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LOGICAL RULES AND THE DETERMINACY OF MEANING

Abstract. The use of conventional logical connectives either in logic, in mathematics, or in both cannot determine the meanings of those connectives. This is because every model of full conventional set theory can be extended conservatively to a model of intuitionistic set plus class theory, a model in which the meanings of the connectives are decidedly intuitionistic and nonconventional. The reasoning for this conclusion is acceptable to both intuitionistic and classical mathematicians. En route, I take a detour to prove that, given strictly intuitionistic principles, classical negation cannot exist.

Keywords: negation, truth value, intuitionistic set theory, use theory of meaning.

For Professor Pavel Materna

1. Preliminaries and Disclosures

I have three things to accomplish. First, I will give definitive and unassailable answers to such questions as "What is a truth value?" and "What is negation?" Second, I will remind you that any model of a conventional (aka classical) set theory can be extended conservatively to a model of a strictly intuitionistic set and class theory. To mathematical logic insiders, this hardly counts as breaking news. However, it is well worth underscoring for the sake of point number three: no widely popular and repeated use of the conventional syntactic rules governing the propositional signs $\wedge$, $\vee$, $\rightarrow$, and $\neg$, known to college freshmen everywhere, determines the meanings of those signs. This remains true even when those rules get added to principles and theorems of conventional transfinite set theory.

In a spirit of full disclosure, I should let you know that I am an intuitionist in lineal intellectual descent from du Bois-Reymond, Brouwer, and Heyting. Except for one small point, which I note *infra*, this biographical

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fact will however not disturb the plain cogency of the argumentation in this essay: conventional mathematicians must acknowledge it as correct throughout. As a practicing intuitionist, I reject categorically the pipe-dream that there are specifically intuitionistic connectives, in other words, that the intuitionist can somehow, by magical means known only to initiates, imbue the signs $\wedge$, $\vee$, $\rightarrow$, and $\neg$ with meanings distinct from those attached somehow to the same signs as employed conventionally. My thoroughgoing rejection of the fantastical 'intuitionistic meanings' is neither premise or presupposition to my reasoning herein. Should I, from time to time, mention 'intuitionistic connectives' or 'intuitionistic truth values,' I am not lapsing into *lingua vulgaris*. I am, for the sake of argument, presuming the defective views of my fell adversaries pro tem–in order to squelch them.

2. What is a Truth Value?

Negation is a truth function, a logical operation. Here is a partial graph of that function.

$$T \mapsto F$$

$$F \mapsto T$$

To what do the letters T and F refer in this childish diagram? Of course, they refer to truth values, to Frege's 'The True' and 'The False,' respectively. And what is a truth value? It is a subset of $\{0\}$, a member of the set $\mathcal{P}(\{0\})$. I shall now prove this.

Theorem: The truth values are (isomorphic to) the subsets of $\{0\}$.

Proof. Let A be any mathematical statement. Map A to $\hat{A}$ where

$$\hat{A} = \{0|A\} = \{x|x = 0 \wedge A\},$$

which is plainly a member of $\mathcal{P}(\{0\})$. This map is surjective because, for $S \,\varepsilon\, \mathcal{P}(\{0\})$, the mathematical statement

$$0 \,\varepsilon\, S$$

maps into the set

$$\widehat{0 \,\varepsilon\, S} = S.$$

Extensionality shows that the map $A \mapsto \hat{A}$ is injective, because for statements A and B,

$$A \leftrightarrow B$$

just in case

$$\hat{A} = \hat{B}.$$

Moreover, the map is a logical isomorphism for, under it, $\wedge$ maps into $\cap$, $\vee$ into $\cup$, $\neg$ into the set operation of complementation, F into $\emptyset$ (the $\subseteq$-least truth value), and T into $\{0\}$ (the $\subseteq$-greatest truth value). (Left as an exercise for the reader.) $\Box$

With this picture of the set of truth values in mind, and a smidgeon of set theory, one can define $\neg$ outright, without circularity: for $p \,\varepsilon\, \mathcal{P}(\{0\})$,

$$\neg p = \bigcup\{q \,\varepsilon\, \mathcal{P}(\{0\}) | q \cap p = \emptyset\}.$$

So, in answer to my original questions, a truth value is just an element of $\mathcal{P}(\{0\})$, and negation is the function taking p into the $\subseteq$-largest member of $\mathcal{P}(\{0\})$ disjoint from it. The interested reader will find further investigation and elaboration of these answers in [McCarty 2018].

