ψ -density point of the complement of A . Let $S = A'$. For any $t > 0$ such that $\psi(2t) \leq 1$, we have

$$\begin{aligned} \frac{m((h(S))' \cap [y_0 - t, y_0 + t])}{2t\psi(2t)} &= \frac{m(h(S') \cap [y_0 - t, y_0 + t])}{2t\psi(2t)} \\ &\geq \frac{m(h(A) \cap [y_0 - t, y_0 + t])}{2t}. \end{aligned}$$

Therefore, h does not preserve ψ -density points. $\square$

COROLLARY 4. *If a homeomorphism $h: [0, 1] \rightarrow [0, 1]$ preserves ψ -density points then it is an absolutely continuous function.*

From Proposition 3, it follows that if homeomorphism h preserves ψ -density points then, for any measurable set $S \subset \mathbb{R}$, $h(S)$ is a measurable set and we need not use the outer measure from Definition 2.

THEOREM 5. *A homeomorphism $h: \mathbb{R} \rightarrow \mathbb{R}$ preserves ψ -density points if and only if h^{-1} is a ψ -continuous function.*

Proof. First, we assume that h preserves ψ -density points. We will show that h^{-1} is a ψ -continuous function at any point. Fix a point y_0 and a set $V \in \mathcal{T}_\psi$ such that $x_0 = h^{-1}(y_0) \in V$. We will show that there exists a set $U \in \mathcal{T}_\psi$ such that $y_0 \in U$ and $h^{-1}(U) \subset V$. Since V is open in $\mathcal{T}_\psi$, for any $x \in V$ we have $x \in \Phi_\psi(V)$. The homeomorphism preserves ψ -density points, so $h(x) \in \Phi_\psi(h(V))$. Hence $h(V)$ is open in $\mathcal{T}_\psi$ and putting $U = h(V)$, we complete the proof of this implication.

Suppose now that h does not preserve ψ -density points. Fix a point $x_0 \in \mathbb{R}$ and a measurable set S such that x_0 is a ψ -dispersion point of S and

$$\limsup_{t \rightarrow 0^+} \frac{m^*(h(S) \cap [h(x_0) - t, h(x_0) + t])}{2t\psi(2t)} > 0.$$

Let $A \supset S$ be a G_δ -set such that $m(A \setminus S) = 0$. Then x_0 is a ψ -dispersion point of A , $h(A)$ is measurable and $h(S) \subset h(A)$. Hence, there exists a sequence

$(a_n)_{n \in \mathbb{N}}$ decreasing to 0 and a number $\alpha > 0$ such that

$$\frac{m\left(h(A) \cap [h(x_0) - a_n, h(x_0) + a_n]\right)}{2a_n\psi(2a_n)} > \alpha$$

for all $n \in \mathbb{N}$. Therefore, there exists a subsequence $(a_{n_k})_{k \in \mathbb{N}}$ of the sequence $(a_n)_{n \in \mathbb{N}}$ such that for any natural k

$$\frac{m\left(h(A) \cap [h(x_0), h(x_0) + a_{n_k}]\right)}{2a_{n_k}\psi(2a_{n_k})} > \frac{\alpha}{2}$$

or

$$\frac{m\left(h(A) \cap [h(x_0) - a_{n_k}, h(x_0)]\right)}{2a_{n_k}\psi(2a_{n_k})} > \frac{\alpha}{2}, \quad (3)$$

For simplicity, we can assume that inequality (3) holds for all elements of the sequence $(a_n)_{n \in \mathbb{N}}$ and for any natural n we have

$$m\left(h(A) \cap [h(x_0) + a_{n+1}, h(x_0) + a_n]\right) > 0.$$

From the properties of measurable sets it follows that for any $n \in \mathbb{N}$ there exists a closed set $B_n \subset h(A) \cap [h(x_0) + a_{n+1}, h(x_0) + a_n]$ such that

$$m(B_n) > m\left(h(A) \cap [h(x_0) + a_{n+1}, h(x_0) + a_n]\right) - \frac{\alpha}{4} \cdot \frac{1}{2^n} \cdot 2a_n\psi(2a_n).$$

