A NEW APPROACH TO ψ -CONTINUITY

MALGORZATA TEREPETA

ABSTRACT. Let $\mathcal{T}_\psi$ be a ψ -density topology for a fixed function ψ . For any topological space X with the topology τ we will consider the family $C(X, \mathbb{R}_\psi)$ of all continuous functions f from (X, τ) into $(\mathbb{R}, \mathcal{T}_\psi)$. The main aim of this paper is to investigate when $C(X, \mathbb{R}_\psi)$ is a ring. This article is based on the results achieved by M. Knox [*A characterization of rings of density continuous functions*, Real Anal. Exchange **31** (2005), 165–177].

We will use the following notations: $\mathbb{R}$ will denote the set of real numbers, S – the σ -algebra of Lebesgue measurable subsets on $\mathbb{R}$, m – the Lebesgue measure on $\mathbb{R}$. For $A \subset \mathbb{R}$ let A' be its complement. Let $\mathcal{C}$ be a family of nondecreasing continuous functions $\psi: (0, \infty) \rightarrow (0, \infty)$ such that $\lim_{t \rightarrow 0^+} \psi(t) = 0$.

Fix $\psi \in \mathcal{C}$. We say that $x \in \mathbb{R}$ is a ψ -density point of a measurable set $A \subset \mathbb{R}$ if

$$\lim_{h \rightarrow 0^+} \frac{m(A' \cap (x - h, x + h))}{2h\psi(2h)} = 0.$$

For any $A \in S$ we denote

$$\Phi_\psi(A) = \{x \in \mathbb{R} : x \text{ is a } \psi\text{-density point of } A\}.$$

From [TW-B, Theorem 1.4] we obtain that the family $\mathcal{T}_\psi = \{A \in S : A \subset \Phi_\psi(A)\}$ is a topology stronger than the natural topology $\mathcal{T}_e$ and weaker than the density topology $\mathcal{T}_d$. We will say that a set is $\mathcal{T}_\psi$ -open ($\mathcal{T}_\psi$ -closed) when it is open (closed) in topology $\mathcal{T}_\psi$. Some properties of the topology $\mathcal{T}_\psi$ are very similar to those of density topology. $\mathcal{T}_\psi$ is invariant under translation, but it does not need be invariant under multiplication.

THEOREM 1 ([F]). *Let $\psi \in \mathcal{C}$. $\mathcal{T}_\psi$ is invariant under multiplication if and only if*

$$\limsup_{x \rightarrow 0^+} \frac{\psi(\alpha x)}{\psi(x)} < \infty \tag{\Delta 2}$$

for any $\alpha \in \mathbb{R}_+$.

2000 Mathematics Subject Classification: 54A10, 28A05.

Keywords: density point, density topology, ψ -density topology, ring, lattice.

To shorten the notation, we will use the following abbreviations: $\mathbb{R}_e = (\mathbb{R}, \mathcal{T}_e)$, $\mathbb{R}_d = (\mathbb{R}, \mathcal{T}_d)$, $\mathbb{R}_\psi = (\mathbb{R}, \mathcal{T}_\psi)$, $\mathbb{R}_{discr} = (\mathbb{R}, \mathcal{T}_{discr})$, where $\mathcal{T}_{discr}$ stands for discrete topology on the real line. Let $C(X, Y)$ be the family of all continuous functions $f: X \rightarrow Y$, where X, Y are topological spaces. If $Y = \mathbb{R}_e$, then we write $C(X)$ instead of $C(X, \mathbb{R}_e)$.

In the articles [FT1] and [FT2] there were discussed some properties of families $C(X, Y)$, where $X, Y \in \{\mathbb{R}_e, \mathbb{R}_d, \mathbb{R}_\psi\}$. It was shown that the family $C(\mathbb{R}_\psi, \mathbb{R}_e)$ consists of constant functions only. It was also proved that the family $C(\mathbb{R}_\psi, \mathbb{R}_\psi)$ (the family of so-called ψ -continuous functions) is not closed under pointwise addition: there exists a ψ -continuous function f such that $f+x$ is not ψ -continuous. Hence, $C(\mathbb{R}_\psi, \mathbb{R}_\psi)$ is not a ring for any function $\psi \in \mathcal{C}$. If the condition $(\Delta 2)$ is not fulfilled for a certain $\alpha > 1$, then even linear function $f(x) = \alpha x$ is not ψ -continuous. In this paper, we will concentrate on the family $C(X, \mathbb{R}_\psi)$, where $X = (X, \tau)$ is an arbitrary topological space. If $f \in C(X, \mathbb{R}_\psi)$, then f is said to be $\tau\psi$ -continuous. There a natural question arose—for which topological space X the family $C(X, \mathbb{R}_\psi)$ can be a ring? We will also discuss the problem if it is a lattice.

