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Perspective of Dimensional Analysis in Medical Science

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Abstract. This paper presents several applications of the dimensional analysis method to problems investigated in medical sciences. The method is used to analyze various complex processes without using formal laws governing the same. It is particularly suitable for a general analysis of fluid transfer (liquids and gases) in the human body. This paper mainly serves as an overview of selected applications, mostly those emerging in the recent years, and includes a discussion of the mathematical fundamentals of dimensional analysis together followed by its critical analysis. Containing detailed calculations of two examples, the paper also serves as training material in the area of the computational method of the dimensional analysis algorithm.

Introduction

The first ideas forecasting the method of dimensional analysis came from Fourier, who proposed the term dimensional homogeneity of physical equations in 1822 (Gibbings, 2011; Tan, 2011). The first person to apply these ideas was Lord Rayleigh in a series of his papers published in the 70s and 80s of the 19th century; his research particularly concerned the processes of fluid flow. Research on fluid viscosity based on dimensional analysis (not quite formalized then) was also conducted by Prandl in the first decade of the 20th century. The method owns its typical form to Buckingham (Buckingham, 1914), who proved that reduction of problem complexity was possible in the form of dimensionless parameters. The first objective of the presented paper is to overview selected applications of dimensional analysis to medical problems. Another objective is to deliver a detailed presentation of a computational procedure for the algorithm, which does not go beyond multiplication operations, hence being universally available. Nevertheless, whoever may perform the calculations, the initial condition required to apply the method is the assistance of an expert in the field. The expert's role is to define the parameters determining the process being studied as completely as possible. The completeness of the set of parameters is crucial since it affects the quality of the final result. The basic feature of the method makes it possible to gain essential knowledge about the reliance of the process on the specified parameters, at the same time omitting the necessity to use complex physical equations. This fact is of importance in the case of flow (e.g. blood transfer in the body), especially for persons whose knowledge in the field of physics and numerical methods is limited. In addition, the method can be used to reduce (sometimes significantly) the problem to a lower number of new parameters. In this respect, the method is similar to statistical dimensionality reduction techniques such as PCA (Principal Component Analysis) and SPA (Successive Projections Algorithm). Finally, when the method is supplemented with a reasonable number of measurements, it is possible to obtain an explicit form of the function – an accurate or almost accurate course of the process.

The following section begins with a discussion of the mathematical fundamentals of the dimensional analysis algorithm and demonstrates the principle using a very simple example of flow through a circular-section tube. Then, it is shown how the algorithm may be employed to analyze systolic/diastolic pressure and stroke volume in the cardiovascular system, with results sourced from papers Stergiopulos et al. (1996) and Westerhof et al. (2010). A subsequent section presents a detailed operation of the algorithm on an example of a bioheat transfer differential equation. Finally, a brief discussion of the limitations of dimensional analysis, including certain complementary information, is presented.

Fundamentals of Dimensional Analysis

Whenever a **dimension** is mentioned, it usually refers to the geometrical concept of dimension, i.e. a single numerical value (scalar) with the information about the minimum number of mutually independent parameters necessary for localization, usually in one-, but also two- or three-dimensional space. This approach (the minimum number, independence) is also effective whenever an attempt is made to describe the essence of a phenomenon, not necessarily of a geometrical nature. On the other hand, dimensional analysis uses dimensions to express the physical meaning of a value, i.e. a dimension resulting directly from the unit associated with the quantity. For instance, a length in a given space may be expressed in meters (m), centimeters (cm), miles (mil), or inches (in). All these units are common for the concept of distance between points selected in the analyzed space. For the purpose of this interpretation, no distinction between the actual units is made, only the aspect of distance (the physical quantity) designated with the letter **L** (length) being used. To determine unit **j** of quantity **W**, notation in square brackets, i.e. $[\mathbf{W}] = \mathbf{j}$ is used. (Lower case will sometimes be used to show the applied system of units.) With reference to the International System of Units, the symbol **L** denotes a measurement given in meters ($[\mathbf{L}]_{\mathbf{SI}} = \mathbf{m}$, **meter**). The SI system provides the so-called base units (dimensions), which can be used to derive other physical units. Apart from length, the basic concepts include the following: **mass** (designation **M**, $[\mathbf{M}]_{\mathbf{SI}} = \mathbf{kg}$, **kilogram**), **time** (designation **T**, $[\mathbf{T}]_{\mathbf{SI}} = \mathbf{s}$, **second**), **temperature** (designation $\boldsymbol{\theta}$, $[\boldsymbol{\theta}]_{\mathbf{SI}} = \mathbf{K}$, **kelvin**), **electric current** (designation **I**, $[\mathbf{I}]_{\mathbf{SI}} = \mathbf{A}$, **ampere**), **luminous intensity** (designation **C**, $[\mathbf{C}]_{\mathbf{SI}} = \mathbf{cd}$, **candela**) and **amount of substance** (designation **N**, $[\mathbf{N}]_{\mathbf{SI}} = \mathbf{mol}$, **mole**). Such base quantities can be used to derive all other physical quantities, e.g. **force** ($[\mathbf{F}]_{\mathbf{SI}} = \mathbf{N}$), which is generally the product of mass and acceleration. Thus, the physical dimension may be expressed as $\mathbf{ML/T^2}$ instead of **F**. If quantity **W** is **dimensionless** (i.e. the quantity cannot be interpreted physically), the notation is $[\mathbf{W}] = \mathbf{1}$.