We put $B = \bigcup_{n=1}^{\infty} B_n \cup \{h(x_0)\}$. Observe that B is a closed set in natural topology. Moreover, for any natural i

$$\begin{aligned} \frac{m\left(B \cap [h(x_0), h(x_0) + a_i]\right)}{2a_i\psi(2a_i)} &= \frac{\sum_{n=i}^{\infty} m(B_n)}{2a_i\psi(2a_i)} \geq \frac{1}{2a_i\psi(2a_i)} \\ &\times \left(\sum_{n=i}^{\infty} m\left(h(A) \cap [h(x_0) + a_{n+1}, h(x_0) + a_n]\right) - \frac{\alpha}{4} \sum_{n=i}^{\infty} 2a_n\psi(2a_n) \frac{1}{2^n} \right) \\ &\geq \frac{m\left(h(A) \cap [h(x_0), h(x_0) + a_i]\right)}{2a_i\psi(2a_i)} - \frac{\frac{\alpha}{4} \cdot 2a_i\psi(2a_i) \sum_{n=i}^{\infty} \frac{1}{2^n}}{2a_i\psi(2a_i)} \geq \frac{\alpha}{4} > 0. \end{aligned}$$

Hence, $h(x_0)$ is not a ψ -dispersion point of the set B .

Since $h^{-1}(B) \subset A$, x_0 is a ψ -dispersion point of $h^{-1}(B)$. Moreover, $h^{-1}(B)$ is closed in natural topology, hence the set $C = \mathbb{R} \setminus h^{-1}(B) \cup \{x_0\} \in \mathcal{T}_\psi$. Simultaneously, $h(C) = \mathbb{R} \setminus B \cup \{h(x_0)\} \notin \mathcal{T}_\psi$, so the function h^{-1} is not ψ -continuous. $\square$

Property 6. If a homeomorphism $h: \mathbb{R} \rightarrow \mathbb{R}$ satisfies the condition

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall A \in \mathcal{L} \quad \forall I \supset A$$

$$\left(m(A) < \delta \, m(I) \, \psi(2m(I)) \implies m(h(A)) < \varepsilon \, m(h(I)) \, \psi(2m(h(I))) \right), \quad (4)$$

then h preserves ψ -density points.

Proof. Suppose that x_0 is a ψ -density point of a set $S \in \mathcal{L}$. Using Property 1, we will show that for any $\varepsilon > 0$ there exists a positive number $\gamma > 0$ such that for each interval $J \ni h(x_0)$ of the length less than γ

$$\frac{m(J \setminus h(S))}{m(J) \, \psi(2m(J))} < \varepsilon.$$

Let $\varepsilon > 0$ and choose δ from condition (4). Since $x_0 \in \Phi_\psi(S)$, from Property 1 it follows that there exists $\lambda > 0$ such that for any $I \ni x_0$ of length less than λ

$$\frac{m(I \setminus S)}{m(I) \, \psi(2m(I))} < \delta. \quad (5)$$

Since h^{-1} is continuous at $h(x_0)$, there is $\gamma > 0$ such that for any interval $J \ni h(x_0)$ of length less than γ , the length of the interval $h^{-1}(J)$ is less than λ .

Let J be an interval containing $h(x_0)$, of the length less than γ . Using (5), we have

$$m(h^{-1}(J) \setminus S) < \delta \, m(h^{-1}(J)) \, \psi(2m(h^{-1}(J))).$$

From condition (4), taking $A = h^{-1}(J) \setminus S$ and $I = h^{-1}(J)$, we obtain

$$m(J \setminus h(S)) = m(h(h^{-1}(J) \setminus S)) < \varepsilon \, m(J) \, \psi(2m(J)).$$

□

THEOREM 7. Let a function $\psi \in \mathcal{C}$ satisfy the condition (2). Suppose that $h: \mathbb{R} \rightarrow \mathbb{R}$ is an absolutely continuous homeomorphism such that for almost all $x \in \mathbb{R}$

$$0 < \alpha \leq h'(x) \leq \beta < \infty. \quad (6)$$

Then h preserves ψ -density points.