Let $\mathbb{R}^X$ denote the space of all real-valued functions on X . With the operations of pointwise addition and multiplication, it is a ring. We can define the partial order on $\mathbb{R}^X$ by $f \leq g$ if

$$\forall x \in X, \quad f(x) \leq g(x).$$

For any two elements of $\mathbb{R}^X$, there exist infimum and supremum of them. We will denote them by $f \wedge g$ and $f \vee g$, respectively, and for any $x \in X$ we have

$$\begin{aligned} (f \wedge g)(x) &= f(x) \wedge g(x), \\ (f \vee g)(x) &= f(x) \vee g(x). \end{aligned}$$

Then $\mathbb{R}^X$ is a lattice. The space $C(X)$ is a subring and a sublattice of $\mathbb{R}^X$ for any space X .

DEFINITION 2. Let $f \in \mathbb{R}^X$. The set

- (a) $Z(f) = \{x \in X: f(x) = 0\}$ is called a zeroset of the function f ,
- (b) $Coz(f) = \{x \in X: f(x) \neq 0\}$ is called a cozeroset of the function f .

For each function $f \in C(X)$ the set $Z(f)$ is closed and $Coz(f)$ is open in the space X .

If Z is a subset of X such that it is a zeroset for a certain function $f \in C(X)$, then we call it a zeroset of a space X . The collection of such sets is denoted by $Z(X)$. It is closed under countable intersection. If $Z \subset X$ satisfies $Z = Z(f)$ for a certain function $f \in C(X, \mathbb{R}_\psi)$, then Z will be called a ψ -zeroset of a space X and the family of such sets will be denoted by $Z_\psi(X)$. So, $Z_\psi(X) = \{Z(f): f \in C(X, \mathbb{R}_\psi)\}$. The families $Coz(X)$ and $Coz_\psi(X)$ are defined analogously.

EXAMPLE 3. If $X = \mathbb{R}_e$, then $C(X, \mathbb{R}_\psi) = \text{Const}$ ([FT1]) and $Z_\psi(X) = \{\emptyset, \mathbb{R}\}$.

Now, we will discuss some properties of the family $C(X, \mathbb{R}_\psi)$ and $\tau\psi$ -continuous functions. Since $\mathcal{T}_e \subset \mathcal{T}_\psi \subset \mathcal{T}_d$, we obtain $C(X, \mathbb{R}_d) \subset C(X, \mathbb{R}_\psi) \subset C(X)$.

LEMMA 4. *Let X, Y be topological spaces and suppose that $X = A \cup B$, where A, B are closed. We assume, that $f: A \rightarrow Y$ and $g: B \rightarrow Y$ are continuous. If $f(x) = g(x)$ for any $x \in A \cap B$, then a function $h: X \rightarrow Y$ defined by the formula*

$$h(x) = \begin{cases} f(x) & \text{for } x \in A, \\ g(x) & \text{for } x \in B \end{cases}$$

is also continuous.

PROPOSITION 5. *For any topological space, X the space $C(X, \mathbb{R}_\psi)$ is a sublattice of $C(X)$.*

PROOF. Let $f, g \in C(X, \mathbb{R}_\psi)$. From inclusion $C(X, \mathbb{R}_\psi) \subset C(X)$ we obtain that f, g are continuous functions from X to $\mathbb{R}_e$, hence the sets

$$A = \{x \in X: f(x) \geq g(x)\} \quad \text{and} \quad B = \{x \in X: f(x) \leq g(x)\}$$

are closed in X and $A \cup B = X$. Let us define the function

$$f \vee g = \begin{cases} f & \text{on } A, \\ g & \text{on } B. \end{cases}$$

As f is $\tau\psi$ -continuous on A , and g is $\tau\psi$ -continuous on B , so $f \vee g$ is $\tau\psi$ -continuous on both sets A and B . Moreover, $f = g$ on $A \cap B$, so from Lemma 4 we have that $f \vee g \in C(X, \mathbb{R}_\psi)$. Analogously, we show that the function

$$f \wedge g = \begin{cases} f & \text{on } B, \\ g & \text{on } A \end{cases}$$

is $\tau\psi$ -continuous on X . We obtain that for any $f, g \in C(X, \mathbb{R}_\psi)$ the functions $f \vee g$ and $f \wedge g$ are the elements of the space $C(X, \mathbb{R}_\psi)$, so $C(X, \mathbb{R}_\psi)$ is a sublattice of $C(X)$. $\square$

For any $f \in C(X, \mathbb{R}_\psi)$ we put

$$f^+ = f \vee \mathbf{0}, \quad f^- = -f \vee \mathbf{0}, \quad |f| = f^+ \vee f^-,$$

where $\mathbf{0}$ denotes the constant function equal to zero. From Proposition 5 we obtain the following.

