ON THE GENERALIZED VARIATION AND GENERALIZED WEAK VARIATION OF MAPS WITH VALUES IN METRIC LINEAR SPACES

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ABSTRACT. In this paper, the class of maps (with values in a metric linear space) of a bounded generalized variation (bounded generalized weak variation) is described. Connections between those kinds of spaces are investigated.

1. Introduction

In the literature one can find many generalizations of a well-known notion of variation of a real valued function of one variable. For example, D. W a t e r m a n introduced an idea of the function of bounded λ -variation in [5], an additive map defined on σ -algebra of subsets of X was considered by S. S a k s in [8] and B. T h o m s o n studied the class of interval functions of bounded S -variation [9]. Another kind generalizations is φ -variation of a map having values in a metric space and defined on a subset of $\mathbb{R}$; in particular, it was considered in [2]. One can also find generalizations of ideas introduced by D. W a t e r m a n and B. T h o m s o n in [3].

In this paper we will introduce a concept of generalized variation in the restricted sense (weak generalized variation in the restricted sense) for a map defined on a nonempty subset E of $\mathbb{R}$ and having values in a metric space (X, d) , which is a more general notion to those considered in [1].

If (X, d) is a metric space then $B^\circ(x, \varepsilon)$, $\overline{B}(x, \varepsilon)$ will denote the open ball with the center x from X and positive radius ε , and the closed ball with the center x in X and positive radius $\varepsilon > 0$, respectively. Under additional assumption of X that is a linear space over a field Φ with operation of addition “+” and multiplying by scalars “ $\cdot$ ”, it is said that d is a translation invariant metric if $d(x+z, y+z) = d(x, y)$ for each x, y and z from X (terminology is taken from [7]).

Let us emphasize that any linear space will always be considered over the field of real or complex numbers, i.e., $\Phi = \mathbb{R}$ or $\Phi = \mathbb{C}$. Moreover, if X is a linear space over Φ , then for a subset A and a subset B of X , the symbol $A + B$ and λA ($\lambda \in \Phi$) will stand to denote the sets of the form:

$$A + B = \{a + b : a \in A \wedge b \in B\}$$

and

$$\lambda A = \{\lambda a : a \in A\}.$$

If $f: E \rightarrow X$ is a map, where E is a nonempty subset of $\mathbb{R}$ and (X, d) is a metric space, then, for a subset A of E , the quantity

$$\sup \left\{ (f(x), f(y)) : x \in A, y \in A \right\}$$

will be said to be d -oscillation of f on A and will be denoted as $\omega_d(f, A)$. In a special case, if $A = \emptyset$, we write $\omega_d(f, A) = 0$.

Throughout this paper, the symbol $\mathcal{R}(E)$ will denote the class of all finite collections of closed and nonoverlapping intervals $\{I_1, \dots, I_n\}$ such that

$$I_i \cap E \neq \emptyset \quad \text{if} \quad i \in \{1, \dots, n\}$$

and

$$\bigcup_{i=1}^n (I_i \cap E) = E.$$

DEFINITION 1.1. Let (X, d) be a metric space and E be a nonempty subset of $\mathbb{R}$. It will be said that a map $f: E \rightarrow X$ has a bounded variation in the restricted sense if the quantity

$$\sup \left\{ \sum_{i=1}^n \omega_d(f, I_i \cap E) : \{I_i : i = 1, \dots, n\} \in \mathcal{R}(E) \right\}$$

is finite. The generalized variation in the restricted sense of f on E will be denoted by $V_d(f, E)$.

We say that a map $f: E \rightarrow X$ has bounded generalized variation in the restricted sense if E can be represented as a countable union of closed sets $\{E_i : i \in \mathbb{N}\}$ such that $V_d(f, E_i) < \infty$ for each $i \in \mathbb{N}$.

DEFINITION 1.2. Let (X, d) be a linear topological space, X^* be dual space to X , and Γ be a nonempty subfamily of X^* . It is said that a map $f: E \rightarrow X$ has a bounded weak variation in the restricted sense on E with respect to Γ , if for each Λ from Γ , the composition $\Lambda \circ f$ has a bounded variation in the restricted sense on E .

A map $f: E \rightarrow X$ has a bounded generalized weak variation in the restricted sense with respect to Γ , if for each $\Lambda \in \Gamma$, the composite function $\Lambda \circ f$ has a bounded generalized variation in the restricted sense on E .

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From now on, we will use the following denotations:

- (1) $dVBG$ – the class of maps of a bounded variation in the restricted sense on E with respect to the metric d in X ;
- (2) $dVBG_*$ – the class of maps of a bounded generalized variation in the restricted sense on E with respect to the metric d in X ;
- (3) $\Gamma WBG(X)$ – the class of maps of a bounded weak variation in the restricted sense on E with respect to the subfamily Γ of X^* ;
- (4) $\Gamma WBG_*(X)$ – the class of maps of a bounded generalized weak variation in the restricted sense on E with respect to the subfamily Γ of X^* .

Let us remark that if $X = \mathbb{R}$ and d_n is Euclidean metric in X , then $d_n VBG$, $d_n VBG_*$ are the classes of functions of a bounded variation in the restricted sense on E , and the classes of functions of a bounded generalized variation in the restricted sense on E , respectively [1]. Instead of denotations $d_n VBG$, $d_n VBG_*$, we will use the symbols VBG , VBG_* , respectively.

2. General properties

At the beginning, some basic properties of maps of a bounded generalized variation in the restricted sense and a bounded generalized weak variation in the restricted sense will be proved. It is not difficult to see that, if (X, d) is a metric space, $E_1 \subset E$ and E_1, E are subsets of $\mathbb{R}$, then for a map $f: E \rightarrow \mathbb{R}$ the inequality

$$V_d(f, E_1) \leq V_d(f, E)$$

holds. To simplify our considerations, from now on, it will be always assumed that X is a real or complex metric linear space [6].