Dimension W is **dimensionally dependent** on dimensions $W_1, W_2, \ldots, W_n$ if there are numbers $1, , 2, \ldots, n$ such that:

$$W = W_1^{\alpha_1} W_2^{\alpha_2} \ldots W_n^{\alpha_n}.$$

Dimensions $W_1, W_2, \ldots, W_n$ create a **dimensionally independent system** provided that no dimension W_i ($i = 1, 2, \ldots, n$) is dimensionally depended on other dimensions in the system.

Many investigated natural phenomena depend on a high number of parameters whose mutual relationships are not always evident or difficult to observe in general. As a method, dimensional analysis is suitable particularly for **reduction of the number of parameters** by their replacement with a smaller number of other parameters, creating a **dimensionally independent** system; additionally, all dimensions in the system are **dimensionless**. This last feature is the reason why the description becomes **independent from the scale of measurement**. In order to provide a simple illustrative example, let us analyze the well-known equation of motion in the gravitational field (Yoon, 2014). Suppose a body is projected vertically at a velocity of V_0, from a height of z_0. The equation of motion in this case is as follows:

$$z = z_0 + V_0 t - \frac{gt^2}{2}. \tag{1}$$

The dimensions on the left and the right side of the formula must be consistent, which is expressed, for the assumed system of symbols, as:

$$L = L + \frac{L}{T}T - \frac{L}{T^2}T^2.$$

Essentially, there must be – and in this case there are – more items, i.e. the dimensions of all additive terms of a physical equation must be dimensionally consistent, according to the so-called **principle of dimensional homogeneity**.

If both sides of equation (1) are divided by z_0 and new (dimensionless) parameters are introduced:

$$z^* = \frac{z}{z_0}, \quad t^* = \frac{V_0 t}{z_0}, \quad a = \frac{V_0^2}{g z_0},$$

a dimensionless form of equation (1) is obtained:

$$z^* = 1 + t^* - \frac{1}{2a}t^{*2}. \tag{2}$$

As a result, equation (2) depends on three dimensionless parameters (z^*, t^*, α), in contrast with equation (1), which depends on five parameters with dimensions (z, z_0, V_0, t, g). Thus, the number of parameters has been reduced from $\mathbf{n = 5}$ to $\mathbf{r = 3}$. As will be shown later, the reduction level of $\mathbf{m = 2}$ results from the fact that equation (1) depends on two base dimensions: L and T. A theoretical explanation of the fact arises from **the Buckingham theorem**. Before that theorem is presented, let us pay attention to the fact that parameter z^* contains information about two output parameters (z and z_0), mixed together by a division of the first one (the motion variable) by the other one (the motion constant). This implies that z^* is a variant of the variable with the corresponding universal constant of $z_0 = 1$. Consequently, equation (2) is independent from the scale of the initial position z_0. By analogy, parameter t^* contains information about parameters z, z_0 and V_0. Parameter t^* may be regarded as a division of time t by quotient z_0/V_0 which has a time dimension, and the value defining the time required to reach height z_0 at a constant velocity of V_0. With equation (2), time is again to some extent universal, i.e. scaled by the "initial time". As a result, equation (2) is also independent from the time scale. In addition, parameter α (the new acceleration constant) in equation (2) is also dimensionless and contains information about the mutual proportions of constants z_0, V_0 and g. To conclude, the initial equation has been restated in a form with a lower number of parameters, in which the parameters are independent from each other and dimensionless. As such, the resulting equation is of a form equivalent to the initial equation – the canonical form in terms of scales of parameters.

Let us now consider a general situation (cf., e.g. Holmes, 2009; Yarin, 2012). Let physical quantity q depend on n physical parameters $p_1, p_2, \ldots, p_n$. For formal purposes, let us assume that parameters p_i ($i = 1, 2, \ldots, n$) depend only on three base dimensions L, M, T. Then:

$$[q] = L^{l_0} M^{m_0} T^{t_0}, \qquad [p_i] = L^{l_i} M^{m_i} T^{t_i},\ i = 1, 2, \ldots, n.$$

In addition, let us assume that a function $f()$ exists such that $q = f(p_1, p_2, \ldots, p_n)$. Any reduction of dimensions consists in analyzing the problem to find out if numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ exist such that:

$$[q] = [p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_n^{\alpha_n}].$$

Then, q is referred to as the **dimensional product** of variables $p_1, p_2, \ldots, p_n$. The existence of numbers $\alpha_1, \ldots, \alpha_n$ is equivalent to finding a solution for a system with three equations (as three dimensions have been assumed):

$$l_1\alpha_1 + l_2\alpha_2 + \ldots + l_n\alpha_n = l_0$$

$$m_1\alpha_1 + m_2\alpha_2 + \ldots + m_n\alpha_n = m_0$$

$$t_1\alpha_1 + t_2\alpha_2 + \ldots + t_n\alpha_n = t_0.$$

The corresponding matrix equation is:

$$Aa = b, \tag{3}$$

where

$$A = \begin{bmatrix} l_1 & \ldots & l_n \\ m_1 & \ldots & m_n \\ t_1 & \ldots & t_n \end{bmatrix}, \quad a = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}, \quad b = \begin{bmatrix} l_0 \\ m_0 \\ t_0 \end{bmatrix}.$$

If system (3) has no solution, the system of parameters $p_1, p_2, \ldots, p_n$ is **dimensionally incomplete**; otherwise, it is dimensionally complete. Dimensional incompleteness means that the system requires new dimensions for the dimension of quantity q to be expressed as a combination of dimensions of the new supplemented system.

