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METZLER CYCLIC ELECTRIC SYSTEMS

ABSTRACT

In applications we meet with positive (nonnegative) systems. A system generated by a linear differential equation with a Metzler-matrix is called a nonnegative system. The paper shows the basic dynamic properties of nonnegative continuous-time systems. Considered was the example of an oscillating electrical circuit with a cyclic Metzler-matrix, enabling the correct physical interpretations.

Key words:
positive systems, Metzler systems, cyclic systems, electrical circuits.

INTRODUCTION

Positive (nonnegative) systems [1–4, 7] and their electrical interpretations were and are the subject of consideration of many works, e.g. [2, 5, 6, 8, 10, 12].

A dynamical system generated by the differential equation $\dot{z}(t) = Mz(t)$ with a Metzler-matrix M is called a nonnegative system, on the ground that $e^{Mt} \geq 0$ for $t \geq 0$. A matrix $M = [m_{ik}] \in R^{n \times n}$ is a Metzler-matrix if the $m_{ik} \geq 0, \quad i \neq k$ (negative elements may occur on the main diagonal of the matrix M). From the inequality $e^{Mt} \geq 0$ it may seem that a real Metzler-matrix M may not have complex eigenvalues. This is not the case. It turns out that in spite of having complex eigenvalues by a Metzler-matrix M, the condition $e^{Mt} \geq 0$ occurs for $t \geq 0$ [8, 9, 11, 12]. This a little surprising property is due to the specific spectrum of a Metzler-matrix.

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The spectrum of a Metzler-matrix (shortly, Metzler spectrum) meets the following condition: $\mathrm{Re}\,\lambda_1 \le \mathrm{Re}\,\lambda_2 \le ..... \le \mathrm{Re}\,\lambda_n = \alpha(M)$, where $\alpha(M) = \max_i \mathrm{Re}\,\lambda_i(M)$, $\lambda_i(M), i = 1, 2, \ldots, n$ are the eigenvalues of the matrix *M*. With every Metzler-matrix *M* a nonnegative matrix can be bound (see, e.g. [9, 11]), which is also a Metzler's matrix, whose eigenvalues lie in the closed circle centered at the zero and the radius equal to the spectral radius (see fig. 1). The sample spectrum of a nonnegative matrix *M* for *n* = 6 is shown in figure 1. The eigenvalue $\lambda_n = \lambda_{\max}$ with the largest real part is a real number.

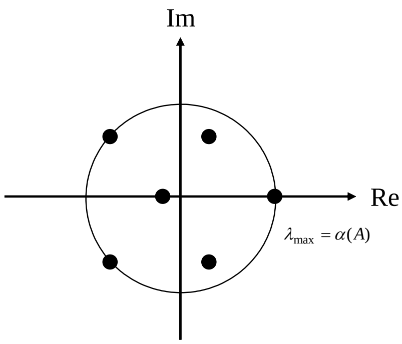

Fig. 1. A sample spectrum of a nonnegative matrix *M* for *n* = 6 [own work]

It is worth remembering that a Metzler-matrix *M* is asymptotically stable if and only if $\max_i \mathrm{Re}\,\lambda_i(M) = \alpha(M) < 0$. Another interesting property is that the characteristic polynomial of a Metzler-matrix *M* is asymptotically stable if and only if all its coefficients are positive (or all negative). The asymptotically stable polynomial has zeros with negative real parts. The question arises: is it possible to built an oscillating (that is, having the coupled complex eigenvalues) electrical circuit of the third order (*n* = 3) for which the state matrix will be a (cyclic) Metzler-matrix *M*?

Building such an electric circuit is cognitively interesting. In fact, the dynamic system $\dot{z}(t) = Mz(t)$ with the Metzler-matrix *M* has the following property: the trajectories $\dot{z}(t) = Mz(t)$ of the system have the form $z(t) = e^{Mt}z(0) \ge 0$ and for $t \ge 0$ are nonnegative, when $z(0) \ge 0$.

Below we will try to resolve this problem. We will begin our the considerations from the prediction of an oscillating electrical *RLC-circuit* of the third order (*n* = 3) for which the appropriate mathematical model $\dot{x}(t) = Ax(t)$ has the spectrum of the cyclic Metzler-matrix (matrix A has the spectrum of the Metzler-matrix, but A rather doesn't have to be the Metzler-matrix). Next, we will find the transformation of the coordinate system $x = Qz, \quad z = Q^{-1}x$ described by the matrix Q such that $M = Q^{-1}AQ$. Therefore, $\dot{z}(t) = Mz(t)$ will be the sought mathematical model of the electrical circuit with the (cyclic) Metzler-matrix M.