THEOREM 2.1. *Let (X, d) be a metric space, where d is a translation invariant metric. If $f_1, f_2 \in dVBG$, then $f_1 + f_2 \in dVBG$.*

Proof. Let us assume that f_1, f_2 belong to $dVBG$ and take $\{I_1, \dots, I_n\}$ from $\mathcal{R}(E)$. For each x, y from $I_i \cap E$, we have

$$\begin{aligned} d((f_1 + f_2)(x), (f_1 + f_2)(y)) &= d(f_1(x) + f_2(x), f_1(y) + f_2(y)) \\ &= d(f_1(x) - f_1(y), f_2(x) - f_2(y)) \\ &\leq d(f_1(x) - f_1(y), 0) + d(f_2(x) - f_2(y), 0) \\ &= d(f_1(x), f_1(y)) + d(f_2(x), f_2(y)) \\ &\leq \omega_d(f_1, I_i \cap E) + \omega_d(f_2, I_i \cap E), \end{aligned}$$

whence

$$\omega_d(f_1 + f_2, I_i \cap E) \leq \omega_d(f_1, I_i \cap E) + \omega_d(f_2, I_i \cap E)$$

if $i \in \{1, \dots, n\}$ and, consequently,

$$V_d(f_1 + f_2, E) \leq V_d(f_1, E) + V_d(f_2, E).$$

From the assumption and the above inequality, we infer that $f_1 + f_2 \in dVBG$. $\square$

A subset E of a metric space is called to be bounded if the diameter of E is finite. Apart from that, in the literature also there appears a definition of bounded subsets in linear topological spaces. According to [7], a subset E of linear topological space is said to be bounded if for each neighborhood V of 0 from X there exists a real positive number s such that $E \subset tV$ for each $t > s$.

To distinguish those two notions, we say that a set E is bounded in the usual sense if it has the finite diameter (in metric space) and a subset E is bounded if it is bounded in the sense given in [7].

Let us remark that in the space Φ with the Euclidean metric definitions of “bounded subset in the usual sense” and “bounded subset” coincide.

Similarly, a map f from E into a metric linear space X is said to be bounded in the usual sense if $f(E)$ is bounded in the usual sense and it is called to be bounded if $f(E)$ is bounded. Moreover, if X is a linear topological space, then a map $f: E \rightarrow X$ is said to be weakly bounded, if for each $\Lambda \in X^*$, the set $\Lambda(f(E))$ is bounded.

THEOREM 2.2. *If (X, d) is a metric space, $E \subset \mathbb{R}$ and $f \in dVBG$, then f is bounded in the usual sense.*

Proof. There exists a partition $\{I_1, \dots, I_n\}$ from $\mathcal{R}(E)$ such that

$$\sum_{i=1}^n \omega_d(f, I_i \cap E) < V_d(f, E) + 1.$$

Without loss of generality, one can assume that

$$\sup\{x : x \in I_i\} \leq \inf\{x : x \in I_{i+1}\},$$

where $i \in \{1, \dots, n-1\}$. If x and y belong to E , then $x \in I_i \cap E$, $y \in I_j \cap E$ for some $i, j \in \{1, \dots, n\}$. If $i \leq j$, then:

$$\begin{aligned} d(f(x), f(y)) &\leq d(f(x), f(x_{i+1})) + d(f(x_{i+1}), f(x_{i+2})) + \dots \\ &\quad \dots + d(f(x_{j-2}), f(x_{j-1})) + d(f(x_{j-1}), f(y)) \\ &\leq \omega_d(f, I_i \cap E) + \omega_d(f, I_{i+1} \cap E) + \dots \\ &\quad \dots + \omega_d(f, I_{j-1} \cap E) + \omega_d(f, I_j \cap E) \\ &\leq V_d(f, E) + 1. \end{aligned}$$

Similarly, one can prove that

$$d(f(x), f(y)) \leq V_d(f, E) + 1 \quad \text{if } i \geq j.$$

Consequently, the diameter of $f(E)$ is finite since x, y have been chosen in arbitrary way, which means that f is bounded in the usual sense. $\square$

THEOREM 2.3. *Let us assume that:*

- (1) (X, d) is a metric space and d is a translation invariant metric in X ;
- (2) if U is a bounded subset in the usual sense of X , then there exists a positive real number M_U such that for each x from U and each $|\beta| \leq 1$ from Φ the inequality $d(\beta x, 0) \leq M_U d(x, 0)$ holds.

Then $\alpha f \in dVBG$ for each α from Φ and each $f \in dVBG$.

PROOF. Let $f \in dVBG$. Then $f(E) - f(E)$ is a bounded subset of X in the usual sense, since, by previous theorem, $f(E)$ is a bounded subset of X in the usual sense. Let us consider a closed interval I such that $I \cap E \neq \emptyset$ and α from Φ , where $\alpha = u + iv$ ($u, v \in \mathbb{R}$). If x, y belong to $I \cap E$, then

$$\begin{aligned}
 d((\alpha f)(x), (\alpha f)(y)) &= d(uf(x) + ivf(x), uf(y) + ivf(y)) \\
 &= d(uf(x) - uf(y), ivf(y) - ivf(x)) \\
 &\leq d(u(f(x) - f(y)), 0) + d(0, iv(f(y) - f(x))) \\
 &\leq d([u](f(x) - f(y)), 0) + d((u - [u])(f(x) - f(y)), 0) \\
 &\quad + d(i[v](f(y) - f(x)), 0) + d(i(v - [v])(f(y) - f(x)), 0) \\
 &\leq |[u]|d(f(x), f(y)) + Md(f(x), f(y)) \\
 &\quad + M|[v]|d(f(x), f(y)) + Md(f(x), f(y)) \\
 &\leq (|[u]| + M(2 + |[v]|))\omega_d(f, I \cap E),
 \end{aligned}$$

where the symbol $[x]$ denotes *entire* of x .