In the discussed case, matrix A (referred to as the **matrix of a dimension**) has 3 rows. Generally, the number of rows is the same as the number of the selected base dimensions, whereas the rank of a matrix (designated as m) implies that the system can be reduced to $r = n - m$ of independent dimensions. These facts constitute **the aforementioned Buckingham theorem**, i.e.:

> Let us assume that formula $q = f(p_1, p_2, \ldots, p_n)$ is dimensionally homogeneous and system $p_1, p_2, \ldots, p_n$, is dimensionally complete. Then, formula q may be reduced as follows:
>
> $$q = Q \cdot F(\pi_1, \pi_2, \ldots, \pi_k),$$
>
> where the system of parameters $\{\pi_i\}$, $i = 1, 2, \ldots, k$ is dimensionally independent and all its elements are dimensionless, whereas Q is the dimensional product of parameters $\{p_j\}$, $j = 1, 2, \ldots, n$ – the product being dimensionally consistent with q. The value of k is within the range $\{0, 1, \ldots, n-1\}$, depending on the system $\{p_j\}$. If $k = 0$, then $q = \alpha Q$ (where α is a constant).

New parameters $\pi_1, \pi_2, \ldots, \pi_k$ are generally called **π-terms**. The procedure used to determine the i^{th} π-term results from the fact that the term is dimensionless and the dimensional product of the so-called **repeating variables**, selected from the set $\{p_1, p_2, \ldots, p_n\}$. These are dimensionally independent representatives of the base dimensions that describe the problem in question, i.e. dimensions L, M, T (the selection is not usually unambiguous, but they must produce a dimensionally independent system). The number of repeating variables is the same as the number of base dimensions, in this case designated as $r_1, r_2, \ldots, r_m$, where $m = n - k$. Let us also assume that $\{z_1, z_2, \ldots, z_k\} = \{p_1, p_2, \ldots, p_n\} \backslash \{r_1, r_2, \ldots, r_m\}$, i.e. the remaining initial variables, except for the repeating variables. If it is assumed, for instance, that the set of base dimensions is $\{L, M, T\}$ ($m = 3$), the following expression can be found for the unknown a, b and c:

$$[\pi_i] = L^0 M^0 T^0, \quad \pi_i = z_i r_1^a r_2^b r_3^c, \quad i = 1, \ldots, k. \tag{4}$$

Example Demonstrating the Algorithm

Yoon (2014) analyzed the case of pressure change of the fluid flowing through a fixed-diameter tube. The model may be applied to the study of blood pressure changes in selected arterial sections (fixed radius and circular section), with an additional simplifying assumption that blood is a homogeneous mixture on the solution-like scale and the flow velocity in the selected portion is constant as it would not be quite possible to analyze the dynamics of white/red blood cells and thrombocytes separately. With these assumptions, the pressure change Δp (the equivalent of q in the Buckingham theorem) depends on four parameters:

- artery diameter, designated as d;
- length of the investigated flow section, designated as x;
- flow velocity, designated as v;
- blood viscosity, designated as b.

Such a study of differential pressure along selected arterial sections is reasonable only for the sections where differential pressure occurs at a fixed diameter of the transfer tube, which is to some extent a simplification, considering the flexibility of artery walls, thick ones in particular. In addition, blood viscosity in an actual system also undergoes dynamic changes depending on whether the systolic or diastolic stage is analyzed (Holsworth & Wright, 2012). Thus, one of the stages must be chosen and the value of viscosity must be averaged for that stage. Blood viscosity is nowadays commonly recognized as a universal risk indicator of cardiac and vascular diseases (Sloop & Garber, 1997), which is directly associated with LDL cholesterol in particular. Despite these potential inconsistencies, it was decided that the chosen example is adequate for the stated purpose due to its simplicity of demonstration.

Parameters (d, x, v, μ) are equivalents of parameters (p_1, p_2, p_3, p_4) in the Buckingham theorem, whereas $n = 4$. If the relationship formula is designated as f in the theorem, we obtain:

$$\Delta p = f(D, x, V, \mu). \tag{5}$$

The dimensions of all the variables in equation (5) may be defined by the three base dimensions L, M, T. Their forms are presented in the Table 1.

Table 1. The dimensions of the variables in equation (5)

Variable	Dimension
Δp	$\mathrm{N/m^2} \Rightarrow L^{-1}M^1T^{-2}$
d	$\mathrm{m} \Rightarrow L^1M^0T^0$
x	$\mathrm{m} \Rightarrow L^1M^0T^0$
v	$\mathrm{m/s} \Rightarrow L^1M^0T^{-1}$
μ	$\mathrm{N\ s/m^2} \Rightarrow L^{-1}M^1T^{-1}$

Hence, component equations (3) are in the following forms:

$$A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix}, \qquad a = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_4 \end{bmatrix}, \qquad b = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}.$$

As the determinant of one of the submatrices is not equal to zero:

$$\det \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix} \neq 0, \quad \text{hence } \operatorname{rank}(A) = 3.$$

