MATHEMATICAL MODELING OF PHENOMENOLOGICAL MATERIAL PROPERTIES – DIFFERENTIAL OPERATOR FORMS OF CONSTITUTIVE EQUATIONS

Mária MINÁROVÁ^{1*}

Abstract

The paper deals with rheological models and creep and relaxation tests on matters which are represented by models. Three models based on two fundamental components (Hooke's elastic and Newton's viscous compounds) are performed. The models originated from several fundamental matters by their parallel or serial connections. The corresponding constitutive equations are derived. The behavior of the models under the creep and relaxation tests is observed and is expressed by corresponding stress – strain formulas and illustrated in the figures.

Address

¹ Department of Mathematics and Constructive Geometry, Faculty of Civil Engineering, Slovak University of Technology, Radlinského 11, 81368, Bratislava, Slovak Republic

* **Corresponding author:** maria.minarova@stuba.sk

Key words

- Linear viscoelastic model,
- creep,
- relaxation,
- constitutive equation,
- rheology.

1 INTRODUCTION

A fundamental law representing the reaction of a body is the dependence between the stress and strain ($\varepsilon \sim \sigma$), which is determined by experimental measurements. Nevertheless, the experiments are realized under certain theoretical concepts concerning the behavior of a material. Uniquely, Hooke's law has been applied to all materials until lately. Subsequently it has been discovered that applying Hooke's law in stress and strain investigations is not sufficient and that for some materials, it is essential to specify the stress and strain relationship ($\varepsilon \sim \sigma$) more precisely by its working diagram. Such an approach has led to one of the main reasons for the development of the theory of plasticity (Dill, 2007). A number of factors appear to be significant. Undoubtedly, not just stress and strain quantities have to be involved in a theoretical investigation, but also the character and the velocity of the stress, the strain over time, the range of the stress, etc. These form the basis of the theory of viscoelasticity.

Viscoelasticity plays an important role in engineering. It is an exact discipline using the analytic and synthetic methods of observing phenomena, and then applying the scientific knowledge in practice. It includes both the elastic and viscous behavior in a matter and is specific for each matter. Contrary to dynamics, the viscoelasticity

theory also implements time derivatives to slow deformation processes, where the effect of the inertial forces can be neglected.

The creep and relaxation theory involved in viscoelasticity theory is also stimulated by the use of new technologies and applications, composite materials, metals, and textiles.

2 ELASTIC, VISCOUS AND STIFF MATERIAL

Soft material known as *viscoelastic* material is characterized by both elastic and viscous behavior. As far as “small” deformations are concerned, an *elastic material* behaves in accordance with Hooke's law, which describes the linear relationship between stress and strain:

$$\varepsilon = \frac{\sigma}{E}, \quad (1)$$

where ε [-] is a deformation (strain), σ [Pa] intensity of the stress, and E [Pa] is Young's elastic module. The deformation occurs immediately as a response to the load.

On the other hand, a *viscous material* under a constant load flows with a constant velocity:

$$\dot{\varepsilon} = \frac{\sigma}{\eta}, \quad (2)$$

Fig. 1 Hooke's elastic matter (left) with constitutive equation (1) and Newton's viscous liquid with constitutive equation (2) and their stress – strain dependence (on graphs).

where η [kg/(m.s)] is the Newton viscosity coefficient. The dot over the entity represents its time derivative.

Whereas (1) represents the relation between the force applied to a matter and its deformation, (2) represents the relation between force applied to a body and its deformation rate.

Remark: In three – dimensional tasks, E and η in (1) and (2) are replaced by tensor operators, $\sigma = \hat{H}\epsilon$, i.e., $K^{(r)}\sigma = \hat{K}_{(s)}\epsilon$, i.e. $Q^{(r)}\epsilon = \hat{Q}_{(s)}\sigma$, where $\hat{H}, \hat{K}, \hat{Q}$ are symmetrical positively definite tensor operators, and K and Q are scalar operators (e.g., Brilla, 1970).

A. What are linear creep and linear relaxation tests?

Creep and relaxation tests are tools for the detection of the response of a material to an external load.