3. There is Only One Negation

This is an intermission feature, ancillary to the main points of the essay. I prove here that the imaginary 'classical negation' is indeed imaginary. It cannot exist. Of course, to prove that, I will need to assume a specifically intuitionistic principle, the Uniformity Principle. After that, I prove, without recourse to Uniformity, there to be exactly one negation operation, despite the tedious cavils of nonstandard logicians, *inter alios.*

Theorem: The 'negation operation' governed by conventional logic does not exist.

Proof. Over $\mathcal{P}(\{0\})$, all assignments of natural numbers in $\mathbb{N}$ to truth-values must be perfectly uniform [Troelstra 1980], in other words, for R any binary mathematical relation,

$$\forall p \,\varepsilon\, \mathcal{P}(\{0\}) \exists n \,\varepsilon\, \mathbb{N}. R(p, n) \rightarrow \exists n \,\varepsilon\, \mathbb{N}\ \forall p \,\varepsilon\, \mathcal{P}(\{0\}). R(p, n).$$

Intuitionists call this *The Uniformity Principle* or **UP** for short. It is a truth, first cousin to Brouwer's *Principle for Numbers* [Troelstra & van Dalen 1988 209], which guides the way to proving Brouwer's famous Continuity Theorem on the reals.

For the nonce, let $\sim$ be a sign for the mythical conventional negation. Were $\sim$ to exist it would have to map $\mathcal{P}(\{0\})$ into $\mathcal{P}(\{0\})$ in a nonconstant fashion, and to satisfy the condition [$\star$]:

$$[\star] \qquad \forall p \,\varepsilon\, \mathcal{P}(\{0\})((\sim p) = T \vee (\sim p) = F).$$

The truth of the formula above is required by the (false) laws of conventional logic: any truth value must be either true or false.

Now, let R be the relation over $\mathcal{P}(\{0\}) \times \mathbb{N}$ where $R(p, n)$ holds just in case

$$[(n = 0 \wedge (\sim p) = F) \vee (n = 1 \wedge (\sim p) = T)].$$

Because of [$\star$], we know that

$$\forall p \,\varepsilon\, \mathcal{P}(\{0\}) \exists n \,\varepsilon\, \mathbb{N}.R(p, n).$$

By **UP**,

$$\exists n \,\varepsilon\, \mathbb{N} \; \forall p \,\varepsilon\, \mathcal{P}(\{0\}).R(p, n).$$

Therefore, there is a single natural number n to which all truth values p are related by R. Since $\sim$ is nonconstant, this is a contradiction. Consequently, 'classical negation' $\sim$ cannot exist. $\square$

Now I prove, without the aid of intuitionistic principles, that there can be only one negation in the intelligible universe. There can be no other. For the proof, it suffices to assume that negation maps F into T, as in the graph above, and that p and $\neg p$ cannot both be true simultaneously.

Theorem: There is a single negation operation only.

Proof. Assume, for *reductio*, that f is a function on $\mathcal{P}(\{0\})$ that obeys the logical laws of negation, to whit, that

$$f(F) = T,$$

and that, for any truth value p, p and $f(p)$ cannot both be true simultaneously:

$$\forall p \,\varepsilon\, \mathcal{P}(\{0\})(p \wedge f(p)) = F.$$

These are the sole assumptions of the proof.

By definition, every truth value p is a subset of $\{0\}$. Hence, it makes sense to ask if $0 \,\varepsilon\, p$ or $0 \,\varepsilon\, f(p)$, and we know that

$$0 \,\varepsilon\, f(p) \leftrightarrow f(p) = T.$$

As we have seen, the intersection of p with $f(p)$ is always $\emptyset$ and $f(F) = T$. Therefore,

$$p = F \leftrightarrow f(p) = T.$$

The very same properties hold of the genuine operation of negation $\neg$. It follows that

$$p = F \leftrightarrow (\neg p) = T \leftrightarrow 0 \,\varepsilon\, (\neg p).$$

Once we have these three rows of true biconditionals, we see that

$$0 \,\varepsilon\, f(p) \leftrightarrow 0 \,\varepsilon\, (\neg p).$$

By Extensionality for subsets of $\{0\}$, the operations $f(p)$ and $\neg p$ are identical, for any $p \,\varepsilon\, \mathcal{P}(\{0\})$. $\square$

Nota Bene: For the last proof, no proprietary intuitionistic principle such as **UP** was employed. Any conventional mathematician should accept the reasoning.