Based on the Buckingham theorem, $k = 1$, $\Delta p = x^{\alpha_1} d^{\alpha_2} v^{\alpha_3} \mu^{\alpha_4} F(\pi_1)$ and π_1 is a dimensionless variable. Let us include d, v, μ in the set of repeating variables, as representatives of dimensions L, T, M, respectively. Hence:

$$\pi_1 = x d^a v^b \mu^c.$$

In the language of dimensions, the equation is equivalent to:

$$L^0 M^0 T^0 = L L^a L^b T^{-b} L^{-c} M^c T^{-c} = L^{a+b-c+1} M^c T^{-b-c}.$$

Thus, $c = 0$, $b = 0$, $a = -1$, hence $\pi_1 = x d^{-1}$. In addition, since dimension Δp is $L^{-1} M^1 T^{-2}$, hence

$$\begin{aligned} [d^{\alpha_1} x^{\alpha_2} v^{\alpha_3} \mu^{\alpha_4}] &= L^{\alpha_1} L^{\alpha_2} L^{\alpha_3} T^{-\alpha_3} L^{-\alpha_4} M^{\alpha_4} T^{-\alpha_4} \\ &= L^{\alpha_1+\alpha_2+\alpha_3-\alpha_4} M^{\alpha_4} T^{-\alpha_3-\alpha_4} \\ &= L^{-1} M^1 T^{-2} \end{aligned}$$

Hence $\alpha_4 = 1$, $\alpha_3 = 1$, $\alpha_2 = -1 - \alpha_1$. Let $\alpha = \alpha_1$. Finally:

$$\Delta p = x^\alpha d^{-1-\alpha} v \mu F(x d^{-1}) = v \mu d^{-1} \pi_1^\alpha F(\pi_1) = v \mu d^{-1} G(\pi_1). \tag{6}$$

Form (6) is a general form of quantity Δp. Any statement with a greater degree of precision **requires experimental data**, which illustrates the universal feature of the dimensional analysis method, its nature is experimental. In this case, experience shows that pressure drop is proportional to flow length, i.e. variable x in the analyzed case. Thus, if it is assumed that $C =$ constant, it can be stated that:

$$\Delta p = C \cdot v \cdot \mu \cdot d^{-1} \cdot x \cdot d^{-1} = C v \mu x d^{-2}.$$

This is a relevant fact as far as the nature of such flow is concerned, i.e. pressure change is inversely proportional to flow cross-section.

The simple example shows that the dimensional analysis method may provide information about relevant features of the process under investigation by highlighting the most important parameters and, at least partially, showing mutual relationships without using classical equations of flow adequate for the process. Obviously, in this specific case, the classical equations and the related numerical methods could well by applied as it is a standard approach in haemodynamics (cf., e.g. Zamir, 2016). These equations provide a much more precise description. However, the following aspects must be considered: firstly, this level of precision is not always necessary; secondly, precision usually leads to a considerable lengthening of calculations and the need to deal with serious numerical methods; thirdly, dimensional analysis may be applied when precise physical laws for the process are unknown (although such laws for hemodynamics are very detailed and consist of various equations of fluid dynamics).

Analysis of Blood Flow in the Cardiovascular System

Dimensional dynamics may be effectively used in a much more complex situations than the example presented above. An application of this kind, to a case of pressure simulation in the cardiovascular flow system is presented in book Westerhof et al. (2010) and paper Stergiopulos et al. (1996). In this case, the heart is a pump and the artery is a transfer line. Understanding the quantitative impact of the heart and artery on blood pressure and flow velocity is essential, for example, in case of hypertension and heart attack. A complete cardiac cycle involves four stages with a pause before the next cycle (Traczyk & Trzebski, 2015; Woźniak, 2003). During the pause, the ventricles (partially) and atria are being filled with blood, which implies a low increase in pressure, but a high increase in ventricular volume. The ventricular systole stage generates a pressure differential between the atria and the ventricle, which implies the flow of blood into the ventricles (stage A in Figure 1), with the ventricles reaching the end-systolic volume (approx. 180–220 cm^3) and end-diastolic pressure – point 2 in Figure 1. The next stage involves the ventricular systole to increase the pressure of blood inside (30 mmHg in the right and 120 mmHg in the left ventricle, volume unchanged – stage B in Figure 1), which causes the ejection of blood into arteries (stroke volume, 70–120 cm^3) and reduces the ventricular volume (stage C in Figure 1). The pressure at the end of the stage (point 1 in Figure 1) is called end-systolic pressure. During the following stage, blood pressure in the ventricles decreases and their volume remains unchanged

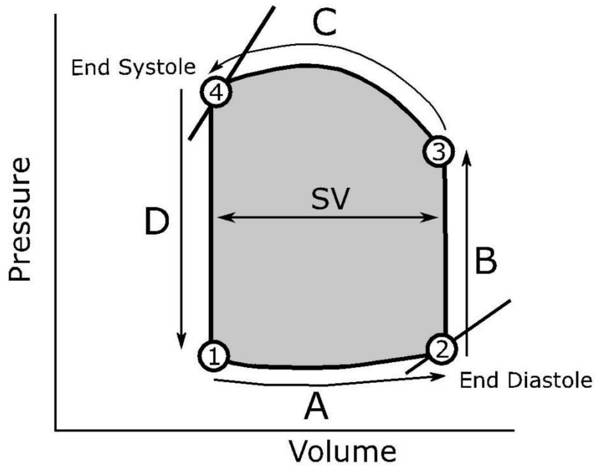

Figure 1. Pressure-volume ratio in the left ventricle during the cycle

(stage D in Figure 1). Simultaneously, the base of the atria lowers, which results in a decrease in the pressure and blood inflow in the atria. Any further decrease of pressure in the ventricles results in repeatedly partial filling, i.e. the pause after four systolic stages is repeated.