COROLLARY 6. *The functions f^+ , f^- , $|f|: X \rightarrow \mathbb{R}_\psi$ are $\tau\psi$ -continuous for any $f \in C(X, \mathbb{R}_\psi)$.*

THEOREM 7. *For any topological space X , the family $Z_\psi(X)$ is a sublattice of $Z(X)$ (partially ordered by inclusion).*

Proof. It is sufficient to show that the union and intersection of two ψ -zerosets of a space X is also a ψ -zeroset of this space. Let $f, g \in C(X, \mathbb{R}_\psi)$. Then, from the above corollary, their absolute values $|f|, |g|$ are also $\tau\psi$ -continuous on X . Hence,

$$|f| \vee |g|, |f| \wedge |g| \in C(X, \mathbb{R}_\psi).$$

Notice that sets

$$Z(f) \cap Z(g) = Z(|f| \vee |g|), \quad Z(f) \cup Z(g) = Z(|f| \wedge |g|)$$

are elements of $Z_\psi(X)$. □

COROLLARY 8. *Let $f \in C(X, \mathbb{R}_\psi)$, $a, b \in \mathbb{R}$, $a \leq b$. Then,*

$$\begin{aligned} \{x \in X: f(x) \leq a\}, \{x \in X: f(x) \geq a\} &\in Z_\psi(X), \\ \{x \in X: f(x) < a\}, \{x \in X: f(x) > a\} &\in \text{Coz}_\psi(X). \end{aligned}$$

Moreover,

$$f^{-1}([a, b]) \in Z_\psi(X), \quad f^{-1}((a, b)) \in \text{Coz}_\psi(X).$$

Proof. Let $\mathbf{a}$ denotes the costant function on X equal to a real number a . From Proposition 5, we have that the functions $f \vee \mathbf{a}$ and $f \wedge \mathbf{a}$ are $\tau\psi$ -continuous on X . The topology $\mathcal{T}_\psi$ is invariant under translation, hence the functions $f \vee \mathbf{a} - \mathbf{a}$ and $f \wedge \mathbf{a} - \mathbf{a}$ are also $\tau\psi$ -continuous on X . We obtain

$$\{x \in X: f(x) \leq a\} = Z(f \vee \mathbf{a} - \mathbf{a}),$$

hence

$$\{x \in X: f(x) \leq a\} \in Z_\psi(X).$$

The set $\{x \in X: f(x) > a\}$ is a complement of $\{x \in X: f(x) \leq a\}$, so it is the ψ -cozeroset of the space X . In the same way, we prove the next conditions. Let us consider the set

$$f^{-1}([a, b]) = \{x \in X: f(x) \geq a\} \cap \{x \in X: f(x) \leq b\}.$$

As $Z_\psi(X)$ is a sublattice of $Z(X)$, then we obtain $f^{-1}([a, b]) \in Z_\psi(X)$. □

THEOREM 9. *Let $f, g \in C(X, \mathbb{R}_\psi)$ and*

$$\text{Coz}(f) \cap \text{Coz}(g) = \emptyset. \tag{1}$$

Then $f + g \in C(X, \mathbb{R}_\psi)$.

Proof. Let x be an arbitrary point of X . Let us consider two cases:

(1) Let $x \in \text{Coz}(f + g)$.

From (1) we obtain that $x \in \text{Coz}(f)$ or $x \in \text{Coz}(g)$. Let us suppose that $x \in \text{Coz}(f)$. Then, $f(x) \neq 0$ and $g(x) = 0$. Hence, there exists the neighbourhood U of x included in $\text{Coz}(f)$ such that $f + g = f$ on U . From this, we have that $f + g$ is $\tau\psi$ -continuous at point x .

(2) Let $x \in Z(f + g)$.

