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CONCEPT OF SEMI-MARKOV PROCESS

ABSTRACT

This paper provides the definitions and basic properties related to a discrete state space semi--Markov process. The semi-Markov process is constructed by the so called Markov renewal process that is a special case the two-dimensional Markov sequence. The Markov renewal process is defined by the transition probabilities matrix, called the renewal kernel and an initial distribution or by another characteristics which are equivalent to the renewal kernel. The counting process corresponding to the semi-Markov process allows to determine concept of the process regularity. In the paper are also shown the other methods of determining the semi-Markov process. The presented concepts are illustrated a simple example.

Key words:
semi-Markov process, probabilities matrix.

INTRODUCTION

The semi-Markov processes were introduced independently and almost simultaneously by P. Levy [9], W. L. Smith [12] and L. Takacs [13] in 1954–1955. The essential developments of semi-Markov processes theory were proposed by R. Pyke [11–13], E. Cinlar [2], Koroluk, Turbin [8, 9], N. Limnios and G. Oprisan [10], D. C. Silvestrov [14]. We present only semi-Markov processes with a discrete state space. A semi-Markov process is constructed by the Markov renewal process which is defined by the renewal kernel and the initial distribution or by another characteristics which are equivalent to the renewal kernel.

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MARKOV RENEWAL PROCESSES

Suppose that $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{N}_0 = \{0, 1, 2, \dots\}$, $\mathbb{R}_+ = [0, \infty)$ and S is a discrete (finite or countable) state space. Let ξ_n be a discrete random variable taking values on S and let ϑ_n be a continuous random variable with values in the set $\mathbb{R}_+$.

Definition 1. *A two-dimensional sequence of random variables* $\{(\xi_n, \vartheta_n)\colon\ n \in \mathbb{N}_0\}$ *is said to be a Markov Renewal Process (MRP) if:*

1) *for all* $n \in \mathbb{N}_0$, $j \in S$, $t \in \mathbb{R}_+$

$$P(\xi_{n+1} = j, \vartheta_{n+1} \le t \mid \xi_n = i, \vartheta_n, \dots \xi_0, \vartheta_0) = P(\xi_{n+1} = j, \vartheta_{n+1} \le t \mid \xi_n = i) \quad (1)$$

with probability 1;

2) *for all* $i, j \in S$, $P(\xi_0 = i, \vartheta_0 = 0) = P(\xi_0 = i)$. (2)

From the definition 1 it follows, that MRP is a homogeneous two-dimensional Markov chain such that its transition probabilities depend only on the discrete component (they do not depend on the second component). A matrix

$$Q(t) = \left[\, Q_{ij}(t)\colon\ i, j \in S \,\right]; \quad (3)$$

$Q_{ij}(t) = P(\xi_{n+1} = j, \vartheta_{n+1} \le t \mid \xi_n = i)$ is called *a renewal matrix.*

A vector $p = [p_i\colon\ i \in S]$, where $p_i = P\{\xi_0 = i\}$ defines an initial distribution of the Markov renewal process. It follows from the definition 1 that the Markov renewal matrix satisfies the following conditions:

1. The functions $Q_{ij}(t)$, $t \ge 0$, $(i, j) \in S \times S$ are not decreasing and right-hand continuous.
2. For each pair $(i, j) \in S \times S$, $Q_{ij}(0) = 0$ and $Q_{ij}(t) \le 1$ for $t \in \mathbb{R}_+$.
3. For each $i \in S$, $\lim\limits_{t \to \infty} \sum_{j \in S} Q_{ij}(t) = 1$.

One can prove that a function matrix $Q(t) = [\, Q_{ij}(t)\colon\ i, j \in S \,]$ satisfying the above mentioned conditions and a vector $p_0 = [\, p_i^{(0)}\colon\ i \in S \,]$ such that $\sum_{i \in S} p_i^{(0)} = 1$ define some Markov renewal process.