Creep test

If we induce the immediate stress σ_0 on a sample piece represented by a rheological model, keep it on a constant level, and observe the increasing deformation $\epsilon(t)$, we can establish the *creep function* $D(t)$:

$$D(t) = \frac{\epsilon(t)}{\sigma_0}. \quad (3)$$

Relaxation test

A relaxation test starts by imposing a deformation on a sample piece represented by a rheological model, keeping it constant, and measuring the resulting stress that is inevitable in order to maintain this constant deformation. The ratio of the measured stress and applied deformation is called the *relaxation modulus* $R(t)$ (Novotný and Hanuška, 1983):

$$R(t) = \frac{\sigma(t)}{\epsilon_0}. \quad (4)$$

The notation *linear* in the subtitle above means that a creep function is not a function of stress and that a relaxation modulus is not a function of a deformation; meanwhile, in the case of nonlinearity it is: $D(t, \tau)$ and $R(t, \epsilon)$. For purely elastic material, where $D(t) = D$ and $R(t) = E$ are constants, it is therefore clear that $D E = I$. In a linear case the superposition principle can be used for stress and deformation values.

B. Relationship between the creep function and the relaxation modulus

When we load a linear viscoelastic material in a time τ_0 by a constant stress σ_0 , then the deformation can be expressed as:

$$\epsilon(t) = \sigma_0 D(t, \tau_0) \text{ for } t > \tau_0, \quad (5)$$

$$\epsilon(t) = \sigma_0 D(t, \tau_0) + \int_{\tau_0}^t D(t, \tau) d\sigma = \sigma_0 D(t, \tau_0) + \int_{\tau_0}^t D(t, \tau) \frac{d\sigma}{d\tau} d\tau, \quad (6)$$

where τ performs the time history; therefore, $D(t, \tau)$ is a function of the actual time and previous time history represented by τ . (Novotný, 2009) This means that the age of the material can also influence its response to the load. When we impose the same load at two different time instances and maintain it within the same length of time on the linear viscoelastic material and obtain two of the same resulting responses, that means that the age of the material does not influence the properties of the material; the material is time invariant, so we write:

$$D(t, \tau) = D(t - \tau). \quad (7)$$

Formula (7) expresses the independence of the deformation – the time curve on the starting instant of the test.

Creep compliance – the response of the material to the load measured from the instant of the load (putting $t_0 = 0$, $\sigma_0 = 0$) is

$$\epsilon(t) = \int_0^t D(t - \tau) \dot{\sigma} d\tau \quad (8)$$

and the relaxation modulus is (Novotný, 2009)

$$\sigma(t) = \int_0^t R(t - \tau) \dot{\epsilon} d\tau. \quad (9)$$

C. Linear viscoelastic material models. Constitutive equations and creep and relaxation tests

When combining two basic rheological components (see Fig. 1: purely elastic material – Hooke's elastic material (H) and purely viscous material – Newton viscous liquid (N)) in various ways, we can obtain various types of linear viscoelastic material models.

The matters (**H**) and (**N**) can be joined serially or on a parallel basis (analogous to electric circuits); each of the matters (**H**) and (**N**) can be used several times within one model. In such a way an infinite number of more or less complicated models can be created. Below we introduce three of them. Afterwards, the constitutive equations of the entire model are created from the constitutive equations of the particular base matters with respect to the type of connection within the composition (Mareš, 2009; Nemec, 1985; Sobotka, 1981).

i. Maxwell rheological model

By serially connecting the two basic matters, we get Maxwell's rheological model; the structural formula is (**M**) = (**H**) – (**N**), see Fig. 2.

Fig. 2 Maxwell rheological model.

When a tensile force P acts as in Fig. 2, the elastic component of the system stretches immediately; the damping viscous component causes a delay. Let us denote ${}_H\sigma$ stress tensor, ${}_H\epsilon$ strain in Hooke's matter (${}_H\sigma$, ${}_H\epsilon$ in one-dimensional case scalars), ${}_N\sigma$, ${}_N\epsilon$ stress (${}_N\sigma$, ${}_N\epsilon$ in 1D), and strain in Newton's liquid. The stress in Maxwell's model is distributed equally within the whole model:

$$\sigma = {}_H\sigma = {}_N\sigma, \quad (10)$$

On the other hand, the overall deformation is equal to the sum of the deformations on the particular components:

$$\epsilon = {}_H\epsilon + {}_N\epsilon. \quad (11)$$

The constitutive equations for the particular components are:

$$\begin{aligned} {}_H\epsilon &= \hat{C}\sigma \\ \frac{\partial}{\partial t} {}_N\epsilon &= \hat{\gamma}\sigma \end{aligned} \quad (12.a,b)$$