It follows that for a partition $\{I_1, \dots, I_n\}$ from $\mathcal{R}(E)$ we have

$$\begin{aligned}
 \sum_{i=1}^n \omega_d(\alpha f, I_i \cap E) &\leq (|[u]| + M(2 + |[v]|)) \cdot \sum_{i=1}^n \omega_d(f, I_i \cap E) \\
 &\leq (|[u]| + M(2 + |[v]|)) \cdot V_d(f, E),
 \end{aligned}$$

whence

$$V_d(\alpha f, E) \leq (|[u]| + M(2 + |[v]|)) \cdot V_d(f, E).$$

It means that $\alpha f \in dVBG$. $\square$

The following example shows that the condition (2) of the above theorem is essential.

EXAMPLE 2.1. Let

$$E_1 = \left\{ \frac{1}{2^{2n}} : n \in \mathbb{N} \right\} \quad \text{and} \quad E_2 = \left\{ \frac{1}{2^{2n-1}} : n \in \mathbb{N} \right\}.$$

Let $g: E_1 \rightarrow \mathbb{R}$ satisfy the following conditions:

- (1) $g\left(\frac{1}{2^{2n}}\right) > g\left(\frac{1}{2^{2n+2}}\right) > 0$ for all $n \in \mathbb{N}$;
- (2) $\lim_{n \rightarrow \infty} g\left(\frac{1}{2^{2n}}\right) = 0$;
- (3) g has bounded variation in the restricted sense on E_1 .

Let $h: E_2 \rightarrow \mathbb{R}$ satisfy the following conditions:

- (1) $0 < h\left(\frac{1}{2^{2n-1}}\right) < h\left(\frac{1}{2^{2n+1}}\right)$ for all $n \in \mathbb{N}$;
- (2) $\lim_{n \rightarrow \infty} h\left(\frac{1}{2^{2n-1}}\right) = \infty$;
- (3) h does not have a bounded variation in the restricted on E_2 .

Additionally, let us assume that

$$\{g(x) : x \in E_1\} \cap \{h(x) : x \in E_2\} = \emptyset.$$

Lastly, we define a function $u: \mathbb{R} \setminus (E_1 \cup E_2) \rightarrow \mathbb{R}$ such that u is one to one, $u(0) = 0$ and

$$u(\mathbb{R} \setminus (E_1 \cup E_2)) \cap (g(E_1) \cup h(E_2)) = \emptyset.$$

The function $k: \mathbb{R} \rightarrow \mathbb{R}$ defined as follows

$$k(x) = \begin{cases} g(x) & \text{if } x \in E_1, \\ h(x) & \text{if } x \in E_2, \\ u(x) & \text{if } x \in \mathbb{R} \setminus (E_1 \cup E_2) \end{cases}$$

is one to one function from $\mathbb{R}$ to $\mathbb{R}$, so the map

$$d(x, y) = |k(x) - k(y)| \quad x, y \in \mathbb{R}$$

is a metric in $\mathbb{R}$.

Now, let us consider the space $\mathbb{R}$ with the metric d and $E = \{\frac{1}{n} : n \in \mathbb{N}\}$, and $f: E \rightarrow \mathbb{R}$, for which $f\left(\frac{1}{n}\right) = \frac{1}{2^{2n}}$, if $n \in \mathbb{N}$.

Then, one can observe that $V_d(f, E) = V(g, E_1) < \infty$, so f has a bounded variation in the restricted sense on E .

On the other hand, $V_d\left(\frac{1}{2}f, E\right) = V(h, E_2) = \infty$ and, consequently, $\frac{1}{2}f \notin dVBG$.

THEOREM 2.4. *If (X, d) is a metric space where d is a translation invariant metric then*

$$f_1 + f_2 \in dVBG_\star \quad \text{for each } f_1, f_2 \quad \text{from } dVBG_\star.$$

Proof. If $f_1 \in dVBG_\star$, then E can be represented in the form $E = \bigcup_{i=1}^{\infty} H_i$, where $V_d(f_1, H_i)$ is finite and H_i are closed for each i from $\mathbb{N}$. For same reason, E can be represented as $E = \bigcup_{i=1}^{\infty} G_i$, where $V_d(f_2, G_i)$ is finite and G_i are closed for all i from $\mathbb{N}$. Thus

$$E = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} (H_i \cap G_j),$$

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$$\begin{aligned} V_d(f_1, H_i \cap G_j) &\leq V_d(f_1, H_i) < \infty, \\ V_d(f_2, H_i \cap G_j) &\leq V_d(f_2, G_j) < \infty. \end{aligned}$$

It follows from the above relations that E is a countable union of closed sets $H_i \cap G_j$ and

$$V_d(f_1 + f_2, H_i \cap G_j) \leq V_d(f_1, H_i \cap G_j) + V_d(f_2, H_i \cap G_j) < \infty$$

for each i, j from $\mathbb{N}$. By Definition 1.1, $f_1 + f_2$ has a bounded generalized variation in the restricted sense on E . $\square$

Applying Theorem 2.3 and the method analogous to that of the proof of Theorem 2.4, one can prove the following

THEOREM 2.5. *If the assumptions from Theorem 2.3 are satisfied, then $\alpha f \in dVBG_\star$, where $\alpha \in \Phi$ and $f \in dVBG_\star$.*

One can observe that, under the assumptions of Theorem 2.3, $dVBG$ and $dVBG_\star$ are linear spaces over the field Φ .

There is a less complicated problem to conclude that

$$\Gamma WBG(X) \quad \text{and} \quad \Gamma WBG_\star(X)$$

are linear spaces over Φ .

THEOREM 2.6. *If X is a linear topological space, Γ is a nonempty subfamily of $X^\star$, then*

$$f_1 + f_2 \in \Gamma WBG(X) \quad \text{and} \quad \alpha f_1 \in \Gamma WBG(X)$$

for α from Φ and f_1, f_2 from $\Gamma WBG(X)$.