From definition of the renewal matrix it follows that

$$P = [\, p_{ij}\colon\ i, j \in S \,], \quad p_{ij} = \lim_{t \to \infty} Q_{ij}(t) \quad (4)$$

is a stochastic matrix. It means that for each pair $(i, j) \in S \times S$ $p_{ij} \ge 0$ and for each $i \in S$, $\sum_{j \in S} p_{ij} = 1$.

It is easy to notice that for each $i \in S$

$$G_i(t) = \sum_{j \in S} Q_{ij}(t) \tag{5}$$

is a probability cumulative distribution function (CDF) on $\mathbb{R}_+$. The definition 1 leads to the interesting and important conclusions $P(\vartheta_0 = 0) = 1$.

For a Markov Renewal Process with an initial distribution p_0 and a renewal kernel $Q(t),\ t \geq 0$ a following equality is satisfied

$$P(\xi_0 = i_0, \xi_1 = i_1, \vartheta_1 \leq t_1, \dots, \xi_n = i_n, \vartheta_n \leq t_n) =$$
$$p_{i_0} Q_{i_0 i_1}(t_1) Q_{i_1 i_2}(t_2) \dots Q_{i_{n-1} i_n}(t_n). \tag{6}$$

For $t_1 \to \infty, \dots, t_n \to \infty$, we obtain

$$P(\xi_0 = i_0, \xi_1 = i_1, \dots, \xi_n = i_n) = p_{i_0} p_{i_0 i_1} p_{i_1 i_2} \cdots p_{i_{n-1} i_n}. \tag{7}$$

It means that a sequence $\{\xi_n : n \in \mathbb{N}_0\}$ is a homogeneous Markov chain with the discrete state space S, defined by the initial distribution $p = [\, p_{i_0} : i_0 \in S \,]$ and the transition matrix $P = [\, p_{ij} : i, j \in S \,]$, where

$$p_{ij} = \lim_{t \to \infty} Q_{ij}(t). \tag{8}$$

The random variables $\vartheta_1, \dots, \vartheta_n$ are conditionally independent if a trajectory of the Markov chain $\{\xi_n : n \in \mathbb{N}_0\}$ is given. It means that

$$P(\vartheta_1 \leq t_1, \vartheta_2 \leq t_2 \dots, \vartheta_n \leq t_n \mid \xi_0 = i_0, \xi_1 = i_1, \dots, \xi_n = i_n) = \tag{9}$$
$$= \prod_{k=1}^{n} P(\vartheta_k \leq t_k \mid \xi_k = i_k, \xi_{k-1} = i_{k-1}).$$

The Markov renewal matrix $Q(t) = [\, Q_{ij}(t) : i, j \in S \,]$ is called continuous if each row of the matrix contains at least one element having continuous component in the Lebesgue decomposition of the probability distribution.

The matrix $Q(t) = [Q_{ij}(t) : i, j \in S]$ with elements

$$Q_{ij}(t) = p_{ij} G_i(t), \quad i \in S,$$

where

$$G_i(t) = c\, I_{[1,\infty)}(t) + (1 - c) \int_0^t h_i(u) du,\ c \in (0,1), \qquad p_{ij} \geq 0,\ \sum_{j \in S} p_{ij} = 1$$

and $h_i(\cdot)$ is a continuous probability density function, is an example of the continuous Markov renewal matrix.

The Markov renewal matrix $Q(t) = [\, Q_{ij}(t): i,j \in S\,]$ with elements

$$Q_{ij}(t) = p_{ij} I_{[1,\infty)}(t), \quad i \in S,$$

where $p_{ij} \geq 0, \ \sum_{j \in S} p_{ij} = 1$ is not continuous Markov renewal matrix. Moreover, in the whole paper we will assume that the Markov renewal matrix $Q(t) = [\, Q_{ij}(t):\ i,j \in S\,]$ is continuous.