En passant, it is well worth asking, in the face of the mathematical fact that there is and can be only one negation, why some logicians think that there is more than one. These clever folks chat merrily about 'nonstandard negation' and 'weak negation.' When they utter such stuff, they seem to be refering to *symbols* in various formal systems of logic, e.g., the symbol $\sim$ in, say, classical logic or some intermediate logic. Were those systems sound in the usual way, we might allow that those symbols, once interpreted, manifest some, but not all, of the mathematical properties we rightly look for in the $\neg$ operation. However, to say that we can create a formal system with a feature that describes X is not to say that the feature does indeed describe X, or even that X exists at all! In the twinkle of an eye, I can create a formal system that codifies and represents properties of transparent cats, e.g., it contains as axioms the claims 'They make mewing noises,' and 'You can see right through them.' The trouble is that there are no transparent cats, just as there are no nonstandard negations.

4. Laws of Logic

Once the truth values and logical operations are identified, one can, from a suitable and obvious definition of valid inference and a little more set theory, determine the laws of logic.

Definition: (*logical validity*) For propositional formulae $\Theta(p, q)$ and $\Psi(p, q)$, the inference

$$\Theta(p, q) \vdash \Psi(p, q)$$

is *logically valid* if and only if, for any mathematical statements A and B,

$$\Theta(\widehat{A, B}) \subseteq \Psi(\widehat{A, B}).$$

To repeat, the map $A \mapsto \hat{A}$ is the canonical function from statements into their truth values. The above *definiens* is a plain, set-theoretic statement describing a truth functional relation on $\mathcal{P}(\{0\})$. From that definition, it is easy to certify various inferences as valid.

Theorem: The following inferential statements are all demonstrable.

1. $\bot \ \vdash \Theta$,
2. $\Theta, \neg\Theta \vdash \bot$,
3. If $\Theta, \Psi \vdash \bot$, then $\Theta \vdash \neg\Psi$, and
4. (under the false assumption that T and F are the sole truth values) If $\Theta, \neg\Psi \vdash \bot$, then $\Theta \vdash \Psi$.

So, set-theoretic mathematics plus obvious definitions determine the rules of deductive logic.

5. Extending Models of Set Theory

In effect, the next questions before us will be, "Can one go the other way? Can the adoption of a particular set of syntactical rules for logical deduction determine, in reverse, the meanings of the connective symbols in those rules?" I will prove that the answer is a resounding "No."

Theorem: Every model of conventional set theory can be extended conservatively to an *intuitionistic* model of set and class theory, one featuring distinctly 'intuitionistic' truth values, ones other than T and F.

Proof. Let M be any model of conventional set theory. We extend it to an intuitionistic model M^τ with topologically-valued sets and classes over the usual order topology τ on Sierpinski space $\{0, 1\}$ where $0 < 1$. M^τ has, therefore, exactly three truth values:

$$\emptyset, \{1\}, \text{ and } \{0, 1\}.$$

In this interpretation, F denotes the $\subseteq$-least value, $\emptyset$, and T denotes the greatest, $\{0, 1\}$.

Here, capital Roman letters from the beginning of the alphabet A, B, etc. range over maps from the domain of M, $|M|$, into open sets in τ. ϕ and ψ are sentences in the multi-sorted, first-order language for sets and classes in which ε is the sole primitive predicate. ϕ and ψ may feature parameters for sets in M and classes in M^τ. As usual in these contexts, the wide brackets on a formula ϕ or other syntactic item,

$$[\![\phi]\!],$$

denotes the semantic value of the formula or item over M^τ. $\mathfrak{I}$ is the interior operation in τ. This time, $\sim$ is ordinary set complement relative to $\{0,1\}$, and $\Rightarrow$ is the usual Heyting implication in τ.

Definition: (*topological model M^τ*)

$[\![A]\!]$ is a function from $|M|$ into τ, for all A.

For $a, b\,\varepsilon\,|M|$, $[\![a]\!]$ is a function from $|M|$ into τ such that

$$[\![a]\!](b) = \begin{cases} T & M \models b\,\varepsilon\,a \\ F & \text{otherwise} \end{cases}$$

For $a, b\,\varepsilon\,|M|$, $[\![a\,\varepsilon\,b]\!] = [\![a]\!](b)$

$[\![a\,\varepsilon\,A]\!] = [\![A]\!](a)$

$[\![\phi \wedge \psi]\!] = [\![\phi]\!] \cap [\![\psi]\!]$

$[\![\phi \vee \psi]\!] = [\![\phi]\!] \cup [\![\psi]\!]$

$[\![\neg\phi]\!] = \mathfrak{I}(\sim [\![\phi]\!])$

$[\![\phi \to \psi]\!] = [\![\phi]\!] \Rightarrow [\![\psi]\!]$

$[\![\exists x.\phi(x)]\!] = \bigcup_{a\,\varepsilon\,|M|}[\![\phi(a)]\!]$

$[\![\forall x.\phi(x)]\!] = \mathfrak{I}(\bigcap_{a\,\varepsilon\,|M|}[\![\phi(a)]\!])$

$[\![\exists A.\phi(A)]\!] = \bigcup_A[\![\phi([\![A]\!])]\!]$

$[\![\forall A.\phi(A)]\!] = \mathfrak{I}(\bigcap_A[\![\phi([\![A]\!])]\!])$

Finally, $M^\tau \models \phi$ if and only if $[\![\phi]\!] = T$.