From (1) we obtain that $Z(f + g) = Z(f) \cap Z(g)$. Let V be a $\mathcal{T}_\psi$ -open neighbourhood of 0. We will show that there exists a set $U \in \tau$ such that $f(U) \subset V$. From the assumption $f, g \in C(X, \mathbb{R}_\psi)$, there exist the open neighbourhoods of x : $U_1, U_2 \in \tau$ such that $f(U_1) \subset V$ and $g(U_2) \subset V$. We will show $(f + g)(U_1 \cap U_2) \subset V$. Let $y \in U_1 \cap U_2$. If $y \in \text{Coz}(f)$, then from (1) we have $y \in Z(g)$, so $(f + g)(y) = f(y) \in V$. If $y \in \text{Coz}(g)$, then $(f + g)(y) = g(y) \in V$. If $y \in Z(f) \cap Z(g)$, then $(f + g)(y) = 0 \in V$. Hence, for any $y \in U_1 \cap U_2$ we obtain $(f + g)(y) \in V$, so $(f + g)(U_1 \cap U_2) \subset V$ and, consequently, $f + g \in C(X, \mathbb{R}_\psi)$. □

LEMMA 10. *Let $f \in C(X, \mathbb{R}_\psi)$. Then $\frac{1}{f}$ is $\tau\psi$ -continuous on $\text{Coz}(f)$.*

As it was mentioned earlier, the space $C(\mathbb{R}_\psi, \mathbb{R}_\psi)$ is not a group under addition. Is it possible to determine when $C(X, \mathbb{R}_\psi)$ is a group for any topological space X ? The next lemmas and propositions will help us to answer this question.

Fix a function $\psi \in \mathcal{C}$. Without loss of generality, we can assume that for any natural number n

$$\psi\left(\frac{1}{2^n}\right) \leq 1 \quad \text{and} \quad \frac{1}{2^{n+1}} + \frac{1}{4^{n+1}} \cdot \psi\left(\frac{1}{2^n}\right) < \frac{1}{2^n}.$$

LEMMA 11. *Let*

$$V = (-\infty, 0] \cup \bigcup_{n=1}^{\infty} \left(\frac{1}{2^{n+1}} + \frac{1}{4^{n+1}} \cdot \psi\left(\frac{1}{2^n}\right), \frac{1}{2^n} \right). \quad (2)$$

Then 0 is a ψ -density point of V .

Proof. Notice that $V' = \bigcup_{n=1}^{\infty} \left[\frac{1}{2^{n+1}}, \frac{1}{2^{n+1}} + \frac{1}{4^{n+1}} \cdot \psi\left(\frac{1}{2^n}\right) \right]$. Let $h > 0$. Then there exists a natural number n for which $h \in \left(\frac{1}{2^{n+1}}, \frac{1}{2^n} \right]$. We estimate the expression

$$\begin{aligned} \frac{m(V' \cap [0, h])}{2h\psi(2h)} &\leq \frac{m\left(V' \cap \left[0, \frac{1}{2^n}\right]\right)}{2 \cdot \frac{1}{2^{n+1}} \cdot \psi\left(2 \cdot \frac{1}{2^{n+1}}\right)} \leq \\ &\frac{\sum_{k=n+1}^{\infty} \frac{1}{4^k} \psi\left(\frac{1}{2^k}\right)}{\frac{1}{2^n} \cdot \psi\left(\frac{1}{2^n}\right)} \leq \frac{\psi\left(\frac{1}{2^n}\right) \sum_{k=n+1}^{\infty} \frac{1}{4^k}}{\frac{1}{2^n} \psi\left(\frac{1}{2^n}\right)} < \frac{1}{2^n}. \end{aligned}$$

Therefore, 0 is a ψ -density point of the set V . □

LEMMA 12. *Assume that W is an arbitrary $\mathcal{T}_\psi$ -open neighbourhood of 0. Then the set*

$$W_0 = ((-\infty, 0] \cap W) \cup \bigcup_{n=1}^{\infty} \left(\left(\frac{1}{2^{n+1}} + \frac{1}{4^{n+1}} \cdot \psi\left(\frac{1}{2^n}\right), \frac{1}{2^n} \right) \cap W - \frac{1}{4^{n+1}} \cdot \psi\left(\frac{1}{2^n}\right) \right). \quad (3)$$

is also a $\mathcal{T}_\psi$ -open neighbourhood of 0.

Proof. Assume that V is the set given by formula (2). Let W be an arbitrary $\mathcal{T}_\psi$ -open neighbourhood of 0. Then the set

$$V \cap W = ((-\infty, 0] \cap W) \cup \bigcup_{n=1}^{\infty} \left(\left(\frac{1}{2^{n+1}} + \frac{1}{4^{n+1}} \cdot \psi\left(\frac{1}{2^n}\right), \frac{1}{2^n} \right) \cap W \right)$$

is also $\mathcal{T}_\psi$ -open neighbourhood of 0. We will show that 0 is a right-hand ψ -density point of the set

$$\bigcup_{n=1}^{\infty} \left(\left(\frac{1}{2^{n+1}} + \frac{1}{4^{n+1}} \cdot \psi\left(\frac{1}{2^n}\right), \frac{1}{2^n} \right) \cap W - \frac{1}{4^{n+1}} \cdot \psi\left(\frac{1}{2^n}\right) \right),$$

hence 0 is a ψ -dispersion point of W'_0 . Let $\varepsilon > 0$. From the assumption, there exists a positive number δ such that for any $x \in (0, \delta)$ we have

$$\frac{m(W' \cap [0, x])}{2x \cdot \psi(2x)} < \frac{\varepsilon}{3}. \quad (4)$$