P r o o f. If f_1, f_2 belong to $\Gamma WBG(X)$ and $\{I_1, \dots, I_n\}$ is a partition from $\mathcal{R}(E)$, then $\Lambda \circ f_1$ and $\Lambda \circ f_2$ belong to VBG . For x, y from I_i we have:

$$\begin{aligned} |\Lambda((f_1 + f_2)(x)) - \Lambda((f_1 + f_2)(y))| &= |(\Lambda f_1(x) - \Lambda f_2(y)) + (\Lambda f_1(y) - \Lambda f_2(y))| \\ &\leq |\Lambda f_1(x) - \Lambda f_2(y)| + |\Lambda f_1(y) - \Lambda f_2(y)| \\ &\leq \omega(\Lambda \circ f_1, I_i \cap E) + \omega(\Lambda \circ f_2, I_i \cap E) \end{aligned}$$

hence

$$\omega(\Lambda \circ (f_1 + f_2), I_i \cap E) \leq \omega(\Lambda \circ f_1, I_i \cap E) + \omega(\Lambda \circ f_2, I_i \cap E).$$

In consequence,

$$\begin{aligned} \sum_{i=1}^n \omega(\Lambda \circ (f_1 + f_2), I_i \cap E) &\leq \sum_{i=1}^n \omega(\Lambda \circ f_1, I_i \cap E) + \sum_{i=1}^n \omega(\Lambda \circ f_2, I_i \cap E) \\ &\leq V(\Lambda \circ f_1, E) + V(\Lambda \circ f_2, E) \end{aligned}$$

which implies that

$$V(\Lambda \circ (f_1 + f_2), E)$$

is finite for each linear functional Λ from Γ . Finally,

$$f_1 + f_2 \in \Gamma WBG(X).$$

Analogously, one can prove that $\alpha f \in \Gamma WBG(X)$ whenever $\alpha \in \Phi$ and $f \in \Gamma WBG(X)$. $\square$

THEOREM 2.7. *Let X be a linear topological space, $E \subset \mathbb{R}$, $\Gamma = X^*$ and $f \in \Gamma WBG(X)$. Then f is weakly bounded.*

PROOF. If $f \in \Gamma WBG(X)$ and $\Lambda \in X^*$, then $\Lambda \circ f$ belongs to VBG . By Theorem 2.2, $\Lambda \circ f$ is bounded in the usual sense in Φ , so it is bounded. Hence, f is weakly bounded, since Λ has been chosen arbitrarily. $\square$

Applying Theorem 2.6 and 2.7, one can prove the following

THEOREM 2.8. *If X is a linear topological space, Γ is a nonempty subset of X^* , then*

$$f_1 + f_2 \in WBG_*(X) \quad \text{and} \quad \alpha f_1 \in WBG_*(X)$$

for each α from Φ and f_1, f_2 from $WBG_*(X)$.

3. Spaces $d\widetilde{V}BG$ and $\Gamma\widetilde{W}BG$

Let us consider a relation “ $\sim$ ” in $dVBG$ defined in the following way:

$$f \sim g \iff V_d(f - g, E) = 0$$

for each $f, g \in dVBG$. One can verify that “ $\sim$ ” is an equivalence relation in $dVBG$. The set of all equivalence classes determined by “ $\sim$ ” will be denoted by $d\widetilde{V}BG$. It can be proved that $d\widetilde{V}BG$ is a metric space equipped with the metric

$$\widetilde{\varrho}(\widetilde{f}, \widetilde{g}) = V_d(f - g, E) \quad (\widetilde{f}, \widetilde{g} \in d\widetilde{V}BG)$$

which is also a translation invariant. It is also not difficult to observe that, under the assumptions of Theorem 2.3, $d\widetilde{V}BG$ is a linear space over Φ , where the operation of addition and multiplying by scalars are defined in a standard way. However, under some additional assumptions of metric d , the following can be proved

THEOREM 3.1. *Let (X, d) be a metric space over Φ , and d a translation invariant metric in X satisfying the condition*

- *there exists a bounded and real valued function ψ defined on $\{\alpha \in \Phi : |\alpha| \leq 1\}$ such that $\lim_{\alpha \rightarrow 0} \psi(\alpha) = 0$ and if α belongs to $\{\alpha \in \Phi : |\alpha| \leq 1\}$ and x in X , then the inequality $d(\alpha x, 0) \leq \psi(\alpha)d(x, 0)$ holds.*

Then $d\widetilde{V}BG$ is a linear topological space over Φ .

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P r o o f. Addition is a continuous operation, since $(d\tilde{V}BG, \tilde{\varrho})$ is a metric space. It is sufficient to show that multiplication by scalars from Φ is a continuous operation; it means that the function $F: \Phi \times d\tilde{V}BG \rightarrow d\tilde{V}BG$, where

$$F(\alpha, f) = \alpha f \quad (\alpha \in \Phi, f \in d\tilde{V}BG)$$

is continuous. Let us choose a sequence $(\alpha_n)_{n=1}^{\infty}$ of scalars and a sequence $(f_n)_{n=1}^{\infty}$ of elements of $d\tilde{V}BG$ which are convergent to α and f , respectively. There exists a positive number M such that:

$$V_d(\alpha(f_n - f), E) \leq M V_d(f_n - f, E),$$

whence $\lim_{n \rightarrow \infty} V_d(\alpha(f_n - f), E) = 0$. On the other hand,

$$V_d((\alpha_n - \alpha)f, E) \leq \psi(\alpha_n - \alpha) V_d(f, E)$$

for sufficiently large n and, in consequence, $\lim_{n \rightarrow \infty} V_d(\alpha_n - \alpha, E) = 0$. Since

$$\tilde{\varrho}(\alpha_n f_n, \alpha f) \leq \tilde{\varrho}(\alpha f_n, \alpha f) + \tilde{\varrho}(\alpha_n f, \alpha f),$$

so $\lim_{n \rightarrow \infty} \tilde{\varrho}(\alpha_n f_n, \alpha f) = 0$ and the proof is complete. $\square$

LEMMA 3.1. *Let us assume that (X, d) is a metric space with translation invariant metric d , and $(f_n)_{n=1}^{\infty}$ converges uniformly to a map f with respect to metric d , where $f_n, f: E \rightarrow X$. Then*

$$\lim_{n \rightarrow \infty} \omega_d(f_n, I \cap E) = \omega_d(f, I \cap E) \quad (1)$$

for each closed interval I .