Let

$$\tau_0 = \vartheta_0, \qquad \tau_n = \vartheta_1 + \vartheta_2 + \cdots + \vartheta_n,\ n \in \mathbb{N}_0, \tag{10}$$
$$\tau_\infty = \lim_{n \to \infty} \tau_n = \sup\{\tau_n:\ n \in \mathbb{N}_0\}.$$

The sequence $\{(\xi_n, \tau_n): n \in \mathbb{N}_0\}$ is two-dimensional Markov chain with transition probabilities

$$P(\xi_{n+1} = j, \tau_{n+1} \leq t \ \mid\ \xi_n = i, \tau_n = h) = Q_{ij}(t-h), \quad i,j \in S \tag{11}$$

and it is also called Markov Renewal Process (MRP) Koroluk [7].

DEFINITION OF DISCRETE STATE SPACE SEMI-MARKOV PROCESS

We shall present a definition and basic properties of a homogeneous semi-Markov process with a countable or finite state space S. The semi-Markov process (SMP) will be determined by the Markov Renewal Process (MRP).

Definition 2. *A stochastic process* $\{N(t):\ t \geq 0\}$ *defined by the formula*

$$N(t) = sup\{n \in \mathbb{N}_0:\ \tau_n \leq t\} \tag{12}$$

is called a counting process corresponding to a random sequence $\{\tau_n: n \in \mathbb{N}_0\}$.

Definition 3. *A discrete state space* S *stochastic process* $\{X(t):\ t \geq 0\}$ *with the piecewise constant and the right continuous sample paths given by*

$$X(t) = \xi_{N(t)} \tag{13}$$

is called a Semi-Markov Process associated with the Markov Renewal Process $\{(\xi_n, \vartheta_n): n \in \mathbb{N}_0\}$ *with the initial distribution* $p = [p_i(0):\ i \in S\,]$ *and the kernel* $Q(t) = [Q_{ij}(t):\ i,j \in S\,],\ \ t \geq 0$.

From definition it follows that

$$X(t) = \xi_n \quad \text{for} \quad t \in [\tau_n, \tau_{n+1}), \quad n \in \mathbb{N}_0. \tag{14}$$

the kernel $Q(t), t \geq 0$ define completely the semi-Markov process. From the definition of SMP it follows that

$$X(\tau_n) = \xi_n \quad \text{for} \quad n \in \mathbb{N}_0. \tag{15}$$

It means that a random sequence $\{X(\tau_n):\ n \in \mathbb{N}_0\}$ is a *homogeneous Markov chain* with a state space S, defined by the initial distribution $p_0 = [\,p_i^0:\ i \in S\,]$ and the stochastic matrix $P = [\,p_{ij}:\ i,j \in S\,]$, where $p_{ij} = \lim\limits_{t\to\infty} Q_{ij}(t)$. The sequence $\{X(\tau_n):\ n \in \mathbb{N}_0\}$ is called an *embedded Markov chain* of the semi-Markov process $\{X(t):\ t \geq 0\}$.

REGULARITY OF SMP

A semi-Markov process $\{X(t):\ t \geq 0\}$ is said to be regular if the corresponding counting process $\{N(t):\ t \geq 0\}$ has a finite number of jumps on a finite period with probability 1:

$$\forall_{t\geq 0} P(N(t) < \infty) = 1. \tag{16}$$

The equality (16) is equivalent to a relation

$$\forall_{t\geq 0} P(N(t) = \infty) = 0. \tag{17}$$

A semi-Markov process $\{X(t) : t \geq 0\}$ is regular if and only if

$$\forall_{t\geq 0} \lim_{n\to\infty} P(N(t) \geq n) = \lim_{n\to\infty} P(\tau_n \leq t) = 0 \ \ [5].$$

If

$E[N(t)] < \infty$, then a semi-Markov process $\{X(t):\ t \geq 0\}$ is regular.