Proof. Since h is an increasing function, it is sufficient to show that if x_0 is a right-hand ψ -density point of a measurable set S , then $h(x_0)$ is a right-hand ψ -density point of a set $h(S)$. The left-hand side case can be shown in the similar way. Let S be a measurable set and $\varepsilon > 0$. We assume that x_0 is a right-hand ψ -density point of S . The function ψ satisfies the condition (2), so there exist numbers $M > 0$ and $\delta_1 > 0$ such that

$$\frac{\psi(t)}{\psi(\alpha t)} < M \quad (7)$$

for all $t < \delta_1$. The point x_0 is a right-hand ψ -density point of the set S , hence there exists a positive number $\delta_2 < \frac{1}{2}\delta_1$ such that for all $t \in (0, \delta_2)$

$$\frac{m([x_0, x_0 + t] \setminus S)}{2t\psi(2t)} < \frac{\varepsilon\alpha}{\beta M}. \quad (8)$$

The fact that h^{-1} is continuous at the point $h(x_0)$ results in the existence of $\delta_3 > 0$ such that $h^{-1}(y) < x_0 + \delta_2$ for any $y \in (h(x_0), h(x_0) + \delta_3)$. To complete the proof it is sufficient to show that for any interval of the form $J = [h(x_0), h(x_0) + y]$, where $y < \delta_3$, we have

$$\frac{m(h(S) \setminus J)}{2m(J)\psi(2m(J))} < \varepsilon.$$

The function h is a homeomorphism, so there exists an interval I which has the following properties: its length is less than δ_2 , so x_0 is the endpoint of the interval $I = h^{-1}(J)$. We have $m(I) < \delta_2$. From the invertibility of h we obtain

$$m(h(S) \setminus J) = m(h(S) \setminus h(I)) = m(h(S \setminus I)).$$

From the absolute continuity of h and (6) we have

$$\alpha m(I) \leq m(h(I)) \leq \beta \cdot m(I)$$

and

$$m(h(S \setminus I)) \leq \beta m(S \setminus I).$$

From this, (7) and (8) it follows that

$$\frac{m(h(S) \setminus J)}{2m(J)\psi(2m(J))} \leq \frac{\beta m(S \setminus I)}{2\alpha m(I)\psi(2\alpha m(I))} = \frac{m(S \setminus I)}{2m(I)\psi(2m(I))} \frac{\beta}{\alpha} \frac{\psi(2m(I))}{\psi(2\alpha m(I))} < \varepsilon.$$

□

COROLLARY 8. *Suppose that $h: \mathbb{R} \rightarrow \mathbb{R}$ is an absolutely continuous homeomorphism such that for almost all $x \in \mathbb{R}$*

$$1 \leq \alpha \leq h'(x) \leq \beta < \infty. \quad (9)$$

Then h preserves ψ -density points for arbitrary function $\psi \in \mathcal{C}$.

We follow the proof of Theorem 7. Notice that if $\alpha \geq 1$ and $\psi \in \mathcal{C}$, then for each $t > 0$ we have

$$\frac{\psi(t)}{\psi(\alpha t)} \leq 1.$$

□

Remark 9. Observe that if a function $\psi \in \mathcal{C}$ does not satisfy the condition (2) then for any $\alpha > 1$

$$\limsup_{x \rightarrow 0^+} \frac{\psi(\alpha x)}{\psi(x)} = \infty.$$

It means that for any $\alpha > 1$ there exists a set $A \in \mathcal{T}_\psi$ such that $\frac{1}{\alpha}A \notin \mathcal{T}_\psi$. Therefore any linear function $f(x) = ax$, with $a > 1$, is not ψ -continuous. By Corollary 8, such the functions preserve ψ -density points.