Applying the time derivative operator to equations (11) and (12.a) and coupling the equations (10), (11), and (12), we get the physical relation between the stress and strain in the form of a partial differential equation

$$\frac{\partial}{\partial t} \epsilon = \frac{\partial}{\partial t} ({}_H\epsilon + {}_N\epsilon) = \hat{C} \frac{\partial \sigma}{\partial t} + \hat{\gamma} \sigma, \quad (13)$$

In a one-dimensional space we get the ordinary differential equation (we have just the scalar E instead of tensor $\hat{C}$ and scalar η instead of tensor $\hat{\gamma}$):

$$\frac{d\epsilon}{dt} = \frac{1}{E} \frac{d\sigma}{dt} + \frac{1}{\eta} \sigma. \quad (14)$$

The deformation of the specimen represented by the Maxwell model can be obtained by the integration of (14) on the interval $\langle t_0, t \rangle$

$$\epsilon(t) = \int_{t_0}^t \left(\frac{1}{E} \dot{\sigma}(\tau) + \frac{1}{\eta} \sigma(\tau) \right) d\tau + \epsilon(t_0). \quad (15)$$

The stress dependence on the strain can be expressed as the solution to the Cauchy problem (14) with the initial deformation $\epsilon(t_0)$, which is of the type $\dot{y} + ay = f(t)$, $y(t_0) = y_0$:

$$\sigma(t) = e^{-\frac{E}{\eta}t} \left(\int_{t_0}^t e^{\frac{E}{\eta}\tau} \dot{\epsilon}(\tau) d\tau + e^{\frac{E}{\eta}t_0} E\epsilon(t_0) \right). \quad (16)$$

Creep test

When we impose a constant stress σ^* on the specimen represented by the Maxwell model in the time $t_0 = 0$ and maintain it until time t , the deformation will run over time according to the formula:

$$\epsilon(t) = \frac{\sigma^*}{\eta} t + \epsilon(t_0) = \frac{\sigma^*}{\eta} t + \frac{\sigma^*}{E}. \quad (17)$$

Fig. 3 Deformation course during creep test done on the Maxwell model.

Relaxation test

When a constant deformation ϵ^* is imposed on the specimen represented by the Maxwell model in the time $t_0 = 0$ and maintained until time t , the stress will change over time according to the formula:

$$\sigma(t) = e^{-\frac{E}{\eta}t} E\epsilon^*. \quad (18)$$

Fig. 4 Course of stress during creep test done using the Maxwell model.

ii. Kelvin – Voigt rheological model

By joining basic matters (**H**) and (**N**) on a parallel basis (by a stiff slab), we obtain the Kelvin–Voigt rheological model, (**V**) = (**H**) | (**N**), see Fig. 5.

Fig. 5 Kelvin - Voigt rheological model.

After the stress (tension) acts as it does in Fig. 5, the deformation within the model is the same in both matters; the stresses are summed:

$$\sigma = \sigma_H + \sigma_N, \quad \varepsilon_H = \varepsilon_N = \varepsilon. \quad (19)$$

By using a constitutive equation for the particular components, we get the relationship between the stress and strain in the Kelvin – Voigt model:

$$\sigma = (\hat{E} + \hat{\eta} \frac{\partial}{\partial t}) \varepsilon. \quad (20)$$

In 1D the constitutive equation of the Kelvin-Voigt model (20) will take the form:

$$\sigma = E\varepsilon + \eta \dot{\varepsilon}. \quad (21)$$

The analytical solution to (21) sets the dependence of the deformation on the stress:

$$\varepsilon(t) = e^{-\frac{E}{\eta}t} \left(\int_0^t e^{\frac{E}{\eta}\tau} \frac{1}{\eta} \sigma(\tau) d\tau + e^{\frac{E}{\eta}t_0} \varepsilon(t_0) \right). \quad (22)$$

Creep test

By imposing a constant value of stress σ^* on the specimen represented by the model, we can run the creep test. Moreover, assuming the initial deformation $\varepsilon(t_0)$, the prescription for the deformation behavior will be as follows:

$$\varepsilon(t) = \frac{\sigma^*}{E} \left(1 - e^{-\frac{E}{\eta}(t-t_0)} \right) + e^{-\frac{E}{\eta}(t-t_0)} \varepsilon(t_0). \quad (23)$$