Theorem: For ϕ and ψ sentences in a standard, two-sorted, first-order language for set and class theory, if $\phi \vdash \psi$ in Heyting's predicate calculus, then $[\![\phi]\!] \subseteq [\![\psi]\!]$ over M^τ.

Proof. Familiar from such literature as [Grayson 1979]. □

6. Internal Mathematics: Main Theorem

What then is the internal mathematics of M^τ?

Lemma: (*absoluteness*) For ϕ a strictly first-order sentence in the language of set theory without variables or parameters for classes, $M^\tau \models \phi$ just in case $M \models \phi$.

Proof. Proof of lemmas like this proceed by induction on formulae ϕ, and are standard in the literature. *Vide* [Bell 1977] or [Grayson 1979]. □

Theorem: The topological model M^τ satisfies Zermelo-Frankel set theory plus an intuitionistic class theory with full Comprehension.

Proof. From the **Lemma** and standard results as in [Grayson 1979]. □

What does the set of truth-values in M^τ look like? As remarked, when viewed externally, there are three 'intuitionistic truth values,' the τ-open sets $\emptyset$, $\{1\}$, and $\{0, 1\}$. The conventional *Tertium non Datur*, $\forall p(p \vee \neg p)$, does not hold over M^τ since, when $[\![p]\!] = \{1\}$, $[\![\neg p]\!] = \emptyset$. Hence, our opponents–those who believe that there are special, intuitionistic meanings for the connective symbols and appeal to those meanings in a vain effort to explain the failure of the *Tertium non Datur* in intuitionistic mathematics–would insist that the meanings in M^τ of connectives such as $\neg$ are definitely intuitionistic.

7. The Determinacy of Meaning

At last to our third and final truth: any strictly syntactic use of the rules for full conventional or classical predicate logic, plus Zermelo-Fraenkel set theory, do not determine the meanings of the logical signs to be classical.

Readers may recall (or heard tell of) an argument Hilary Putnam constructed back in 1975, the Twin Earth argument [Putnam 1975]. The fabled conclusion of that argument was that meanings aren't in the head, more precisely, that the narrowly individuated mental contents associated in a speaker with a word do not determine the meaning and referent of that word. On behalf of that conclusion, Putnam imagined a possible world he named 'Twin Earth.' It is just like our own world in time up to 1750, including the mental contents of speakers, apart from the important fact that all the water-looking liquid on Twin Earth is not ordinary H_2O, but a sensory

Doppelgänger with chemical formula XYZ. The latter chemical looks and tastes and feels just like water but, as far as anyone living before 1750 could tell, it is not water, not H_2O. The point of the fantasy is that the Twin-Earthians entertain exactly the same mental contents as our ancestors on Earth did when using the term 'water,' but that term does not pick out, on Twin Earth, real water. Hence, since the Twin Earth scenario is possible, the mental contents of whole tribes of speakers do not suffice to determine the meanings and referents of their words.

I am far from endorsing the details of this famous *Gedankenexperiment.* I wish merely to draw a high-level analogy between it and the demonstration that closes this essay. The firm and repeated use of conventional rules governing classical logic, conceived as syntax and not with their (nonexistent) meanings attached, do not suffice to determine the meanings and referents of the signs deployed in those rules. Why? Not because some imaginary Twin Earth might exist. Rather because a model M^τ is proved to exist–one that agrees with conventional model M on its strictly first-order, set-theoretic portion, logical rules included. Hence, conventional logicians and mathematicians can use their rules as much as they want in developing conventional set theory, indeed all of it. Yet, for all they know, the real model for their theorizing is nothing but the classical portion of our M^τ. That is possible because what is real is possible. There is no way that conventional logicians and mathematicians can tell the difference, since M and M^τ cannot differ on strictly set-theoretic statements, thanks to absoluteness. However, in M^τ, the $\neg$ sign denotes not the function whose entire graph is thought to be

$$T \mapsto F$$

$$F \mapsto T$$

but the wholly distinct topological or intuitionistic operation on three-valued τ,

$$\emptyset, \{1\}, \text{ and } \{0,1\}.$$

that takes, for example, $\{1\}$ into $\emptyset$. So, meanings aren't in the rules either.

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Citations

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