The pressure-volume ratio in the ventricles is referred to as elastance, designated as $E(t)$, where t is the time during the cycle. This will serve as the main quantity characterizing heart activity and a measure of heart output. Let $E_{\min}$ and $E_{\max}$ refer to the elastance of end-systolic pressure and end-diastolic pressure (inclinations of the tangents relative to the curve in Figure 1 at points 4 and 2). Inclination $E_{\max}$ is a good indicator of heart inotropy, as it is generally changing. A higher value of $E_{\max}$ means greater inotropy with a simultaneous shift of the $E(t)$ cycle graph to the left along the volume axis. This implies that it is possible to analyze a hypothetical curve $E_{\max}(V)$, which intersects the V axis at point V_d. The physical meaning of V_d is not completely understood (Blaudszun & Morel, 2011). In the heart model designed in Stergiopulos et al. (1996), heart activity is parametrized by six parameters. Apart from the three already mentioned, i.e. $E_{\max}$, $E_{\min}$ and V_d, there are venous filling pressure P_V, heart cycle T and time T_p of elastance $E_{\max}$. It is not necessary to know a precise function $E(t)$ because the function standardized by $E_{\max}$ delivers a universal shape of the graph, independent from the medical condition of the heart, in which case the shape is not a process variable. Another element of the flow system comes in the form of arteries, which, together with the heart, create a closed hydraulic circuit. The system is defined by the commonly known

Windkessel model (cf. Catanho et al., 2012; Westerhof et al., 2010), which associates blood pressure in the artery with the following: arterial compliance (the tendency to increase length and volume with increasing pressure, i.e. to resist recoil toward the original dimensions when force is applied – the reciprocal of elastance, designation C), peripheral vascular resistance (the ratio of pressure differential along a given section to flow velocity, designation R_p), and blood inertia force (momentum of the blood flowing out of the left ventricle toward the aorta). Aortic characteristic impedance (designated as Z_p) will be referred to as the sum of external factors reducing the ejection of blood into the aorta. To conclude, aortic flow is characterized by C, R_p and Z_p. From the perspective of dimensional analysis, there are $n = 9$ variables. For dependent variables, let us choose systolic pressure, diastolic pressure (designated as P_s, P_d) and stroke volume (designated as SV). For base dimensions, non-strictly SI-based quantities are selected – time (T), length (L) and force (F). Hence $m = 3$, which implies the $r = n - m = 6$ structure of dimensionless parameters. A classical dimensional analysis performed in Stergiopulos et al. (1996) results in the following three functions:

$$\begin{aligned}
\frac{P_s}{P_v} &= \Phi_1\left(\frac{Z_c}{R_p}, \frac{R_pC}{T}, CE_{\min}, \frac{E_{\max}}{E_{\min}}, \frac{E_{\min}V_d}{P_v}, \frac{T_p}{T}\right),\\
\frac{P_d}{P_v} &= \Phi_2\left(\frac{Z_c}{R_p}, \frac{R_pC}{T}, CE_{\min}, \frac{E_{\max}}{E_{\min}}, \frac{E_{\min}V_d}{P_v}, \frac{T_p}{T}\right),\\
\frac{SVE_{\min}}{P_v} &= \Phi_3\left(\frac{Z_c}{R_p}, \frac{R_pC}{T}, CE_{\min}, \frac{E_{\max}}{E_{\min}}, \frac{E_{\min}V_d}{P_v}, \frac{T_p}{T}\right).
\end{aligned}$$

Further analysis at that stage required experimental observations, showing that the influence of T_p/T on P_s/P_v, P_d/P_v and $SV \cdot E_{\min}/P_v$ is negligible as is the influence of Z_c/R_p on P_s/P_v. In addition, parameter $V_d \cdot E_{\min}/P_v$ does not affect P_s/P_v as Z_c/R_p does not affect SV/V_d and $E_{\max}/E_{\min}$ does not affect $SV \cdot E_{\min}/P_v$. As a result, the functional relations are simplified:

$$\begin{aligned}
\frac{P_s}{P_v} &\cong \Phi_1\left(\frac{R_pC}{T}, CE_{\min}, \frac{E_{\max}}{E_{\min}}\right),\\
\frac{P_d}{P_v} &\cong \Phi_2\left(\frac{R_pC}{T}, CE_{\min}, \frac{E_{\max}}{E_{\min}}\right),\\
\frac{SVE_{\min}}{P_v} &\cong \Phi_3\left(\frac{R_pC}{T}, CE_{\min}, \frac{E_{\min}V_d}{P_v}\right).
\end{aligned}$$

Parameters R_pC/T and $CE_{\min}$ include the ventricular and aortic variables and appear in all the functions. This proves that pressure and flow are affected simultaneously by the activity of the heart and the artery. The obtained form allowed the authors Stergiopulos et al. (1996) to conduct experiments concerning the influence of individual parameters on the systolic/diastolic pressure and the stroke volume (cf. Westerhof et al., 2010 – Table 29.1). The paper concluded that a 10% increase in compliance would result in a 1% decrease in systolic pressure and 2% increase in diastolic pressure. The authors of the presented solution did not provide any explicit form of function Φ_1, Φ_2, Φ_3. Obviously, it could be obtained through a sufficiently large number of experiments and the use of selected regression and/or approximation methods. In addition, the authors drew attention to the fact that the obtained form leads directly to the Frank-Starling law, which states that during systole the amount of blood flowing out of the heart increases in response to an increase in the volume of blood flowing into the heart.

Other examples of how dimensional analysis is applied to medical problems can be found in papers published by Rohlf & Tenti, (2001) and Burleson et al. (1995).