COROLLARY 10. *Let $f: I \rightarrow \mathbb{R}$. If there are numbers α, β such that*

$$0 < \alpha < \frac{f(x) - f(y)}{x - y} < \beta < \infty \quad (10)$$

for any $x, y \in I$, $x \neq y$, then f is ψ -continuous for each function $\psi \in \mathcal{C}$ satisfying (2).

Moreover, if $\beta \leq 1$, then f is ψ -continuous for any function $\psi \in \mathcal{C}$.

PROOF. From (10) the function f is strictly monotonic and continuous, so $f: I \rightarrow f(I)$ has the inverse function $g = f^{-1}$. This function satisfies the condition

$$0 < \frac{1}{\beta} < \frac{g(u) - g(v)}{u - v} < \frac{1}{\alpha} < \infty$$

for all $u, v \in f(I)$, $u \neq v$. As it fulfils the local Lipschitz condition on $f(I)$, so g is absolutely continuous. Its derivative g' is bounded from above and below by positive numbers, hence g preserves ψ -density points (Theorem 7). From Theorem 5 the function $g^{-1} = f$ is ψ -continuous.

If $\beta \leq 1$, then for each function $\psi \in \mathcal{C}$, the function g preserves ψ -density points (we use Corollary 8), whence f is ψ -continuous. $\square$

COROLLARY 11. *Let $\psi \in \mathcal{C}$ fulfil the condition (2). If a function $h: \mathbb{R} \rightarrow \mathbb{R}$ is a homeomorphism and h, h^{-1} satisfy the local Lipschitz condition, then h, h^{-1} are ψ -continuous.*

It is well-known fact that every function $f(x) = x^\tau$ on the interval $[0, \infty)$, where $\tau > 0$, is density continuous. We will examine, whether such functions are ψ -continuous.

THEOREM 12. *Consider the function $f: [0, \infty) \ni x \mapsto x^\tau$, where $\tau > 0$. If*

$$\liminf_{x \rightarrow 0^+} \frac{\psi(2x)}{\psi(2f(x))} > 0, \quad (11)$$

then f is ψ -continuous at $x = 0$ for each $\psi \in \mathcal{C}$.

PROOF. We will show that if 0 is not a right-hand ψ -dispersion point of a measurable set A , then $f(0) = 0$ is not a right-hand ψ -dispersion point of $f(A)$. Suppose that there exists a number $\alpha > 0$ and a sequence $(y_n)_{n \in \mathbb{N}}$ decreasing to zero such that for all natural n

$$\frac{m(A \cap [0, y_n])}{2y_n \psi(2y_n)} > \alpha. \quad (12)$$

Consider the interval $[0, y_1]$. Let $x_k^{(1)} = \frac{y_1}{2^k}$ for $k = 0, 1, 2, \dots$. Notice that there exists a number k_0 such that

$$m\left(A \cap \left[\frac{1}{2}x_{k_0}^{(1)}, x_{k_0}^{(1)}\right]\right) > x_{k_0}^{(1)} \alpha \psi(2y_1). \quad (13)$$