OSCILLATING ELECTRICAL CIRCUIT OF THE THIRD ORDER (*n* = 3) WITH METZLER SPECTRUM

Let's consider the electric RLC-circuit of the third order (n = 3) shown in figure 2. Given are the circuit parameters R, L, C_1, C_2, and the instantaneous voltage u (t) of the control source. The relevant currents are marked with the symbols $i(t)$, $i_1(t)$ and $i_2(t)$.

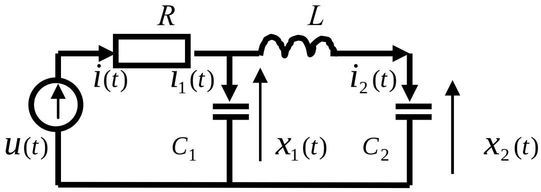

Fig. 2. The electric circuit RLC of the third order (n = 3) [own work]

Using the following equalities:

$$i_1(t) = i(t) - i_2(t), \quad C_2 \frac{dx_2(t)}{dt} = i_2(t), \quad L \frac{di_2(t)}{dt} = x_1(t) - x_2(t) \tag{1}$$

we can describe the circuit in figure 2 with the following state equation:

$$\begin{bmatrix} \frac{dx_1(t)}{dt} \\ \frac{dx_2(t)}{dt} \\ \frac{di_2(t)}{dt} \end{bmatrix} = \begin{bmatrix} \frac{-1}{RC_1} & 0 & \frac{-1}{C_1} \\ 0 & 0 & \frac{1}{C_2} \\ \frac{1}{L} & \frac{-1}{L} & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ i_2(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{RC_1} \\ 0 \\ 0 \end{bmatrix} u(t). \tag{2}$$

Therefore, we have the state matrix A of the system (2) and the control matrix B with the following forms:

$$A = \begin{bmatrix} \frac{-1}{RC_1} & 0 & \frac{-1}{C_1} \\ 0 & 0 & \frac{1}{C_2} \\ \frac{1}{L} & \frac{-1}{L} & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} \frac{1}{RC_1} \\ 0 \\ 0 \end{bmatrix} \tag{3}$$

For further considerations, we take *u* = 0 (the system without control) and by *x* (*t*) we denote the status of the system (2), provided that $x(t) = [x_1(t) \;\; x_2(t) \;\; i_2(t)]^T$. Therefore, the equality (2) for *u* = 0 takes the following form:

$$\dot{x}(t) = Ax(t), \quad x(t) \in R^3, \tag{4}$$

where the matrix *A* is given in (3).

The characteristic equation of the state matrix *A* of the system (2) is the equation of the third order (*n* = 3) and it has the following form:

$$s^3 + \frac{1}{RC_1} s^2 + \left(\frac{1}{C_1 L} + \frac{1}{C_2 L}\right) s + \frac{1}{RC_1 C_2 L} = 0. \tag{5}$$

Next, we will determine the parameters of the electrical circuit in figure 2, so that the system will be the oscillating electrical circuit (that is, it will have the pair of coupled eigenvalues). For example, when

$$RC_1 = 0.3478 \qquad C_2 L = 3.8333 \qquad C_1 L = 0.4121, \tag{6}$$

then the equation (5) takes the form

$$s^3 + 2.8750\, s^2 + 2.6875\, s + 0.7500 = 0 \tag{7}$$

and the equation (7) has the following roots (which are the eigenvalues of the matrix *A*):

$$s_1 = -0.5 \qquad s_{2,3} = -1.1875 \pm 0.2997\, j \qquad j^2 = -1. \tag{8}$$

Let's determine more precisely the parameters of the circuit in figure 2, taking in (6) *L* = 1. Therefore, we have

$$L = 1 \qquad R = 0.3478/0.4\,121 = 0.8440 \qquad C_2 = 3.8333 \qquad C_1 = 0.4121. \tag{9}$$

The root $s_1 = -0.5$ is a real number, and the roots $s_{2,3} = -1.1875 \pm 0.2997\ j$ are the pair of coupled complex numbers. The real parts of the roots are negative and therefore, the system with the matrix *A* is asymptotically stable (exponentially stable). In the domain of complex numbers the polynomial (7) has various roots that are the eigenvalues of the matrix *A.* In this case, the *A* has only linear elementary dividers. Therefore, the complex canonical Jordan's form *J* of the matrix *A* is a diagonal matrix.

The roots (8) of the equation (7) are the eigenvalues of the state matrix of the system (2) with the parameters (9). The roots (8) represent the Metzler spectrum of the system (2), but the state matrix *A* of the system (2) is not the Metzler--matrix.