P r o o f. If I is an interval for which $I \cap E \neq \emptyset$, then for $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ and $x \in I \cap E$, then $d(f_n(x), f(x)) < \frac{\varepsilon}{4}$. For x, y from $I \cap E$ we get:

$$\begin{aligned} d(f(x), f(y)) &\leq d(f_n(x), f(x)) + d(f_n(y), f(y)) + d(f_n(x), f_n(y)) \\ &< \frac{\varepsilon}{2} + \omega_d(f_n, I \cap E). \end{aligned}$$

Since x, y were taken arbitrarily, then

$$\omega_d(f, I \cap E) \leq \frac{\varepsilon}{2} + \omega_d(f_n, I \cap E) < \varepsilon + \omega_d(f_n, I \cap E).$$

For the same reason, one can infer that

$$\omega_d(f_n, I \cap E) \leq \frac{\varepsilon}{2} + \omega_d(f, I \cap E) < \varepsilon + \omega_d(f, I \cap E).$$

We have proved that the inequality

$$|\omega_d(f, I \cap E) - \omega_d(f_n, I \cap E)| < \varepsilon$$

holds for each n not less than n_0 , which implies (1). $\square$

Applying Lemma 3.1 in the classical way, one can prove the following

THEOREM 3.2. *If (X, d) is a metric space with translation invariant metric and $(f_n)_{n=1}^\infty$ converges uniformly to f with respect to metric d , where $f_n, f: E \rightarrow X$ for all n from $\mathbb{N}$, then*

$$V_d(f, E) \leq \liminf_{n \rightarrow \infty} V_d(f_n, E).$$

THEOREM 3.3. *If (X, d) is a complete metric space with translation invariant metric, then $d\tilde{V}BG$ is complete metric space.*

PROOF. Let $(f_n)_{n=1}^\infty$ be a Cauchy sequence in $d\tilde{V}BG$ and ε be a positive real number. There exists n_0 such that $V_d(f_n - f_m, E) < \varepsilon$ for m, n not less than n_0 . For each partition $\{I_1, \dots, I_k\}$ from $\mathcal{R}(E)$, the inequality

$$\sum_{i=1}^k \omega_d(f_n - f_m, I_i \cap E) < \frac{\varepsilon}{2} \quad (2)$$

holds. Let us put

$$F_n(x, y) = f_n(x) - f_n(y) \quad x, y \in E.$$

By (2), the sequence $(F_n(x, y))_{n=1}^\infty$ satisfies Cauchy's condition in a complete space X , so it converges to an element $a(x, y) \in X$. Let us denote by F a function from $E \times E$ to X such that $F(x, y) = a(x, y)$ for $x, y \in E$. We can define a function f from E to X such that

$$f(x) - f(y) = F(x, y) \quad \text{for each } x, y \text{ from } E.$$

It is sufficient to prove that f belongs to $d\tilde{V}BG$. According to (2), the sequence $(F_n)_{n=1}^\infty$ converges uniformly to a function F with respect to metric d on E . Using Lemma 3.1 and the inequality (2), one can get

$$\sum_{i=1}^k \omega_d(f_n - f, I_i \cap E) < \varepsilon$$

for any partition $\{I_1, \dots, I_k\}$ from $\mathcal{R}(E)$. Hence, the sequence $(V_d(f_n - f, E))_{n=1}^\infty$ converges to zero and, consequently, the sequence $(V_d(f_n, E))_{n=1}^\infty$ satisfies Cauchy's condition in a complete space $\mathbb{R}$. Thus $(V_d(f_n, E))_{n=1}^\infty$ is bounded, and by Theorem 3.2, $V_d(f, E) < \infty$. Finally, f belongs to $d\tilde{V}BG$. $\square$

DEFINITION 3.1 ([7]). It is said that a linear topological space X is an F -space, if it is metrizable by a complete and translation invariant metric. Additionally, if X is a locally convex F -space, then it is called a Fréchet space.

THEOREM 3.4. *If X is a linear space with translation invariant and complete metric d , which satisfies the condition of Theorem 3.1, then $d\tilde{V}BG$ is an F -space.*

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P r o o f. By Theorem 3.2, $d\widetilde{V}BG$ is complete and, from Theorem 3.1, it is a linear topological space. According to Definition 3.1, it is an F -space. $\square$

THEOREM 3.5. *Let (X, d) be a metric space with translation invariant and complete metric d , where d satisfies the additional condition*

- *for each α from $[0, 1]$ and $x \in X$, the inequality $d(\alpha x, 0) \leq \alpha d(x, 0)$ holds.*

Then $d\widetilde{V}BG$ is a Fréchet space.

P r o o f. It is sufficient to show that $d\widetilde{V}BG$ is locally convex, since it is an F -space by Theorem 3.4. Denote by $\mathcal{B}$ the family

$$\mathcal{B} = \left\{ B^\circ(0, \varepsilon) \subset d\widetilde{V}BG : \varepsilon > 0 \right\}.$$

We will prove that each of open balls from $\mathcal{B}$ is convex. Let us fix a positive real number ε , and take $\tilde{f}, \tilde{g}$ from $B^\circ(0, \varepsilon)$ and α, β from $[0, 1]$ such that $\alpha + \beta = 1$. Let $\{I_1, \dots, I_n\}$ be an arbitrary partition of E from $\mathcal{R}(E)$. If x, y belong to $I_i \cap E$, then

$$\begin{aligned} d(\alpha f(x) + \beta g(x), \alpha f(y) + \beta g(y)) &= d(\alpha f(x) - \alpha f(y), \beta g(y) - \beta g(x)) \\ &\leq d(\alpha f(x) - \alpha f(y), 0) + d(0, \beta g(y) - \beta g(x)) \\ &\leq \alpha d(f(x), f(y)) + \beta d(g(x), g(y)) \\ &\leq \alpha \omega_d(f, I_i \cap E) + \beta \omega_d(g, I_i \cap E), \end{aligned}$$

so

$$\omega_d(\alpha f + \beta g, I_i \cap E) \leq \alpha \omega_d(f, I_i \cap E) + \beta \omega_d(g, I_i \cap E)$$

and, consequently,

$$\begin{aligned} \sum_{i=1}^n \omega_d(\alpha f + \beta g, I_i \cap E) &\leq \sum_{i=1}^n \alpha \omega_d(f, I_i \cap E) + \sum_{i=1}^n \beta \omega_d(g, I_i \cap E) \\ &\leq \alpha V_d(f, E) + \beta V_d(g, E). \end{aligned}$$