Let N stands for a natural number for which $\frac{1}{2^N} \leq \min(\delta, \frac{\varepsilon}{3})$ and $h \in (0, \frac{1}{2^N})$. There is $n \geq N$ such that $h \in [\frac{1}{2^{n+1}}, \frac{1}{2^n})$. Let us notice that for any $k \geq n+1$ we have

$$\begin{aligned} m\left(W_0 \cap \left[\frac{1}{2^{k+1}}, \frac{1}{2^k}\right)\right) &= m\left(\left[\frac{1}{2^{k+1}} + \frac{1}{4^{k+1}} \cdot \psi\left(\frac{1}{2^k}\right), \frac{1}{2^k}\right) \cap W \right. \\ &\quad \left. - \frac{1}{4^{k+1}} \cdot \psi\left(\frac{1}{2^k}\right)\right) \\ &= m\left(\left[\frac{1}{2^{k+1}} + \frac{1}{4^{k+1}} \cdot \psi\left(\frac{1}{2^k}\right), \frac{1}{2^k}\right) \cap W\right) \\ &= m\left((V \cap W) \cap \left[\frac{1}{2^{k+1}}, \frac{1}{2^k}\right)\right). \end{aligned}$$

Hence,

$$\begin{aligned} m\left(W'_0 \cap \left[\frac{1}{2^{k+1}}, \frac{1}{2^k}\right)\right) &= m\left((V' \cup W') \cap \left[\frac{1}{2^{k+1}}, \frac{1}{2^k}\right)\right) \\ &\leq m\left(V' \cap \left[\frac{1}{2^{k+1}}, \frac{1}{2^k}\right)\right) + m\left(W' \cap \left[\frac{1}{2^{k+1}}, \frac{1}{2^k}\right)\right). \end{aligned} \quad (5)$$

Moreover,

$$\begin{aligned} &m\left(W'_0 \cap \left[\frac{1}{2^{n+1}}, h\right]\right) \\ &\leq m\left(V' \cap \left[\frac{1}{2^{n+1}}, h\right]\right) + m\left(W' \cap \left[\frac{1}{2^{n+1}}, h + \frac{1}{4^{n+1}} \cdot \psi\left(\frac{1}{2^n}\right)\right]\right) \\ &\leq m\left(V' \cap \left[\frac{1}{2^{n+1}}, h\right]\right) + m\left(W' \cap \left[\frac{1}{2^{n+1}}, h\right]\right) + \frac{1}{4^{n+1}} \cdot \psi\left(\frac{1}{2^n}\right). \end{aligned} \quad (6)$$

From (5) and (6) we obtain

$$m(W'_0 \cap [0, h]) \leq m(V' \cap [0, h]) + m(W' \cap [0, h]) + \frac{1}{4^{n+1}} \cdot \psi\left(\frac{1}{2^n}\right).$$

From the above and (4), we have inequalities

$$\begin{aligned} \frac{m(W'_0 \cap [0, h])}{2h \cdot \psi(2h)} &\leq \frac{m(W' \cap [0, h])}{2h \cdot \psi(2h)} + \frac{m(V' \cap [0, \frac{1}{2^n}])}{2 \cdot \frac{1}{2^{n+1}} \cdot \psi(\frac{2}{2^{n+1}})} + \frac{\frac{1}{4^{n+1}} \cdot \psi(\frac{1}{2^n})}{\frac{1}{2^n} \cdot \psi(\frac{1}{2^n})} \\ &< \frac{\varepsilon}{3} + \frac{\sum_{k=n}^{\infty} \frac{1}{4^{k+1}} \psi(\frac{1}{2^k})}{\frac{1}{2^n} \psi(\frac{1}{2^n})} + \frac{1}{2^n} < \varepsilon \end{aligned}$$

and we obtain, that 0 is a ψ -dispersion point of W'_0 . $\square$

Now, we construct the functions which enable us to solve the main problem of this paper.

LEMMA 13. *Let $\psi \in \mathcal{C}$ fulfil the condition $(\Delta 2)$.*

Suppose that $f \in C(X, \mathbb{R}_\psi)$ is a nonnegative function and its zeroset is not open. Let g_n be a linear mapping from $[\frac{1}{2^{n+1}} - \frac{1}{4^{n+2}} \cdot \psi(\frac{1}{2^{n+1}}), \frac{1}{2^{n+1}}]$ onto $[\frac{1}{2^{n+1}}, \frac{1}{2^{n+1}} + \frac{1}{4^{n+1}} \cdot \psi(\frac{1}{2^n})]$, $n \in \mathbb{N}$.