Indeed, if for all k

$$m\left(A \cap \left[\frac{1}{2}x_k^{(1)}, x_k^{(1)}\right]\right) \leq x_k^{(1)} \alpha \psi(2y_1),$$

then

$$\frac{m(A \cap [0, y_1])}{2y_1 \psi(2y_1)} = \frac{\sum_{k=0}^{\infty} m\left(A \cap \left[\frac{1}{2}x_k^{(1)}, x_k^{(1)}\right]\right)}{2y_1 \psi(2y_1)} \leq \frac{\alpha \psi(2y_1) \sum_{k=0}^{\infty} x_k^{(1)}}{2y_1 \psi(2y_1)} \leq \alpha,$$

contrary to (12). We put $a_1 = x_{k_0}^{(1)}$ and obtain

$$\frac{m\left(A \cap \left[\frac{1}{2}a_1, a_1\right]\right)}{2a_1 \psi(2a_1)} \geq \frac{m\left(A \cap \left[\frac{1}{2}a_1, a_1\right]\right)}{2a_1 \psi(2y_1)} > \frac{\alpha}{2}.$$

There exists $n_2 \in \mathbb{N}$ such that $y_{n_2} < a_1$. Analogously, we can find $a_2 \in (0, y_{n_2}]$ such that

$$\frac{m\left(A \cap \left[\frac{1}{2}a_2, a_2\right]\right)}{2a_2 \psi(2a_2)} > \frac{\alpha}{2}.$$

By induction we define the sequence $(a_n)_{n \in \mathbb{N}}$ decreasing to zero such that for all $n \in \mathbb{N}$ we have

$$\frac{m\left(A \cap \left[\frac{1}{2}a_n, a_n\right]\right)}{2a_n \psi(2a_n)} > \frac{\alpha}{2}.$$

Notice that for any $n \in \mathbb{N}$ the derivative of the function f on the interval $[\frac{1}{2}a_n, a_n]$ fulfils the condition

$$f'(x) \geq \frac{\tau a_n^{\tau-1}}{2^{\tau-1}},$$

so

$$m\left(f\left(A \cap \left[\frac{1}{2}a_n, a_n\right]\right)\right) \geq \frac{\tau a_n^{\tau-1}}{2^{\tau-1}} \cdot m\left(A \cap \left[\frac{1}{2}a_n, a_n\right]\right).$$

Let us estimate the following expression

$$\begin{aligned}
 \frac{m\left(f(A) \cap [0, a_n^\tau]\right)}{2a_n^\tau \psi(2a_n^\tau)} &= \frac{m\left(f(A) \cap f([0, a_n])\right)}{2a_n^\tau \psi(2a_n^\tau)} \\
 &= \frac{m\left(f(A \cap [0, a_n])\right)}{2a_n^\tau \psi(2a_n^\tau)} \\
 &\geq \frac{m\left(f(A \cap [\frac{1}{2}a_n, a_n])\right)}{2a_n^\tau \psi(2f(a_n))} \\
 &\geq \frac{\tau a_n^{\tau-1}}{2^{\tau-1}} \cdot \frac{m\left(A \cap [\frac{1}{2}a_n, a_n]\right)}{2a_n^\tau \psi(2f(a_n))} \\
 &\geq \frac{\tau}{2^{\tau-1}} \cdot \frac{m\left(A \cap [\frac{1}{2}a_n, a_n]\right)}{2a_n \psi(2a_n)} \cdot \frac{\psi(2a_n)}{\psi(2f(a_n))} \\
 &> \frac{\tau}{2^{\tau-1}} \cdot \frac{\alpha}{2} \cdot \frac{\psi(2a_n)}{\psi(2f(a_n))}.
 \end{aligned}$$

Since

$$\liminf_{n \rightarrow \infty} \frac{\psi(2a_n)}{\psi(2f(a_n))} > 0,$$

hence

$$\limsup_{x \rightarrow 0^+} \frac{m(f(A) \cap [0, x])}{2x \psi(2x)} > 0.$$

We obtain that 0 is not a ψ -dispersion point of the set $f(A)$. We have shown that if 0 is a ψ -dispersion point of the set $f(A)$, then 0 is a ψ -dispersion point of the set A . Hence f is ψ -continuous at 0. $\square$

COROLLARY 13. *Let $\psi \in \mathcal{C}$. Consider the function $f: [0, \infty) \ni x \mapsto x^\tau$, where $\tau \geq 1$. Then f is ψ -continuous on $[0, \tau^{-1}/(\tau-1)]$.*

In fact, for any $x \in (0, \tau^{-1}/(\tau-1)]$ for the derivative $f'(x)$ we have $0 < f'(x) \leq 1$, so, from Corollary 8, f is ψ -continuous at x . Continuity at 0 follows from the last theorem.