Thus

$$V_d(\alpha f + \beta g, E) \leq \alpha V_d(f, E) + \beta V_d(g, E) < \alpha \varepsilon + \beta \varepsilon = \varepsilon.$$

and finally, $\alpha \tilde{f} + \beta \tilde{g}$ belongs to $B^\circ(0, \varepsilon)$. $\square$

At the end of this section, we will define a new space, which will be denoted by the symbol $\Gamma\widetilde{W}BG(X)$. Recall that if X is a linear topological space and $\Gamma \subset X^*$, then $\Gamma\widetilde{W}BG(X)$ denotes the class of all maps from E to X , which has a bounded weak variation in the restricted sense on E with respect to subfamily Γ .

We start with the following

THEOREM 3.6. *If X is a linear topological space, Γ is a nonempty subset of X^* , then the formula*

$$\sigma_\Gamma(f, g) = \min \left\{ \sup \left\{ V(\Lambda \circ (f - g), E) : \Lambda \in \Gamma \right\}, 1 \right\}$$

defines a pseudometric in $\Gamma WBG(X)$.

Proof. It is clear that $\sigma_\Gamma(f, f) = 0$. For f, g, h from $\Gamma WBG(X)$ and any continuous linear functional Λ over X from Γ

$$V(\Lambda \circ (f - g), E) = V(\Lambda \circ (g - f), E)$$

and

$$V(\Lambda \circ (f - g), E) \leq V(\Lambda \circ (f - h), E) + V(\Lambda \circ (h - g), E).$$

Due to the above remarks and the definition of σ_Γ we have

$$\sigma_\Gamma(f, g) = \sigma_\Gamma(g, f) \quad \text{and} \quad \sigma_\Gamma(f, g) \leq \sigma_\Gamma(f, h) + \sigma_\Gamma(h, g).$$

□

Now, following the above theorem, the relation “ $\sim_\Gamma$ ” defined as

$$f \sim_\Gamma g \iff \sigma_\Gamma(f, g) = 0$$

is an equivalence relation in $\Gamma WBG(X)$. Then, the symbol $\widetilde{\Gamma WBG}(X)$ will denote the set of all equivalence classes determined by “ $\sim_\Gamma$ ”. With respect to those remarks, if X is a linear topological space, then the formula

$$\tilde{\sigma}(\tilde{f}, \tilde{g}) = \sigma_\Gamma(f, g) \quad (\tilde{f}, \tilde{g} \in \widetilde{\Gamma WBG}(X)) \quad (3)$$

defines a translation invariant metric in $\widetilde{\Gamma WBG}(X)$.

THEOREM 3.7. *Let X be a linear topological space and $\Lambda \in X^*$. If $f_n, f: E \rightarrow X$ for each $n \in \mathbb{N}$ and $(f_n)_{n=1}^\infty$ converges uniformly to f on E , then*

$$V(\Lambda \circ f, E) \leq \liminf_{n \rightarrow \infty} V(\Lambda \circ f_n, E). \quad (4)$$

Proof. In the first step, we prove that

$$\omega(\Lambda \circ f, I \cap E) = \lim_{n \rightarrow \infty} \omega(\Lambda \circ f_n, I \cap E) \quad (5)$$

for each interval I such that $I \cap E \neq \emptyset$. If ε is a positive real number, then $\Lambda^{-1}((-\varepsilon, \varepsilon))$ is a neighborhood of 0. By assumptions, there exists $n_0 \in \mathbb{N}$ such that

$$f_n(x) - f(x) \in \Lambda^{-1} \left(\left(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2} \right) \right)$$

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for x from $I \cap E$ and $n \geq n_0$. Owing to this,

$$\begin{aligned} \left| |\Lambda f_n(x) - \Lambda f_n(y)| - |\Lambda f(x) - \Lambda f(y)| \right| &\leq \left| \Lambda(f_n(x) - f(x)) + \Lambda(f(x) - f_n(y)) \right| \\ &\leq |\Lambda f_n(x) - \Lambda f(x)| + |\Lambda f_n(x) - \Lambda f(x)| \\ &< \varepsilon \end{aligned}$$

for arbitrary x, y from $I \cap E$ and n not less than n_0 . Thus

$$|\omega(\Lambda \circ f_n, I \cap E) - \omega(\Lambda \circ f, I \cap E)| < \varepsilon$$

for $n \geq n_0$ and, consequently, (5) is true.