If $g: X \rightarrow \mathbb{R}_\psi$ is defined by the following formula

$$g(x) = \begin{cases} \frac{1}{16} \cdot \psi\left(\frac{1}{2}\right) + f(x) & \text{for } x \in f^{-1}\left(\left[\frac{1}{2} - \frac{1}{16} \cdot \psi\left(\frac{1}{2}\right), \infty\right)\right), \\ \frac{1}{4^{n+1}} \cdot \psi\left(\frac{1}{2^n}\right) + f(x) & \text{for } x \in f^{-1}\left(\left[\frac{1}{2^{n+1}}, \frac{1}{2^n} - \frac{1}{4^{n+1}} \cdot \psi\left(\frac{1}{2^n}\right)\right]\right), \\ & n \geq 2 \\ (g_n \circ f)(x) & \text{for } x \in f^{-1}\left(\left[\frac{1}{2^{n+1}} - \frac{1}{4^{n+2}} \cdot \psi\left(\frac{1}{2^{n+1}}\right), \frac{1}{2^{n+1}}\right]\right), \\ & n \geq 2 \\ 0 & \text{for } x \in Z(f), \end{cases} \quad (7)$$

then g is $\tau\psi$ -continuous on X .

P r o o f. If $f \in C(X, \mathbb{R}_\psi)$ and condition $(\Delta 2)$ is fulfilled, then g_n is ψ -continuous, hence $g_n \circ f$ is $\tau\psi$ -continuous on X for any $n \in \mathbb{N}$. From (7) we have that g is $\tau\psi$ -continuous on $X \setminus Z(f)$ and $\text{Int } Z(f)$. We will show that g is $\tau\psi$ -continuous at each point of the set $Z(f) \setminus \text{Int } Z(f)$. Let W be any $\mathcal{T}_\psi$ -open neighbourhood of 0. Then, the set W_0 defined in Lemma 12, is the $\mathcal{T}_\psi$ -open neighbourhood of 0. Let $a \in Z(f) \setminus \text{Int } Z(f)$. Function f is $\tau\psi$ -continuous, so there exists a neighbourhood $U \in \tau$ of a such that $f(U) \subset W_0$. We will prove that $g(U) \subset W$. Let $x \in U$.

- (a) If $x \in Z(f)$, then $g(x) = 0$. So, $g(x) \in W$.
- (b) If $x \notin Z(f)$, then $f(x) \in W_0$ and there exists $N \in \mathbb{N}$ such that

$$f(x) \in \left(\frac{1}{2^{N+1}} + \frac{1}{4^{N+1}} \cdot \psi\left(\frac{1}{2^N}\right), \frac{1}{2^N} \right) \cap W - \frac{1}{4^{N+1}} \cdot \psi\left(\frac{1}{2^N}\right).$$

Hence,

$$f(x) + \frac{1}{4^{N+1}} \cdot \psi\left(\frac{1}{2^N}\right) \in \left(\frac{1}{2^{N+1}} + \frac{1}{4^{N+1}} \cdot \psi\left(\frac{1}{2^N}\right), \frac{1}{2^N} \right) \cap W \subset W. \quad (8)$$

On the other hand,

$$\begin{aligned} f(x) &\in \left(\frac{1}{2^{N+1}}, \frac{1}{2^N} + \frac{1}{4^{N+1}} \cdot \psi\left(\frac{1}{2^N}\right) \right) \cap \left(W - \frac{1}{4^{N+1}} \cdot \psi\left(\frac{1}{2^N}\right) \right) \\ &\subset \left(\frac{1}{2^{N+1}}, \frac{1}{2^N} + \frac{1}{4^{N+1}} \cdot \psi\left(\frac{1}{2^N}\right) \right). \end{aligned}$$

Therefore, on the set $f^{-1}\left(\frac{1}{2^{N+1}}, \frac{1}{2^N} + \frac{1}{4^{N+1}} \cdot \psi\left(\frac{1}{2^N}\right)\right)$ we have

$$g(x) = f(x) + \frac{1}{4^{N+1}} \cdot \psi\left(\frac{1}{2^N}\right).$$

From this and (8), we get $g(x) \in W$.