Example Application of Dimensional Analysis to Differential Equations

The Pennes Bioheat Equation (Pennes, 1948) describing the passive aspect of heat transfer in the human body is a second-order partial differential equation of two variables (r, t). Nevertheless, the equation includes eight physical constants, so it eventually depends on as many as eleven parameters (cf., e.g. Fiala, 1998):

$$k\left(\frac{\partial^2 T_t}{\partial r^2} + \frac{\omega}{r}\frac{\partial T_t}{\partial r}\right) + q_m + q_b w_b c_b (T_b - T_t) = \rho c_t \frac{\partial T_t}{\partial t}. \tag{7}$$

The solution of the equation is almost always found as a result of applying numerical methods to differential equations. The potential use of dimensional analysis as a complete replacement of numerical solution appears doubtful, but it may prove helpful when analyzing mutual relationships of parameters. Moreover, the case will be recalculated in more detail as a more complex presentation of the procedure for the Buckingham Theorem.

No matrix notation will be used in this case (as it may be unknown to some readers); an explicit expansion will be given instead.

Generally, the unknown tissue temperature T_t ($[T_t]_{\mathrm{SI}} = \mathrm{K}$) is a function of arterial blood temperature (T_b, $[T_b]_{\mathrm{SI}} = \mathrm{K}$), density of tissue and blood (ρ, ρ_b, $[\rho]_{\mathrm{SI}} = [\rho_b]_{\mathrm{SI}} = \mathrm{kg/m^3}$), the specific heat of tissue and blood (c_t, c_b, $[c_t]_{\mathrm{SI}} = [c_b]_{\mathrm{SI}} = \mathrm{J/kg{\cdot}K}$), metabolic level ($q_m$, $[q_m]_{\mathrm{SI}} = \mathrm{W/m^3}$), blood perfusion ($w_b$, $[w_b]_{\mathrm{SI}} = \mathrm{m^3/m^3s} = 1/\mathrm{s}$), tissue radius ($r$, $[r]_{\mathrm{SI}} = \mathrm{m}$), time ($t$, $[t]_{\mathrm{SI}} = \mathrm{s}$), and tissue thermal conductivity (k, $[k]_{\mathrm{SI}} = \mathrm{W/m{\cdot}K}$). The base dimensions in the equation are as follows: length (L), mass (M), time (T) and temperature (θ); thus, $m = 4$. Therefore, the application of dimensional analysis results in seven dimensionless variables. The following parameters will be chosen as repeating variables: r representing L, ρ representing M, t representing T_t and c_t representing θ. Then their dimensions are presented in Table 2.

Table 2. The dimensions of the variables

Variable	Dimension
ρ	$\mathrm{kg/m^3} \Rightarrow L^{-3}M^1T^0\theta^0$
r	$\mathrm{m} \Rightarrow L^1M^0T^0\theta^0$
T	$\mathrm{s} \Rightarrow L^0M^0T^1\theta^0$
c_t	$\mathrm{J/kg{\cdot}K} \Rightarrow L^2M^0T^{-2}\theta^{-1}$

For generators of subsequent terms π_i, let us assume (specify) the following generator-term pairs: (k, π_1), (T_t, π_2), (q_m, π_3), (q_b, π_4), (w_b, π_5), (c_b, π_6), (T_b, π_7). A general multiplier for each term π_i is:

$$r^a \rho^b t^c c_t^d$$

for certain integers a, b, c and d. Hence the multiplier dimension is:

$$L^a \frac{M^b}{L^{3b}} T^c \frac{L^{2d}}{\theta^d T^{2d}} = L^{a+2d-3b} M^b T^{c-2d} \theta^{-d}.$$

Now, let us calculate individual terms π_i (by repeating the same pattern 7 times).

$\boldsymbol{\pi_1}$: $$\frac{ML}{QT^3}L^{a+2d-3b}M^bT^{c-2d}\theta^{-d} = L^{a+2d-3b+1}M^{b+1}T^{c-2d-3}\theta^{-d-1} = L^0M^0T^0\theta^0$$

The following system of equations is obtained:

$$\begin{aligned}
&a+2d-3b+1=0 \\
&b+1=0 && \Rightarrow b=-1 \\
&c-2d-3=0 \\
&-d-1=0 && \Rightarrow d=-1
\end{aligned}$$

Hence the first equation gives $a=-2$ and the third equation gives $c=1$. Finally:

$$\pi_1 = k\cdot r^{-2}\rho^{-1}t^1c_t^{-1}.$$

$\boldsymbol{\pi_2}$: $$M^{b+1}T^{c-2d-3}\theta^{-d-1} = L^0M^0T^0\theta^0$$

The following system of equations is obtained:

$$\begin{aligned}
&a+2d-3b=0 \\
&b=0 && \Rightarrow b=0 \\
&c-2d=0 \\
&1-d=0 && \Rightarrow d=1
\end{aligned}$$

Hence the third equation gives $c=2$ and the first equation gives $a=-2$. Finally:

$$\pi_2 = T_t\cdot r^{-2}\rho^0t^2c_t^1.$$

$\boldsymbol{\pi_3}$: $$\frac{M}{T^3L}L^{a+2d-3b}M^bT^{c-2d}\theta^{-d} = L^{a+2d-3b-1}M^{b+1}T^{c-2d-3}\theta^{-d} = L^0M^0T^0\theta^0$$

The following system of equations is obtained:

$$\begin{aligned}
&a+2d-3b-1=0 \\
&b+1=0 && \Rightarrow b=-1 \\
&c-2d-3=0 \\
&-d=0 && \Rightarrow d=0
\end{aligned}$$