Every semi-Markov process with a finite state space S is regular [7].

OTHER METHODS OF DETERMINING SEMI-MARKOV PROCESS

The semi-Markov process was defined by the initial distribution p and renewal kernel $Q(t)$ which determine the Markov Renewal Process. There are other ways of determining semi-Markov process. They are presented, among others, by Koroluk and Turbin [7], Limnios and Oprisan [10], Grabski [5, 6]. Some definitions of semi-Markov process enable its construction. First, we introduce the concepts and symbols that will be necessary for further considerations. For $P(\xi_{n+1} = j, \xi_n = i) > 0$ we define a function

$$F_{ij}(t) = P(\vartheta_{n+1} \le t \mid \xi_n = i, \xi_{n+1} = j), \quad i,j \in S,\ t \ge 0. \tag{18}$$

Notice that

$$F_{ij}(t) = P(\vartheta_{n+1} \le t \mid \xi_{n+1} = j, \xi_n = i) = \frac{P\{\vartheta_{n+1} \le t,\ \xi_{n+1} = j,\ \xi_n = i\}}{P(\xi_{n+1} = j,\ \xi_n = i)} =$$

$$= \frac{P(\vartheta_{n+1} \le t,\ \xi_{n+1} = j \mid \xi_n = i)}{P(\xi_{n+1} = j \mid \xi_n = i)} = \frac{Q_{ij}(t)}{p_{ij}} \quad \text{for} \quad i,j \in S, \qquad t \ge 0.$$

The function

$$F_{ij}(t) = P(\tau_{n+1} - \tau_n \le\ t \mid X(\tau_n) = i, X(\tau_{n+1}) = j) = \frac{Q_{ij}(t)}{p_{ij}} \tag{19}$$

is a cumulative probability distribution (CDF) of some random variable which is denoted by T_{ij} and it is called a *holding time* in state i, if the next state will be j. From (19) we have

$$Q_{ij}(t) = p_{ij} F_{ij}(t). \tag{20}$$

The function

$$G_i(t) = P(\tau_{n+1} - \tau_n\ \le t \mid X(\tau_n) = i) = \sum_{j \in S} Q_{ij}(t) \tag{21}$$

is a cumulative probability distribution of a random variable T_i that is called a *waiting time* in state i when a successor state is unknown.

It follows from (20) that a semi-Markov process with the discrete state space can be defined by the transition probabilities matrix of an embedded Markov chain: $P = [p_{ij}\colon\ i,j \in S]$ and the matrix of the holing times CDF $F(t) = [F_{ij}(t)\colon i,j \in S]$. Therefore a triple $(p, P, F(t))$ determines the homogenous SMP with the discrete space S. This method of determining SMP is convenient, for Monte-Carlo simulation of the SMP sample path.

From the Radon-Nikodym theorem it follows that there exist the functions $a_{ij}(x),\ \ x \ge 0,\ i,j \in S$ such that

$$Q_{ij}(t) = \int_0^t a_{ij}(t)\, dG_i(x). \tag{22}$$

Since

$$Q_{ij}(t) = P(\vartheta_{n+1} \le t, \xi_{n+1} = j \mid \xi_n = i) =$$
$$= \int_0^t P(\xi_{n+1} = j \mid \xi_n = i, \vartheta_{n+1} = x) dP(\vartheta_{n+1} x \mid \xi_n = i) =$$
$$= \int_0^t P(\xi_{n+1} = j \mid \xi_n = i, \vartheta_{n+1} = x)\, dG_i(x)$$

then

$$a_{ij}(x) = P(\xi_{n+1} = j \mid \xi_n = i,\ \vartheta_{n+1} = x). \tag{23}$$

The function $a_{ij}(x)$, $x \ge 0$ represents the transition probability from the state i to state j under condition that duration of the state i is equal to x. From (22) it follows that matrices

$$a(x) = [\, a_{ij}(x)\colon\ i,j \in S\,] \text{ and } G(x) = [\, \delta_{ij}\, G_i(x)\colon\ i,j \in S\,]$$

determine the kernel $Q(t) = [\, Q_{ij}(t)\colon i,j \in S\,]$. Therefore a triple $(p, a(x), G(x))$ defines the continuous time semi-Markov process with a discrete state space S.