Now, we will select the cyclic Metzler-matrix *M* with the spectrum (8). Practically, one can firstly choose the cyclic matrix *M*, then choose the parameters of the electrical circuit (2), so that the characteristic polynomial (5) of the electrical circuit was equal to the characteristic polynomial of the cyclic Metzler matrix *M*. Below we present the procedure of conduct with regard to the demonstration of the 3 x 3 matrix *M*. For example below is presented the procedure for matrix *M* with size 3 x 3.

METZLER-MATRIX *M*

Let's consider the Metzler-matrix *M* with the following form:

$$M = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ \frac{1}{16} & \frac{1}{16} & \frac{-7}{8} \end{bmatrix} = \begin{bmatrix} -1.0000 & 1.0000 & 0.0000 \\ 0.0000 & -1.0000 & 1.0000 \\ 0.0625 & 0.0625 & -0.8750 \end{bmatrix}. \tag{10}$$

The matrix *M* of the form (10) was considered in [9] and it is the cyclic matrix. Its characteristic polynomial has the following form:

$$\det[sI - M] = s^3 + 2.8750\ s^2 + 2.6875\ s + 0.7500\ . \tag{11}$$

The polynomial (11) is the same as the polynomial (7) and therefore, it has the following roots (see. (8)):

$$s_1 = -0.5 \qquad s_{2,3} = -1.1875 \pm 0.2997\ j \qquad j^2 = -1. \tag{12}$$

Let's consider the matrix *A* given in (3) with the parameters (9) and the matrix *M* circumscribed with the equality (10). It's easy to see notice that the matrices *A* and *M* are similar to each other, because they have the same elementary dividers.

Matrices similar to each other have the same characteristic polynomials. However, matrices with identical characteristic polynomials do not have to be similar to each other. Matrices with identical characteristic polynomials are similar to each other, when they are cyclic matrices. The necessary and sufficient condition of cyclic matrices similarities is the equality of their characteristic polynomials. A matrix is cyclical if and only if its characteristic polynomial is equal to the minimal polynomial. A cyclic matrix is similar to a Frobenius matrix. The invariant factors of cyclic matrices are identically equal to 1 except for one factor, which is the minimal polynomial. The characteristic polynomial of a cyclic matrix is equal to the minimum polynomial that uniquely specifies the elementary dividers, and therefore the canonical Jordan's form.

The matrix *M* has the same canonical Jordan's form *J* as the matrix *A.* Below we will provide an algorithm for finding the transformation matrix of Q such that $Q^{-1}AQ = M$.

ALGORITHM FOR FINDING TRANSFORMATION MATRIX OF Q

Below we will show the way to find the transformation matrix of Q (by similarity) such that $Q^{-1}AQ = M$:

1. We bring the matrix *M* to its canonical Jordan's form *J* (*J* may be complex; only the order of frames setting in the matrix *J* is important). That is, we are looking for a such *S* that, $S^{-1}MS = J$. If *M* is given in (10), we have (in the following matrices 'I' is the imaginary unit, in MATLAB software such symbol of the imaginary unit is used; the calculations were carried out in MATLAB)

$$S = \begin{bmatrix} -0.9363 & -0.9363 & -0.8729 \\ 0.1756 - 0.2807i & 0.1756 + 0.2807i & -0.4364 \\ 0.0512 + 0.1052i & 0.0512 - 0.1052i & -0.2182 \end{bmatrix};$$
$$J = \begin{bmatrix} -1.1875 + 0.2997i & 0 & 0 \\ 0 & -1.1875 - 0.2997i & 0 \\ 0 & 0 & -0.5000 \end{bmatrix}. \tag{13}$$

2. Then, we propose a specific electrical circuit of the third order (n = 3) (but such, that the transient oscillations could arise in it), e.g. the circuit in figure 2.
3. For the chosen circuit in figure 2, we compose the state equation with the state matrix A (in our case the equality (2)) with the state set by us, e.g. $x(t)=[x_1(t)\ \ x_2(t)\ \ i_2(t)]^T$
4. In the next step, we choose the circuit parameters so that the characteristic polynomial (5) of the matrix A had the form (11) and the roots (12). Such parameters of the circuit in figure 2 are set down in the equality (9).
5. Next, we reduce the matrix A given in (3) with the parameters (9) to the canonical Jordan's form J looking for the matrix P such that $P^{-1}AP=J$, while the J has the same form as in the equation $S^{-1}MS=J$ (the matrices A and M have the same elementary dividers in the domain of complex numbers and, therefore, the same Jordan's frames. These frames need to be set in both cases of matrixes A and M in the same order). Therefore, we have

$$P=\begin{bmatrix} 0.8107 & 0.8107 & -0.6714 \\ 0.1113+0.0500i & 0.1113-0.0500i & -0.3427 \\ -0.5638-0.1000i & -0.5638+0.1000i & 0.6571 \end{bmatrix};$$

$$J=\begin{bmatrix} -1.1875+0.2997i & 0 & 0 \\ 0 & -1.1875-0.2997i & 0 \\ 0 & 0 & -0.5000 \end{bmatrix}. \tag{14}$$