Example: Creep test on the specimen represented by the Kelvin – Voigt model

Starting at the time $t_0 = 0$ with the initial deformation $\varepsilon(t_0) = 0$, by imposing a constant stress σ^* on t_0 on the specimen represented by the Kelvin – Voigt model and keeping it until $t = t_1$ (see Fig. 6), we obtain the formula for the change in the deformation over time:

$$\varepsilon(t) = \frac{\sigma^*}{E} \left(1 - e^{-\frac{E}{\eta}t} \right), \quad 0 \leq t \leq t_1. \quad (23a)$$

From (23), we also get the formula for the case when the stress stops immediately – drops to zero in time t_1 , and the specimen stays unburdened for the rest of the time:

$$\varepsilon(t) = e^{-\frac{E}{\eta}(t-t_1)} \varepsilon(t_1). \quad (23b)$$

Fig. 6 Course of deformation and deformation rate during the creep test done on the specimen represented by the Maxwell model.

Relaxation test

A relaxation test on the specimen represented by the Kelvin-Voigt model is impossible to realize because to reach an immediate constant non-zero deformation ε^* , the infinite stress is needed.

iii. Zener rheological model

When we join Hooke's matter to the Maxwell model on a parallel basis, we get the Zener rheological model, (**Z**) = (**H**) | [(**H**) - (**N**)], see Fig. 7. When we index all the items on the left segment of the model by 1 and every item on the right segment by 2, in accordance with the two previous models, we can write:

$$\varepsilon_1 = \varepsilon_2 = \varepsilon, \quad (24.a)$$

$$\varepsilon_1 = \varepsilon_{2N} + \varepsilon_{2H}, \quad (24.b)$$

$$\sigma = \sigma_1 + \sigma_2, \quad (24.c)$$

$$\sigma_1 = E_1 \varepsilon_1, \quad (24.d)$$

$$\sigma_2 = \eta_{2N} \dot{\varepsilon}_{2N}. \quad (24.e)$$

Fig. 7 Zener rheological model.

Because from (24a–c) it follows that $\sigma_2 = \sigma - \sigma_1 = \sigma - E_1 \varepsilon_1$ and, $\dot{\varepsilon}_1 = \dot{\varepsilon}_{2N} + \dot{\varepsilon}_{2H} = \frac{\sigma_2}{\eta_{2N}} + \frac{\dot{\sigma}_2}{E_{2H}}$, then the constitutive equation of the Zener rheological model will take the form

$$\dot{\varepsilon} = \frac{\sigma - E_1 \varepsilon}{\eta_{2N}} + \frac{\dot{\sigma} - E_1 \dot{\varepsilon}}{E_{2H}}. \quad (25)$$

Rearranging (25) into a form when the stress and strain will be separated on the opposite sides of the equation, we get the differential dependence between them:

$$\dot{\varepsilon} \left(\eta_{2N} + E_1 \frac{\eta_{2N}}{E_{2H}} \right) + E_1 \varepsilon_1 = \sigma + \frac{\eta_{2N}}{E_{2H}} \dot{\sigma}. \quad (26)$$

By solving (26), the linear ordinary differential equation with respect to $\sigma(t)$ (as the index of η does not fulfill the distinguishing function, we can skip it further), we get the formula for $\sigma(t)$ dependence on $\varepsilon(t)$:

$$\sigma(t) = e^{-\frac{E_2}{\eta} t} \left(\int_0^t e^{\frac{E_2}{\eta} \tau} \left((E_1 + E_2) \dot{\varepsilon}(\tau) + \frac{E_1 E_2}{\eta} \varepsilon(\tau) \right) d\tau + e^{\frac{E_2}{\eta} t_0} \sigma(t_0) \right). \quad (27)$$

Eventually, by solving (26), the linear ordinary differential equation with respect to $\varepsilon(t)$, we get the direct formula for the time course of $\varepsilon(t)$ depending on $\sigma(t)$:

$$\varepsilon(t) = e^{-\frac{E_1 E_2}{(E_1 + E_2) \eta_2} t} \left(\int_0^t e^{\frac{E_1 E_2}{(E_1 + E_2) \eta_2} \tau} \left(\dot{\sigma}(\tau) + \frac{E_2}{\eta_2} \sigma(\tau) \right) \frac{1}{E_1 + E_2} d\tau + e^{\frac{E_1 E_2}{(E_1 + E_2) \eta_2} t_0} \varepsilon(t_0) \right). \quad (28)$$