Hence the first equation gives $a=-2$ and the third equation gives $c=3$. Finally:

$$\pi_3 = q_m\cdot r^{-2}\rho^{-1}t^3c_t^0.$$

π_4: $\frac{M}{L^3}L^{a+2d-3b}M^bT^{c-2d}\theta^{-d} = L^{a+2d-3b-3}M^{b+1}T^{c-2d}\theta^{-d}$
$= L^0M^0T^0\theta^0$

The following system of equations is obtained:

$$\begin{aligned}
&a + 2d - 3b - 3 = 0 && \\
&b + 1 = 0 && \Rightarrow b = -1 \\
&c - 2d = 0 && \\
&-d = 0 && \Rightarrow d = 0
\end{aligned}$$

Hence the third equation gives $c = 0$ and the first equation gives $a = 0$. Finally:

$$\pi_4 = q_b \cdot r^0\rho^{-1}t^0c_t^0.$$

π_5: $\frac{1}{T}L^{a+2d-3b}M^bT^{c-2d}\theta^{-d} = L^{a+2d-3b}M^bT^{c-2d-1}\theta^{-d} = L^0M^0T^0\theta^0$

The following system of equations is obtained:

$$\begin{aligned}
&a + 2d - 3b = 0 && \\
&b = 0 && \Rightarrow b = 0 \\
&c - 2d - 1 = 0 && \\
&-d = 0 && \Rightarrow d = 0
\end{aligned}$$

Hence the first equation gives $a = 0$ and the third equation gives $c = 1$. Finally:

$$\pi_5 = w_b \cdot r^0\rho^0t^1c_t^0.$$

π_6: $\frac{L^2}{T^2\theta}L^{a+2d-3b}M^bT^{c-2d}\theta^{-d} = L^{a+2d-3b+2}M^bT^{c-2d-2}\theta^{-d-1}$
$= L^0M^0T^0\theta^0$

The following system of equations is obtained:

$$\begin{aligned}
&a + 2d - 3b + 2 = 0 && \\
&b = 0 && \Rightarrow b = 0 \\
&c - 2d - 2 = 0 && \\
&-d - 1 = 0 && \Rightarrow d = -1
\end{aligned}$$

Hence the third equation gives $c = 0$ and the first equation gives $a = 0$. Finally:

$$\pi_6 = c_b \cdot r^0\rho^0t^0c_t^{-1}.$$

π_7: $\theta \cdot L^{a+2d-3b} M^b T^{c-2d} \theta^{-d} = L^{a+2d-3b} M^b T^{c-2d} \theta^{-d} = L^0 M^0 T^0 \theta^0$

The following system of equations is obtained:

$$\begin{array}{ll} a + 2d - 3b = 0 & \\ b = 0 & \Rightarrow b = 0 \\ c - 2d = 0 & \\ 1 - d = 0 & \Rightarrow d = 1 \end{array}$$

Hence the third equation gives $c = 2$ and the first equation gives $a = -2$. Finally:

$$\pi_7 = T_b \cdot r^{-2} \rho^0 t^2 c_t^1.$$

After reduction, the following forms of the seven terms π are obtained:

$$\begin{aligned} \pi_1 &= \frac{k}{c_t \cdot \rho} \cdot \frac{t}{r^2}, \\ \pi_2 &= c_t \cdot T_t \cdot \frac{t^2}{r^2}, \\ \pi_3 &= \frac{q_m}{\rho} \cdot \frac{t^3}{r^2}, \\ \pi_4 &= \frac{q_b}{\rho}, \\ \pi_5 &= w_b \cdot t, \\ \pi_6 &= \frac{c_b}{c_t}, \\ \pi_7 &= c_t \cdot T_b \cdot \frac{t^2}{r^2}. \end{aligned}$$

A general solution may be expressed as follows:

$$\pi_2 = f(\pi_1, \pi_3, \pi_4, \pi_5, \pi_6, \pi_7), \tag{8}$$

where $f()$ is an unknown function.

In addition, $\pi_2 = \pi_7 \cdot \frac{T_t}{T_b}$. Hence (8) can be written mathematically as:

$$T_t = T_b \cdot g(\pi_1, \pi_3, \pi_4, \pi_5, \pi_6, \pi_7), \tag{9}$$

where $g()$ is an unknown function. Any further reduction would require an experimental study of how terms $\pi_1, \pi_3, \pi_4, \pi_5, \pi_6, \pi_7$ affect temperature T_t, similarly to the example relating to blood flow between the heart and the artery. To obtain a specific form of function $g()$, it would be necessary to take many measurements of parameters $\pi_1, \pi_3, \pi_4, \pi_5, \pi_6, \pi_7$, use regression and/or approximation methods and precisely compare the results with the solutions obtained from equation (7) with the use of numerical methods. Obviously, if an analytical form that provides correct results was obtained, it would be very useful. However, the complexity of the problem in the example in question advises caution while anticipating any success. On the other hand, for problems described by differential equations, dimensional analysis may sometimes be successfully used to replace an equation with a simpler one, one that would be much easier to calculate. The benefit in this case is solely in the numerical aspect. It is possible to restate the equation when the number of problem-related parameters does not exceed a certain level, which allows dimensional analysis to reduce the number to one or two. A classic example of the procedure is applied to the equation of diffusion (e.g. oxygen transfer from lungs into blood), as can be found in particular in (Holmes, 2009).