In conclusion, three equivalent ways of determining the semi-Markov process are presented in this section:
– by pair $(p, Q(t))$;
– by triple $(p, P, F(t))$;
– by triple $(p, a(x), G(x))$.

It should be added that there exist other ways to define of semi-Markov process [7, 10]. Presented here ways of defining SMP seem to be most useful in applications.

CONNECTION BETWEEN SEMI-MARKOV AND MARKOV PROCESS

A discrete state space and continuous time semi-Markov process is a generalisation of that kind of Markov process. The Markov process can be treated as a special case of the semi-Markov process.

Theorem 1. *Every homogeneous Markov process* $\{X(t)\colon t \ge 0\}$ *with the discrete space* S *and the right-continuous trajectories keeping constant values on the half-intervals, given by the transition rate matrix* $\Lambda = [\lambda_{ij}\colon\ i,j \in S]$, $0 < -\lambda_{ii} = \lambda_i < \infty$ *is the semi-Markov process with the kernel* $Q(t) = [\, Q_{ij}(t)\colon\ i,j \in S\,]$, *where*

$$Q_{ij}(t) = p_{ij}\left(1 - e^{-\lambda_i t}\right), \quad t \geq 0, \quad p_{ij} = \frac{\lambda_{ij}}{\lambda_i} \quad \text{for} \quad i \neq j, \quad p_{ii} = 0. \tag{24}$$

P r o o f [5]. From definition of HMP transition rates we have:

$$p_{ij}(h) = P\{X(t+h) = j | \, X(t) = i\} = \begin{cases} \lambda_{ij}h + o(h) & \text{for} \quad i \neq j \\ 1 + \lambda_{ii}h + o(h) & \text{for} \quad i = j \end{cases},$$

where

$$\lim_{h \to 0} \frac{o(h)}{h} = 0.$$

The process having stepwise, right-continuous trajectories is separable. Since, for some $h > 0$, we have

$$\begin{aligned} &P(X(kh) = j, \quad X(s) = i, \quad s \in (0, (k-1)h) \mid X(0) = i) = \\ &= P(\{X(kh) = j\} \cap \bigcap_{r=1}^{k-1} X(rh) = i \mid X(0) = i) = \\ &= P(X(kh) = j \mid X((k-1)h) = i) \cdot P(X((k-1)h) = i \mid X((k-2)h) = i) \ldots \\ &\ldots \cdot P(X(h) = i \mid X(0) = i) = p_{ij}(h) p_{ii}^{k-1}(h), \end{aligned}$$

$$\begin{aligned} &p_{ij} = P(X(\tau_{n+1}) = j \mid X(\tau_n) = i) = P(X(\tau_1) = j | X(0) = i) = \\ &= \lim_{h \to 0} P(\bigcup_{k=1}^{\infty} X(kh) = j, \quad X(s) = i \ \text{ for } \ s \in (0, (k-1)h) | X(0) = i) = \\ &= \lim_{h \to 0} \sum_{k=1}^{\infty} p_{ij}(h) p_{ii}^{k-1}(h) = \lim_{h \to 0} \frac{p_{ij}(h)}{1 - p_{ii}(h)} = \lim_{h \to 0} \frac{\lambda_{ij}h + o(h)}{1 - [1 + \lambda_{ii}h + o(h)]} = -\frac{\lambda_{ij}}{\lambda_{ii}} = \frac{\lambda_{ij}}{\lambda_i}. \end{aligned}$$