Creep test

When we impose a constant stress σ^* on the specimen represented by the Zener model in the time t_0 and maintain it until time t and skip the index on η , the deformation will run over time according to the formula:

$$\varepsilon(t) = \frac{\sigma^*}{E_1} \left(1 - e^{-\frac{E_1 E_2}{(E_1 + E_2) \eta} (t - t_0)} \frac{1}{E_1} \right) + e^{-\frac{E_1 E_2}{(E_1 + E_2) \eta} (t - t_0)} \varepsilon(t_0). \quad (29)$$

Relaxation test

When a constant deformation ε^* is imposed on the specimen represented by the Zener model in the time $t_0 = 0$ and maintained until time t , the stress will change over time according to the formula:

Fig. 8 Creep test realized on a specimen represented by the Zener rheological model (beginning at $t_0 = 0$ with a constant stress imposed and dropping the stress instantly in a t_1).

$$\sigma(t) = E_1 \varepsilon^* \left(1 - e^{-\frac{E_2}{\eta} (t - t_0)} \right) + e^{-\frac{E_2}{\eta} (t - t_0)} \sigma(t_0). \quad (30)$$

Fig. 9 Relaxation test realized on a specimen represented by the Zener rheological model (with a constant deformation initiated at $t_0 = 0$).

A number of other rheological models can be created as models of real types of material by various connections of the fundamental masses (Hooke's and Newton's) in complex models; then the constitutional equations can be derived according to the basic rules of parallel and serial connections.

Fig. 10 The causes of the viscoelastic behavior of the materials – a liquid presence (www.ce.berkeley.edu).

Fig. 11 Motivation for using rheology in building physics (www.ce.berkeley.edu).

3 CONCLUSION

The theory of mathematical and mechanical modeling of types of material whose properties are time dependent is an important branch of mechanical modeling. Knowing the actual properties of types of material enhances the possibilities of a better understanding and the proper and optimal utilization of a wide range of time-proven materials, as well as types of actual new composite material nowadays (Brilla, 1970).

This paper performs rheological models that involve elastic and viscous matters. The corresponding governing equations for the creep and relaxation tests are derived. Accordingly, the solutions in one dimension are provided.

Other, more complex, methods for modeling the visco-elastic properties of configurations of types of real materials by using the application of “weakly singular kernels” was theoretically developed by Sumec (1983); in 1985, Sumec and Potůček extended the theory by practical applications of two and three-dimensional structural elements.

Acknowledgement

This work was supported by the APVV – 0184 – 10 grant.

REFERENCES

- Brilla, J. (1970) *Variational Methods in Linear Anisotropic Viscoelasticity*. Zeitschrift für Angewandte Mathematik und Mechanik (Journal of Applied Mathematics and Mechanics), Sonderheft, Vol. 50, Nos. 1-4, pp. 102–103 ISSN: 0044-2267
- Dill, E. H. (2007) *Continuum Mechanics: Elasticity, Plasticity, Viscoelasticity*. CRC Press, Taylor & Francis Group, New York, ISBN-10: 0849397790
- Mareš, T. (2009) *Úvod do reologie – reologické modely měkkých tkání (Introduction to rheology – rheological models of soft tissues)*, Matmodel.cz, studijní materiály v ČVUT v Praze, FS, Ústav mechaniky, biomechaniky a mechatroniky (Matmodel.cz, study materials of CTU, FCE, Prague, Department of Mechanics, Biomechanics and Mechatronics).
- Nemec, J. (1985) *Foreword to Biomechanics*. Academia, Prague.
- Novotný, B. – Hanuška, A. (1983) *Theory of Layered Half Spaces*. VEDA Publishing house, Bratislava.
- Novotný, C. (2009) *Viskoelastická (Viscoelasticity)*. Matmodel.cz, studijní materiály ČVUT v Praze, FS, Ústav mechaniky, biomechaniky a mechatroniky (Matmodel.cz, study materials of CTU, FCE, Prague, Department of Mechanics, Biomechanics and Mechatronics).
- Sobotka, Z. (1981) *Rheology of Mass and structures*. Academia, Prague.
- Sumec, J. (1983) *Mechanics - Mathematical Modeling of Materials Whose Physical Properties are Time – Dependent*. Internal Research Report, III-3-4/9.4 USTARCH-SAV Bratislava.
- Sumec, J. – Potůček, M. (1985) *State of Stress – Strain of Structural Elements and Systems with Linear Viscoelastic Materials*. Internal Research Report III-3-4/9.5, ÚSTARCH SAV, Bratislava.
- www.ce.berkeley.edu/~paulmont/241/creep.pdf.