We obtain that $g(U) \subset W$. Hence the function g is $\tau\psi$ -continuous at a and $g \in C(X, \mathbb{R}_\psi)$. $\square$

PROPOSITION 14. *Suppose that all assumptions of Proposition 13 are fulfilled and g is the function defined by the formula (7). Then $g - f: X \rightarrow \mathbb{R}_\psi$ is not $\tau\psi$ -continuous on the set $Z(f) \setminus \text{Int } Z(f)$.*

Proof. Let $h = g - f$. Then

$$h(x) = \begin{cases} \frac{1}{16} \cdot \psi\left(\frac{1}{2}\right) & \text{for } x \in f^{-1}\left(\left[\frac{1}{2} - \frac{1}{16} \cdot \psi\left(\frac{1}{2}\right), \infty\right)\right), \\ \frac{1}{4^{n+1}} \cdot \psi\left(\frac{1}{2^n}\right) & \text{for } x \in f^{-1}\left(\left[\frac{1}{2^{n+1}}, \frac{1}{2^n} - \frac{1}{4^{n+1}} \cdot \psi\left(\frac{1}{2^n}\right)\right]\right), \\ & n \geq 2 \\ (g_n \circ f - f)(x) & \text{for } x \in f^{-1}\left(\left[\frac{1}{2^{n+1}} - \frac{1}{4^{n+2}} \cdot \psi\left(\frac{1}{2^{n+1}}\right), \frac{1}{2^{n+1}}\right]\right), \\ & n \geq 2 \\ 0 & \text{for } x \in Z(f). \end{cases} \quad (9)$$

We will show that h is not $\tau\psi$ -continuous on the set $Z(f) \setminus \text{Int } Z(f)$. Let U be a $\mathcal{T}_\psi$ -open set defined as follows:

$$U = (-\infty, 0] \cup \bigcup_{n=1}^{\infty} \left(\frac{1}{2^{n+1}}, \frac{1}{2^n} - \frac{1}{4^{n+1}} \cdot \psi\left(\frac{1}{2^n}\right) \right).$$

Put

$$C = h^{-1}\left(\left\{\frac{1}{4^{n+1}} \cdot \psi\left(\frac{1}{2^n}\right)\right\}_{n=1}^{\infty}\right).$$

Let $a \in Z(f) \setminus \text{Int } Z(f)$. Suppose that C is closed. Then, the set

$$\begin{aligned} A &= (X \setminus C) \cap f^{-1}(U) \\ &= (X \setminus C) \cap \left(Z(f) \cup \bigcup_{n=n_0}^{\infty} f^{-1}\left(\frac{1}{2^{n+1}}, \frac{1}{2^n} - \frac{1}{4^{n+1}} \cdot \psi\left(\frac{1}{2^n}\right)\right) \right) \end{aligned}$$

is the open neighbourhood of the point a and A is the subset of $Z(f)$. We obtain that $a \in A$ and $a \notin \text{Int } Z(f)$, hence,

$$A \cap \text{Coz}(f) \neq \emptyset.$$

We have a contradiction.

From the above, C is not closed. The set $Y = \left\{\frac{1}{4^{n+1}} \cdot \psi\left(\frac{1}{2^n}\right)\right\}_{n=1}^{\infty}$ is $\mathcal{T}_\psi$ -closed (as a set of measure 0), but its preimage is not closed in X . Hence, the function h is not $\tau\psi$ -continuous at a . $\square$

The next theorem presents the main result of this paper.

THEOREM 15. *Let $\psi \in \mathcal{C}$ fulfil the condition $(\Delta 2)$. The following conditions are equivalent:*

- (1) $C(X, \mathbb{R}_\psi) = C(X, \mathbb{R}_{discr})$.
- (2) $C(X, \mathbb{R}_\psi)$ is a ring.
- (3) $C(X, \mathbb{R}_\psi)$ is closed under multiplication.

(4) $C(X, \mathbb{R}_\psi)$ is a group.

(5) $Z(f)$ is open for any function $f \in C(X, \mathbb{R}_\psi)$.

Proof. Implications (2) $\Rightarrow$ (3) and (2) $\Rightarrow$ (4) follow from the definition of a ring. Implication (1) $\Rightarrow$ (2) is obvious, because $C(X, \mathbb{R}_{discr})$ is a ring.

Let us consider the implication (3) $\Rightarrow$ (4). We will show that $C(X, \mathbb{R}_\psi)$ is closed under addition. Let $f, g \in C(X, \mathbb{R}_\psi)$. If $f_1(x) = e^x$, $g_1(x) = \ln x$ and $(\Delta 2)$ is fulfilled, then the functions f_1, g_1 are ψ -continuous [FT2, Theorem 11]. Then compositions $(f_1 \circ f)(x) = e^{f(x)}$ and $(g_1 \circ g)(x) = e^{g(x)}$ are $\tau\psi$ -continuous. From the assumption, $C(X, \mathbb{R}_\psi)$ is closed under multiplication, so

$$\begin{aligned} e^{f(x)} \cdot e^{g(x)} &= e^{f(x)+g(x)} \in C(X, \mathbb{R}_\psi), \\ \ln e^{f(x)+g(x)} &\in C(X, \mathbb{R}_\psi), \\ f(x) + g(x) &\in C(X, \mathbb{R}_\psi). \end{aligned}$$

We obtain that $C(X, \mathbb{R}_\psi)$ is closed under addition, so it is a group.