If $L = \liminf_{n \rightarrow \infty} V(\Lambda \circ f_n, E)$ and $L = \infty$ then (4) is satisfied. Now, let us assume that $L < \infty$ and choose an arbitrary partition $\{I_1, \dots, I_m\}$ from $\mathcal{R}(E)$ and a positive real number η . There exists a subsequence $(f_{n_k})_{k=1}^\infty$ such that

$$V(\Lambda \circ f_{n_k}, E) < L + \eta \quad \text{for } k \in \mathbb{N}.$$

Then,

$$\sum_{i=1}^m \omega(\Lambda \circ f_{n_k}, I_i \cap E) < L + \eta,$$

and according to the previous part of the proof

$$\sum_{i=1}^m \omega(\Lambda \circ f, I_i \cap E) \leq L + \eta.$$

Since $\{I_1, \dots, I_m\}$ and η has chosen arbitrarily then

$$V(\Lambda \circ f, E) \leq \liminf_{n \rightarrow \infty} V(\Lambda \circ f_n, E).$$

□

THEOREM 3.8. *If X is a linear topological space, Γ is a nonempty subset of X^* , $f_n, f: E \rightarrow X$ for $n \in \mathbb{N}$ and $(f_n)_{n=1}^\infty$ converges uniformly to f on E , then*

$$\tilde{\sigma}(\tilde{f}, \tilde{0}) \leq \liminf_{n \rightarrow \infty} \tilde{\sigma}(\tilde{f}_n, \tilde{0}).$$

Proof. By the last theorem, the inequality

$$V(\Lambda \circ f, E) \leq \liminf_{n \rightarrow \infty} V(\Lambda \circ f_n, E)$$

holds for each Λ from Γ . Thus

$$V(\Lambda \circ f, E) \leq \liminf_{n \rightarrow \infty} \sup\{V(\Lambda \circ f_n, E) : \Lambda \in \Gamma\},$$

so

$$\sup\{V(\Lambda \circ f, E) : \Lambda \in \Gamma\} \leq \liminf_{n \rightarrow \infty} \sup\{V(\Lambda \circ f_n, E) : \Lambda \in \Gamma\}$$

for each Λ from Γ . Since

$$\min\left\{\sup\{V(\Lambda \circ f, E) : \Lambda \in \Gamma\}, 1\right\} \leq \min\left\{\liminf_{n \rightarrow \infty} \sup\{V(\Lambda \circ f_n, E) : \Lambda \in \Gamma\}, 1\right\} \quad (6)$$

and

$$\min\left\{\liminf_{n \rightarrow \infty} \sup\{V(\Lambda \circ f_n, E) : \Lambda \in \Gamma\}, 1\right\} = \liminf_{n \rightarrow \infty} \left(\min\left\{\sup\{V(\Lambda \circ f_n, E) : \Lambda \in \Gamma\}, 1\right\} \right) \quad (7)$$

then (6) and (7) give the required inequality. $\square$

From now on, for a family $\Gamma = \{\Lambda\}$, where Λ is a functional from X^* , we will write $\Lambda\widetilde{W}BG$ instead of $\Gamma\widetilde{W}BG$.

THEOREM 3.9. *If (X, d) is a metric space with translation invariant metric d and Λ is a linear functional over X such that there exists a positive real number $M > 0$ that is $Md(x, 0) \leq |\Lambda x|$ for each x from X , then $\Lambda\widetilde{W}BG(X)$ is a complete space.*

P r o o f. Let $(\widetilde{f}_n)_{n=1}^\infty$ be a Cauchy sequence in $\Lambda\widetilde{W}BG(X)$.

Then, for a positive real number ε there exists a positive integer n_0 such that

$$V(\Lambda \circ (f_n - f_m), E) < \varepsilon,$$

i.e.,

$$\sum_{i=1}^k \omega(\Lambda \circ (f_n - f_m), I_i \cap E) < \varepsilon$$

for m, n not less than n_0 and for each sequence $\{I_1, \dots, I_k\}$ from $\mathcal{R}(E)$.

Let us denote

$$F_n(x, y) = f_n(x) - f_n(y) \quad x, y \in E.$$

If $\{I_1, \dots, I_k\}$, where $k \in \mathbb{N}$, is a partition of E from $\mathcal{R}(E)$, then

$$\sum_{i=1}^k \omega(\Lambda \circ (f_n - f_m), I_i \cap E) < \varepsilon,$$

which implies that

$$|\Lambda(F_n(x, y)) - \Lambda(F_m(x, y))| < \varepsilon,$$

where $n, m \geq n_0$. Whence $(\Lambda(F_n(x, y)))_{n=1}^\infty$ is a Cauchy sequence in a complete space Φ for each x, y from E . Let us denote by f a function from E to X such that

$$\Lambda(f(x) - f(y)) := \lim_{n \rightarrow \infty} \Lambda(F_n(x, y)).$$

One can observe that if $x, y \in E$, then the inequality

$$|\Lambda(F_n(x, y)) - \Lambda(F_m(x, y))| \leq \varepsilon$$

holds, and from properties of Λ

$$Md(F_n(x, y) - (f(x) - f(y)), 0) \leq |\Lambda(F_n(x, y)) - \Lambda(f(x) - f(y))| \leq \varepsilon$$

for each x, y from E and n not less than n_0 . It implies that $(F_n)_{n=1}^\infty$ converges uniformly to the map $F(x, y) = f(x) - f(y)$. By Theorem 3.6,

$$V(\Lambda \circ f, E) \leq \liminf_{n \rightarrow \infty} V(\Lambda \circ f_n, E),$$

whence $V(\Lambda \circ f, E)$ is finite. As a result we get that f belongs to $\Lambda\widetilde{WBG}(X)$. In addition,

$$V(\Lambda \circ (f_n - f), E) \leq \varepsilon,$$

so

$$\lim_{n \rightarrow \infty} \widetilde{\sigma}(f_n, f) = 0,$$

and the proof is complete. $\square$

THEOREM 3.10. $\Gamma\widetilde{WBG}(X)$ is a locally convex space for each subfamily Γ of X^* .

Proof. By simple calculus one can show that $B^\circ(0, \frac{1}{n})$ are convex for each n from $\mathbb{N}$, so the proof is omitted. $\square$

From Theorem 3.9 and Theorem 3.10 it immediately follows

THEOREM 3.11. $\Lambda\widetilde{WBG}(X)$ is a Fréchet space for each space X and functional Λ from X^* , which satisfies the conditions of Theorem 3.9.