For $i = j$ we obtain

$$p_{ii} = 1 - \sum_{j \neq i} p_{ij} = 1 - \sum_{j \neq i} \frac{\lambda_{ij}}{\lambda_i} = -\frac{\sum_{j \in S} \lambda_{ij}}{\lambda_i} = 0.$$

Let $t = lh$. Similar way we get

$$\begin{aligned} &Q_{ij}(t) = P(X(\tau_{n+1}) = j, \tau_{n+1} - \tau_n \ t \mid X(\tau_n) = i) = \lim_{h \to 0} \sum_{k=1}^{l} p_{ij}(h) p_{ii}^{k-1}(h) = \\ &\lim_{h \to 0} \frac{p_{ij}(h)[1 - p_{ii}^{l}(h)]}{1 - p_{ii}(h)} = \frac{\lambda_{ij}}{\lambda_i} \lim_{h \to 0} [1 - p_{ii}^{l}(h)] = p_{ij} \lim_{h \to 0} [1 - (1 + \lambda_{ii}h + o(h))^{l}]. \end{aligned}$$

If $h \to 0$, then $l \to \infty$. Since

$$Q_{ij}(t) = p_{ij}[1 - \lim_{l\to\infty}\left(1 + \lambda_{ii}\frac{t}{l}\right)^{l}] =$$
$$= p_{ij}\left\{1 - \left[\lim_{l\to\infty}\left(1 + \frac{1}{\frac{l}{\lambda_{ii}t}}\right)^{\frac{l}{\lambda_{ii}t}}\right]^{\lambda_{ii}\,t}\right\} = p_{ij}\left(1 - e^{-\lambda_i\,t}\right).$$

From this theorem it follows that the length of interval $[\tau_n,\ \tau_{n+1})$ given states at instants τ_n and τ_{n+1} is a random variable having an exponential distribution with parameter independent of state at the moment τ_{n+1}:

$$F_{ij}(t) = P(\tau_{n+1} - \tau_n\ \le t \mid X(\tau_n) = i, X(\tau_{n+1}) = j) = 1 - e^{-\lambda_i\,t},\quad t \ge 0.$$

As we know, the function $F_{ij}(t)$ is a cumulative probability distribution of a holding time in the state i, if the next state is j. Let us recall that the function

$$G_i(t) = \sum_{j\in S} Q_{ij}(t) = 1 - e^{-\lambda_i\,t},\quad t \ge 0$$

is a CDF of a waiting time in the state i. For the Markov process, holding times T_{ij}, $i,j \in S$ and waiting times T_i, $j \in S$ have the identical exponential distributions with parameters $\lambda_i = \frac{1}{E(T_i)}$, $i \in S$ that do not depend on state j.

ILLUSTRATIVE EXAMPLE

Now we take under consideration the reliability model of a renewable two--component series system under assumption that the times to failure of booth components denoted as ζ_1, ζ_2 are exponentially distributed with parameters λ_1 and λ_2 but the renewal (repair) times of components are the non-negative random variables η_1, η_2 with arbitrary distributions defined by CDF $F_{\eta_1}(t)$, $F_{\eta_2}(t)$. We suppose that the above considered random variables and their copies are mutually independent. First we have to determine the process states:

$1 -$ $\mathit{the\ system\ renewal\ after\ the\ failure\ of\ the\ first\ component}$ $(\mathit{down\ state})$;

$2 -$ $\mathit{the\ system\ renewal\ after\ the\ failure\ of\ the\ second\ component}$ $(\mathit{down\ state})$;

$3 -$ $\mathit{work\ of\ the\ system\ \ both\ components\ are\ up}$.