We determine the matrix Q in the following way. The canonical Jordan's form J is the same for the both matrixes A and $M.$ From the equality $S^{-1}MS=J=P^{-1}AP$ we have $Q^{-1}AQ=M$ where $Q=PS^{-1}$. In other words, the state matrix A of the circuit in figure 2 is similar to the Metzler-matrix M with the transformation matrix Q. The state variables are related to the following equations: $x=Qz,\quad z=Q^{-1}x$. In the case under consideration and taking into account (13) and (14) we obtain

$$Q=\begin{bmatrix} -0.5025 & 1.0897 & 2.9076 \\ -0.0114 & 0.2444 & 1.1274 \\ 0.3134 & -0.7101 & -2.8450 \end{bmatrix},\ Q^{-1}=\begin{bmatrix} 1.0776 & 10.6013 & 5.3022 \\ 3.2870 & 5.3061 & 5.4619 \\ -0.7017 & -0.1566 & -1.1307 \end{bmatrix}. \tag{15}$$

POSITIVITY *z* (0)

From the equality $z = Q^{-1}x \geq 0$ and (15), we have got (compare with (1) and (2))

$$\begin{array}{l} 1.0776\ x_1 + 10.6013\ x_2\ + 5.3022\ i_2 \geq 0 \\ 3.2870\ x_1 + 5.3061\ x_2\ \ + 5.4619\ i_2 \geq 0, \\ -0.7017\ x_1 - 0.1566\ x_2 - 1.1307\ i_2 \geq 0 \end{array} \tag{16}$$

where $x(0) = [x_1(0)\ \ x_2(0)\ \ i_2(0)]^T = [x_1\ \ \ x_2\ \ \ i_2]^T$.

For example, the inequalities (16) are met, if the appropriate initial conditions of the system (2) will take the following form: $x_1 = 1,\ \ x_2 = 1,\ \ i_2 = -1$. The initial conditions *x* (0) satisfying the inequality (16) imply the inequality $z(0) = Q^{-1}x(0) \geq 0$. Therefore, $z(t) = e^{Mt}z(0) \geq 0$ for $t \geq 0$.

SUMMARY

This article shows that it is possible to build an oscillating electrical system with a state cyclic Metzler matrix. In the paper the electrical circuit with the Metzler spectrum was considered. The considerations were carried out on the example of the circuit of the third order (*n* = 3) in figure 2. The electrical circuit in figure 2 is described by the state equation (2), while the state of the system is the vector $x(t) = [x_1(t)\ \ x_2(t)\ \ i_2(t)]^T$. Given were the steps to find the transformation matrix of *Q* by similarity in such a way that $Q^{-1}AQ = M$ would be the cyclic Metzler-matrix. Therefore, the electric circuit in figure 2 has been described by the state equation $\dot{z}(t) = Mz(t)$, in which the state is the vector $z(t) = Q^{-1}x(t)$, $x(t) = [x_1(t)\ \ x_2(t)\ \ i_2(t)]^T$, the *Q* is given in (15). The trajectories of the system $\dot{z}(t) = Mz(t)$ have the form $z(t) = e^{Mt}z(0) \geq 0$ and for $t \geq 0$ are nonnegative, when $t \geq 0$.

It seems that it is possible to make generalization of the carried out considerations on the systems of a higher order than the third (*n* = 3). It is important that the appropriate electrical circuit had the Metzler spectrum. The assumption of the matrix cyclicality is needed to ensure that the equality of the corresponding characteristic

polynomials was sufficient to the similarity of the corresponding matrixes. In the absence of the assumptions about the matrix cyclicality, the equality of elementary divisors (or the equality of the invariant factors of the appropriate matrices) must to be considered.

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CYKLICZNE OBWODY ELEKTRYCZNE METZLERA

STRESZCZENIE

Układy dodatnie (nieujemne) można spotkać w wielu zastosowaniach. Układem nieujemnym jest układ generowany równaniem różniczkowym liniowym z macierzą Metzlera. W artykule przedstawiono podstawowe własności dynamiczne układów nieujemnych z czasem ciągłym. Rozważono przykład elektrycznego obwodu oscylacyjnego z cykliczną macierzą Metzlera umożliwiający odpowiednie interpretacje fizyczne.

Słowa kluczowe:
układy dodatnie, układy Metzlera, układy cykliczne, obwody elektryczne.