Further Remarks on the Limitations of Dimensional Analysis

Generally, in very complex processes that depend on many parameters, where the mutual dependency may be (and usually is) relatively unclear (i.e. when it is difficult to distinguish dimensionally independent repeating variables with absolute certainty), reduction by means of dimensional analysis usually gives no cognitively relevant results. An ineffective reduction of the number of variables is not the only drawback of dimensional analysis. Another undoubted disadvantage is the non-discrimination of vector and scalar quantities (e.g. position in space is described numerically by three scalars x, y and z, whereas a parameter in dimensional analysis is usually understood as a single quantity), which may in practice also lead to an unnecessary increase in the number of new parameters (Madrid & Alhama, 2006). These as well as other drawbacks of dimensional analysis are the reason why method is often aided by other dimensional reduction methods, with statistical methods among the most popular ones (Madrid & Alhama, 2006; Rajan et al., 2009).

Conclusion

Dimensional analysis can serve as an analytical tool in the medical sciences, primarily in the first stage of modeling complex processes. It enables the user to extract important parameters for the model being built. The development of a full analytical model usually requires additional experiments (which in the case of medicine are unfortunately often invasive) and interaction with experts in various fields (in the case of medicine these are usually chemists, physicists, mathematicians and computer scientists)

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REFERENCES

Blaudszun, G., & Morel, D. R. (2011). Relevance of the volume-axis intercept, V0, compared with the slope of end-systolic pressure-volume relationship in response to large variations in inotropy and afterload in rats. *Experimental Physiology*, *96*(11), 1179–1195.

Buckingham, E. (1914). On physically similar systems; illustrations of the use of dimensional equations. *Physical Review*, *4*, 345–376.

Burleson, A. C., N'Guyen, T., Fontaine, A., Levine, R. A., & Yoganathan, A. P. (1995). A model based on dimensional analysis for noninvasive quantification of valvular regurgitation under confined and impinging conditions: in vitro pulsatile flow validation. *Ultrasound in Medicine and Biology*, *21*(7), 899–911.

Catanho, M., Sinha, M., & Vijayan, V. (2012). *Model of Aortic Blood Flow Using the Windkessel Effect* (Report). University of California San Diego. Retrieved from http://isn.ucsd.edu/classes/beng221/problems/2012/BENG221_Project%20-%20Catanho%20Sinha%20Vijayan.pdf

Fiala, D. (1998). *Dynamic Simulation of Human Heat Transfer and Thermal Comfort* (PhD Thesis). De Montfort University. Retrieved from https://core.ac.uk/download/pdf/2749294.pdf

Gibbings, J. C. (2011). *Dimensional Analysis*. Berlin Heidelberg: Springer-Verlag.

Holmes, M. H. (2009). *Introduction to the Foundations of Applied Mathematics*. Texts in Applied Mathematics 56. New York: Springer-Verlag. doi: 10.1007/978-0-387-87765-5

Holsworth, R. E., & Wright, J. W. (2012). *Blood Viscosity: The Unifying Parameter In Cardiovascular Disease Risk*. Retrieved from https://www.holisticprimarycare.net/topics/topics-a-g/functional-medicine/1297-blood-viscosity-the-unifying-parameter-in-cardiovascular-diseaserisk.html

Madrid, C. F., & Alhama, F. (2006). Discrimination: A fundamental and necessary extension of classical dimensional analysis theory. *International Communications in Heat and Mass Transfer*, *33*, 287–294.

Pennes, H. H. (1948). Analysis of Tissue and Arterial Blood Temperatures in the Resting Human Forearm. *Journal of Applied Physiology*, 1, 93–122.

Rajan, K., Suh, C., & Mendez, P. F. (2009). Principal Component Analysis and Dimensional Analysis as Materials Informatics Tools to Reduce Dimensionality in Materials Science and Engineering. *Statistical Analysis and Data Mining*, *1*(6), 361–371.

Rohlf, K., & Tenti, G. (2001). The role of the Womersley number in pulsatile blood flow a theoretical study of the Casson model. *Journal of Biomechanics*, *34*(1), 141–148.

Sloop, G. D., & Garber, D. W. (1997). The effects of low-density lipoprotein and high-density lipoprotein on blood viscosity correlate with their association with risk of atherosclerosis in humans. *Clinical Science*, *92*, 473–479.

Stergiopulos, N., Meister, J. J., & Westerhof, N. (1996). Determinants of stroke volume and systolic and diastolic aortic pressure. *American Journal of Physiology*, *270*(6), H2050-H2059.

Tan, Q. M. (2011). *Dimensional Analysis. With Case Studies in Mechanics*. Berlin Heidelberg: Springer-Verlag.

Traczyk, W., & Trzebski, A. (2015). *Fizjologia człowieka z elementami fizjologii stosowanej i klinicznej*. Warszawa: PZWL.

Westerhof, N., Stergiopulos, N., & Noble, M. I. M. (2010). *Snapshots of Hemodynamics: An Aid for Clinical Research and Graduate Education* (2nd ed.). Berlin Heidelberg: Springer-Verlag.

Woźniak, W. (2003). *Anatomia człowieka. Podręcznik dla studentów i lekarzy.* Wrocław: Urban & Partner.

Yarin, L. P. (2012). *The Pi-Theorem, Experimental Fluid Mechanics.* Berlin Heidelberg: Springer-Verlag.

Yoon, H. (2014). *Dimensional Analysis.* Course Notes, Fluid Mechanics, University of Iowa. Retrieved from http://user.engineering.uiowa.edu/~fluids/posting/schedule/Example/Dimensional%20Analysis_11-03-2014.pdf

Zamir, M. (2016). *Hemo-Dynamics.* Biological and Medical Physics, Biomedic Engineering. Berlin Heidelberg: Springer-Verlag.