Remark 14. If ψ satisfies (2), then $f(x) = x^\tau$, where $\tau \geq 1$, is ψ -continuous on $[0, \infty)$.

THEOREM 15. *Let $\tau \in (0, 1)$ and $f(x) = x^\tau$ for $x \in [0, \infty)$. If*

$$\lim_{x \rightarrow 0^+} \frac{\psi(2x)}{\psi(2f(x))} = 0, \tag{14}$$

then f is not ψ -continuous at $x = 0$.

Proof. We will construct an interval set

$$A = \bigcup_{n=1}^{\infty} [a_n, b_n], \quad 0 < a_n < b_n < a_{n-1}, \quad \lim_{n \rightarrow \infty} b_n = 0$$

such that 0 is a ψ -dispersion point of the set A and 0 is not a ψ -dispersion point of the set $f^{-1}(A)$.

Let c_1 be a point from $(0, 1)$ such that $\psi(2c_1) < \frac{1}{2}$. From the assumption of a function ψ , there exists $d_1 \in (0, c_1)$ such that for any $x \in (0, d_1]$

$$\frac{\psi(2x)}{\psi(2f^{-1}(x))} > \frac{1}{c_1}.$$

Since

$$\frac{d_1}{2d_1\psi(2d_1)} \geq \frac{1}{2\psi(2c_1)} > 1 > c_1,$$

there exists a point a_1 such that

$$\frac{m([a_1, d_1])}{2d_1\psi(2d_1)} = c_1.$$

Let us consider the function

$$f_1: [a_1, c_1] \ni x \mapsto \frac{m([a_1, x])}{2x\psi(2x)}.$$

It is a continuous function attaining the value c_1 . The set $f_1^{-1}(\{c_1\})$ is closed in natural topology and not empty. Let $b_1 = \min f_1^{-1}(\{c_1\})$. Notice that for any $x \in (a_1, b_1)$,

$$\frac{m([a_1, x])}{2x\psi(2x)} < c_1 \quad \text{and} \quad \frac{m([a_1, b_1])}{2b_1\psi(2b_1)} = c_1.$$

Let c_2 be a point from $(0, a_1)$ such that

$$\frac{c_2}{2a_1\psi(2a_1)} < c_1.$$

In an analogous way we can find $0 < a_2 < b_2 < d_2 < a_1$ such that

$$\frac{\psi(2x)}{\psi(2f^{-1}(x))} > \frac{1}{c_2}; \quad \text{for each } x \in (0, d_2],$$

$$\frac{m([a_2, b_2])}{2b_2\psi(2b_2)} = c_2 \quad \text{and} \quad \frac{m([a_2, x])}{2x\psi(2x)} < c_2 \quad \text{for } x \in (a_2, b_2).$$

By induction we construct the sequence of intervals $([a_n, b_n])_{n \in \mathbb{N}}$ and the decreasing sequence $(c_n)_{n \in \mathbb{N}}$ which have the following properties:

- (1) $0 < a_n < b_n \leq c_n$,
- (2) $\frac{m([a_n, b_n])}{2b_n\psi(2b_n)} = c_n$,

- (3) $\frac{\psi(2b_n)}{\psi(2f^{-1}(b_n))} > \frac{1}{c_n}$,
 (4) $\frac{m([a_n, x])}{2x \psi(2x)} < c_n$ for $x \in (a_n, b_n)$,
 (5) $\frac{c_{n+1}}{2a_n \psi(2a_n)} < c_n$.