A renewal kernel is given by the rule

$$Q(t) = \begin{bmatrix} 0 & 0 & Q_{13}(t) \\ 0 & 0 & Q_{23}(t) \\ Q_{31}(t) & Q_{32}(t) & 0 \end{bmatrix}.$$

Using the assumptions we calculate all elements of this matrix.

$$Q_{13}(t) = F_{\eta_1}(t), \qquad Q_{23}(t) = F_{\eta_2}(t);$$

$$Q_{31}(t) = P(\zeta_1 \le t, \qquad \zeta_1 > \zeta_2) = \iint\limits_{D_{13}} \lambda_1 e^{-\lambda_1 x}\, \lambda_2 e^{-\lambda_2 y} dx\, dy,$$

where

$$D_{31} = \{(x,y):\ x \le t, \qquad y > x\}.$$

Thus

$$Q_{31}(t) = \int_0^t \lambda_1 e^{-\lambda_1 x}\, e^{-\lambda_2 x} dx = \frac{\lambda_1}{\lambda_1 + \lambda_2}\left(1 - e^{-(\lambda_1+\lambda_2)\, t}\right).$$

In the same way we obtain

$$Q_{32}(t) = \frac{\lambda_2}{\lambda_1 + \lambda_2}\left(1 - e^{-(\lambda_1+\lambda_2)\, t}\right).$$

So the model the reliability model of a renewable two-component series system was constructed.

CONCLUSIONS

The main goal of this paper is to present and explain the basic concepts of the semi-Markov process theory. Expected next publications will cover applications of the semi-Markov process in the reliability and operation of the ships systems. The peper provides the definitions and basic properties related to a discrete state space semi-Markov process. The semi-Markov process is constructed by the so--called Markov renewal process. The Markov renewal process is defined by the transition probabilities matrix, called the renewal kernel, and by an initial disribution. It follows from the definition of the SMP that future states of the process and their sojourn times do not depend on past states and their sojourn times if a present state is known. Let us add that the initial distribution p and the kernel $Q(t)$ define completely the SMP. Moreover two equivalent ways of determining the semi-Markov process are presented in this section:

– by triple $(p, P, F(t))$;
– by triple $(p, a(x), G(x))$.

In the paper connection between semi-markov and markov process is shown. A discrete state space and continuous time semi-Markov process is a generalisation of that kind of Markov process. The Markov process can be treated as a special case of the semi-Markov process. From presented here theorem it follows that the length of interval $[\tau_n, \tau_{n+1})$ given states at instants τ_n and τ_{n+1} is a random variable having an exponential distribution with parameter independent of state at the moment τ_{n+1}:

$$F_{ij}(t) = P(\tau_{n+1} - \tau_n \le t \mid X(\tau_n) = i, X(\tau_{n+1}) = j) = 1 - e^{-\lambda_i t}, \quad t \ge 0.$$

The function denoting a CDF of a waiting time T_i, in the state i has also identical exponential distributions with parameters $\lambda_i = \frac{1}{E(T_i)}$:

$$G_i(t) = \sum_{j \in S} Q_{ij}(t) = 1 - e^{-\lambda_i t}, \quad t \ge 0.$$

It means that the semi-Markov processes allow to construct the models for a wider class of problems.

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KONCEPCJA PROCESU SEMI-MARKOWA

STRESZCZENIE

Artykuł przedstawia definicje i podstawowe cechy procesu semi-Markowa dyskretnego stanu przestrzeni. Proces semi-Markova jest zbudowany przez tzw. proces odnawiania Markova, który jest specjalnym przypadkiem dwuwymiarowego ciągu Markova. Proces odnawiania Markova jest zdefiniowany przez macierz prawdopodobieństw przejściowych, zwaną jądrem odnawiania, i początkowy rozkład lub przez inne charakterystyki, które są równe jądru odnawiania. Proces obliczeniowy odpowiadający procesowi semi-Markova pozwala na określenie koncepcji regularności procesu. W artykule przedstawiono również pozostałe metody określania procesu semi-Markova. Przedstawione koncepcje są zaprezentowane na prostym przykładzie.

Słowa kluczowe:
proces semi-Markova, macierz prawdopodobieństw.