4. Comparison of spaces: $dVBG$, $dVBG_\star$, $\Gamma WBG(X)$, $\Gamma WBG_\star(X)$

In this section, we will try to establish the connections between $dVBG$ and $\Gamma WBG(X)$, where (X, d) is a linear and metric space with translation invariant metric d and Γ is a subset of X^* . Immediately from definitions of $dVBG$, $dVBG_\star$, it follows that $dVBG \subset dVBG_\star$. If X is a linear topological space then it is clear that $\Gamma WBG(X) \subset \Gamma WBG_\star(X)$ for each subfamily Γ of X^* .

From now on, we will consider only such metric spaces, for which every bounded subset in the usual sense in X is bounded in the sense of linear topological spaces.

THEOREM 4.1. *Let (X, d) be a metric linear space with a translation invariant metric, Γ be a nonempty subfamily of X^* . If for each bounded subset U of X the set U_d of the form*

$$U_d = \left\{ \frac{x}{d(x, 0)} : x \in U \setminus \{0\} \right\}$$

is bounded, then $dVBG \subset \Gamma WBG(X)$ and $dVBG_\star \subset \Gamma WBG_\star(X)$.

Proof. We only prove the first inclusion, since the second inclusion is a simple consequence of the first one.

Let us choose f from $dVBG$, Λ from Γ and a partition $\{I_1, \dots, I_n\}$, where $n \in \mathbb{N}$, from $\mathcal{R}(E)$. Since Λ is a continuous and linear functional in metric linear space X , so it is bounded, that means $\Lambda(W)$ is bounded in Φ for each bounded subset W of X . Since f belongs to $dVBG$, then, by Theorem 2.2, $f(E)$ is bounded in X in the usual sense. By assumption of X , $f(E)$ is bounded in X and hence $f(E) - f(E)$ is a bounded subset of X , too. Then

$$A = \left\{ \frac{f(x) - f(y)}{d(f(x), f(y))} : f(x) \neq f(y) \wedge x, y \in E \right\}$$

is bounded in X , so there exists a positive real number M such that

$$|\Lambda f(x) - \Lambda f(y)| \leq M d(f(x), f(y))$$

for all vectors from A . If x, y belong to $I_i \cap E$, then

$$|\Lambda f(x) - \Lambda f(y)| = |\Lambda(f(x) - f(y))| \leq M d(f(x), f(y)) \leq M \omega_d(f, I_i \cap E).$$

Thus

$$\omega(\Lambda \circ f, I_i \cap E) \leq M \omega_d(f, I_i \cap E)$$

and

$$\sum_{i=1}^n \omega(\Lambda \circ f, I_i \cap E) \leq M \cdot \sum_{i=1}^n \omega_d(f, I_i \cap E) \leq M \cdot V_d(f, E).$$

Since $\{I_1, \dots, I_n\}$ and Λ have been chosen in an arbitrary way, so

$$V(\Lambda \circ f, E) \leq M \cdot V_d(f, E)$$

and, consequently, $f \in \Gamma WBG_\star(X)$. □

THEOREM 4.2. *If $(X, \|\cdot\|_X)$ is a normed space, Γ is a nonempty subfamily of X^* , then*

$$dVBG \subset \Gamma WBG(X) \quad \text{and} \quad dVBG_\star \subset \Gamma WBG_\star(X),$$

where $d(x, y) = \|x - y\|_X$ for each x, y from X .

Let us see that the inclusions in Theorem 4.1 might be proper.

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EXAMPLE 4.1. Let us consider the space $L^p([0, 1])$ of all measurable functions f from $[0, 1]$ to $\mathbb{C}$ with respect to Lebesgue measure λ and such that $\int_0^1 |f|^p d\lambda < \infty$, where $p \in (0, 1)$. Then the function d defined in the following way

$$d(f, g) = \int_0^1 |f(x) - g(x)|^p d\lambda, \quad f, g \in L^p([0, 1])$$

is a metric in $L^p([0, 1])$ (for details, see [7]).

For a real number η , let g_η denote the functions from $[0, 1]$ into $[0, 1]$, where

$$g_\eta(x) = \eta x \quad \text{and} \quad x \in [0, 1].$$

If $f: \mathbb{R} \rightarrow L^p([0, 1])$, where $f(\eta) = g_\eta$, then $\omega_d(f, [n, n+1]) \geq 1$ for all n from $\mathbb{N}$. Hence $V(f, \mathbb{R}) = \infty$, that means f does not belong to $dVBG_\star$. One can show that $WBG_\star(L^p([0, 1]))$ consists of all mappings from $[0, 1]$ to X .

DEFINITION 4.1 ([4]). It is said that the metrics d_1 and d_2 are uniformly equivalent in X if there exist positive real numbers α and β such that

$$\alpha d_1(x, y) \leq d_2(x, y) \leq \beta d_1(x, y)$$

for each x, y from X .

THEOREM 4.3. If X is a linear space and d, d' are uniformly equivalent and translation invariant metrics in X , then

$$\begin{aligned} dVBG &= d'VBG, & WBG((X, d)) &= WBG((X, d')), \\ dVBG_\star &= d'VBG_\star, & WBG_\star((X, d)) &= WBG_\star((X, d')). \end{aligned}$$

Proof. By assumptions, there exist some positive real numbers M_1 and M_2 such that

$$M_1 d(x, y) \leq d'(x, y) \leq M_2 d(x, y)$$

for all x, y from X . If I is a closed interval such that $E \cap I \neq \emptyset$, then

$$M_1 \omega_d(f, I \cap E) \leq \omega_{d'}(f, I \cap E) \leq M_2 \omega_d(f, I \cap E).$$

By the above observation, we can obtain

$$M_1 V_d(f, E) \leq V_{d'}(f, E) \leq M_2 V_d(f, E),$$

that means $f \in dVBG$ if $f \in d'VBG$.

Let us assume that f belongs to $WBG((X, d))$. If Λ is a continuous functional over X with respect to metric d' then, by assumptions, it is also continuous over X with respect to metric d , so $V(\Lambda \circ f, E) < \infty$ for all linear functionals Λ , which are continuous with respect to metric d' . Hence,

$$WBG((X, d)) \subset WBG((X, d')).$$

The inverse inclusion can be proved in the same way. □

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