Next, we show the implication (1) $\Rightarrow$ (5). If $C(X, \mathbb{R}_\psi) = C(X, \mathbb{R}_{discr})$, then for any function $f \in C(X, \mathbb{R}_\psi)$ preimage of any set open in $\mathbb{R}_{discr}$ is open in X . So, the set $f^{-1}(\{0\}) = Z(f)$ is open in X .

We will prove the implication (5) $\Rightarrow$ (1) now. Observe that $f \in C(X, \mathbb{R}_{discr})$ if and only if $f^{-1}(\{y\}) \in \tau$ for any $y \in \mathbb{R}$, that means, f is locally constant. We will show that any function $f \in C(X, \mathbb{R}_\psi)$ with open zeroset belongs to the family $C(X, \mathbb{R}_{discr})$. Take an arbitrary function $f \in C(X, \mathbb{R}_\psi)$ and a point $y \in \mathbb{R}$. If there is no x_0 such that $y = f(x_0)$, then $f^{-1}(\{y\}) = \emptyset \in \tau$. Consider the case when $y = f(x_0)$ for a certain $x_0 \in X$. Put $g(x) = f(x) - f(x_0)$, $x \in X$. Then $g \in C(X, \mathbb{R}_\psi)$ and the set $Z(g) = \{x \in X : f(x) = f(x_0)\}$ is open from (5) and not empty as $x_0 \in Z(g)$. Hence, we obtain that $f^{-1}(\{y\}) = Z(g) \in \tau$ and $f \in C(X, \mathbb{R}_{discr})$. As $\mathbb{R}_\psi \subset \mathbb{R}_{discr}$, so $C(X, \mathbb{R}_\psi) \supset C(X, \mathbb{R}_{discr})$ and we have the equality $C(X, \mathbb{R}_\psi) = C(X, \mathbb{R}_{discr})$.

Let us consider the implication (4) $\Rightarrow$ (1). Suppose that $C(X, \mathbb{R}_\psi) \neq C(X, \mathbb{R}_{discr})$. Then there exists a function $f \in C(X, \mathbb{R}_\psi)$ which zeroset is not open in X . From Proposition 14, there is a function $g \in C(X, \mathbb{R}_\psi)$ such that $g - f$ is not $\tau\psi$ -continuous on $Z(f) \setminus \text{Int } Z(f)$. Hence, $C(X, \mathbb{R}_\psi)$ is not a group.

The proof of the last implication completes the proof of theorem. $\square$

COROLLARY 16. *The identity function $f(x) = x$ is ψ -continuous. Its zeroset is not open in $\mathcal{T}_\psi$, because it is a set of measure zero. Hence, $C(\mathbb{R}_\psi, \mathbb{R}_\psi)$ is not a group, so, it is not a ring. Analogously, we can show that $C(\mathbb{R}_d, \mathbb{R}_d)$ and $C(\mathbb{R}_d, \mathbb{R}_\psi)$ are not groups.*

Similar results for the family $C(X, \mathbb{R}_d)$ have been obtained by Michelle KNOX in [K].

A NEW APPROACH TO ψ -CONTINUITY

REFERENCES

- [F] FILIPCZAK, M.: σ -ideals, topologies and multiplication, Bull. Soc. Sci. Lett. Łódź **52** (2002), 11–16.
- [FT1] FILIPCZAK, M.—TEREPETA, M.: *On continuity concerned with ψ -density topologies*, Tatra Mt. Math. Publ. **34** (2006), 29–36.
- [FT2] FILIPCZAK, M.—TEREPETA, M.: *ψ -continuous functions*, Rend. Circ. Mat. Palermo **58** (2009), 245–255.
- [K] KNOX, M. L.: *A characterization of rings of density continuous functions*, Real Anal. Exchange **31** (2005), 165–177.
- [TW-B] TEREPETA, M.—WAGNER-BOJAKOWSKA, E.: *ψ -density topology*, Rend. Circ. Mat. Palermo **48** (1999), 451–476.

Received October 23, 2007

Center of Mathematics and Physics
Łódź Technical University
al. Politechniki 11
PL-90-924 Łódź
POLAND
E-mail: ttrp@poczta.onet.pl