The point 0 is a ψ -dispersion point of the set $A = \bigcup_{n=1}^{\infty} [a_n, b_n]$. Indeed, let us take $x \in (0, b_1)$. There exists a natural number n such that $x \in (b_{n+1}, b_n]$. If $x \in (b_{n+1}, a_n]$, then

$$\begin{aligned} \frac{m(A \cap [0, x])}{2x \psi(2x)} &= \frac{m(A \cap [0, b_{n+1}])}{2x \psi(2x)} \\ &\leq \frac{m([a_{n+1}, b_{n+1}])}{2b_{n+1} \psi(2b_{n+1})} + \frac{b_{n+2}}{2b_{n+1} \psi(2b_{n+1})} \leq 2c_{n+1}. \end{aligned}$$

If $x \in (a_n, b_n]$, then

$$\begin{aligned} \frac{m(A \cap [0, x])}{2x \psi(2x)} &\leq \frac{b_{n+1} + m([a_n, x])}{2x \psi(2x)} \\ &\leq \frac{b_{n+1}}{2a_n \psi(2a_n)} + \frac{m([a_n, x])}{2x \psi(2x)} < 2c_n. \end{aligned}$$

For $n \rightarrow \infty$ the sequence $(c_n)_{n \in \mathbb{N}}$ tends to zero, so $0 \in \Phi_{\psi}(A')$. For completeness of the proof it remains to show that 0 is not a ψ -dispersion point of the set $f^{-1}(A)$. Denote, for the simplicity, by α the number $\frac{1}{\tau}$ and observe that

$$f^{-1}(A) = \bigcup_{n=1}^{\infty} [a_n^{\alpha}, b_n^{\alpha}].$$

For any $b > a > 0$

$$b^{\alpha} - a^{\alpha} > b^{\alpha} - ab^{\alpha-1} = (b-a)b^{\alpha-1}.$$

Therefore, for any $n \in \mathbb{N}$, we have

$$\begin{aligned} \frac{m(f^{-1}(A) \cap [0, b_n^{\alpha}])}{2b_n^{\alpha} \psi(2b_n^{\alpha})} &\geq \frac{b_n^{\alpha} - a_n^{\alpha}}{2b_n^{\alpha} \psi(2b_n^{\alpha})} \\ &> \frac{b_n - a_n}{2b_n \psi(2b_n)} \cdot \frac{\psi(2b_n)}{\psi(2b_n^{\alpha})} \\ &= c_n \cdot \frac{\psi(2b_n)}{\psi(2f^{-1}(b_n))} > 1. \end{aligned}$$

□

COROLLARY 16. *Functions $f(x) = x^{\tau}$ for $\tau \in (0, 1)$ are not ψ -continuous at zero for $\psi = \text{id}$. In particular, the function $f(x) = x^2$ is ψ -continuous and its inverse function is not ψ -continuous for $\psi = \text{id}$.*

Remark 17. From Theorem 12 it follows that the function $f(x) = x^\tau$ for $\tau \in (0, 1)$ is ψ -continuous at zero if ψ satisfies (11). It is not ψ -continuous at zero if ψ satisfies (14). A natural question arises whether such a function is ψ -continuous at zero if ψ satisfies $\liminf_{x \rightarrow 0^+} \frac{\psi(2x)}{\psi(2f(x))} = 0$. Another natural question is if $f(x) = x^\tau$, where $\tau \geq 1$, is not ψ -continuous on $[\tau^{-1/(\tau-1)}, \infty)$ for ψ which does not satisfy the condition (2).

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Małgorzata Filipczak
Institute of Mathematics
Łódź University
ul. Stefana Banacha 22
PL-90-238 Łódź
POLAND
E-mail: malfil@math.uni.lodz.pl

Małgorzata Terepeta
Center of Mathematics and Physics
Łódź Technical University
al. Politechniki 11
PL-90-924 Łódź
POLAND
E-mail: ttrp@poczta.onet.pl