LOCATION OF OBJECTS WITH A HIGH DEGREE OF INNER PRECISION

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Abstract

Let us have two coordinate systems in a plane. In the first one (e.g. JTSK) the accuracy of the coordinate estimators is significantly lower than the accuracy in the other system (the inner system of an object). The task is to determine the coordinate estimators of the characteristic points of the object in the first system (with a significantly lower degree of accuracy) in such a way that the relative accuracy of the characteristic points is the same as the relative accuracy in the inner system.

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1 INTRODUCTION

The location of every new engineering work (a nuclear plant, tunnel, dam, etc.) must be determined in a state coordinate system, i.e., the coordinates of the characteristic points of the structure must be determined. The requirements for precision in a state coordinate system are usually less demanding than the requirements in the internal system of the new structure. In the following text, for the sake of simplicity, it will be assumed that the internal system is error-free. Corresponding problems see also in [2], [3], [5], [7].

Thus the task is to find such coordinate estimators of the characteristic points in the state system so that the dispersions of the estimators of the distances among the characteristic points are zeros.

2 NOTATION

Let $\mathbf{a} = (\mathbf{a}'_1, \dots, \mathbf{a}'_k)'$ be the coordinate vector of the characteristic points $P_1, \dots, P_k$, of the new structure, $\mathbf{a}_i$ be the 2D coordinate vector of the i -th characteristic point P_i , and $i = 1, \dots, k$. Let $\hat{\Theta}$ be the estimator of the coordinates of points $B_1, \dots, B_N$ in the state coordinate system in the neighbourhood of the new structure, which enables us to determine the state coordinates of the characteristic points $P_1, \dots, P_k$. The fact that the mean value $E(\hat{\Theta})$ of the estimator $\hat{\Theta}$ is Θ

and its covariance matrix $\text{Var}(\hat{\Theta})$ is $\mathbf{W}$, is denoted as $\hat{\Theta} \sim_{2N} (\Theta, \mathbf{W})$. The real value of the coordinate vector of the point P_i in the state coordinate system is $\mathbf{b}_i$. The preliminary estimator $\tilde{\mathbf{b}}_i$ of the vector $\mathbf{b}_i$ is based on the observation vector $\mathbf{Y} \sim_n (\mathbf{D}\Theta + \mathbf{F}\mathbf{b}, \Sigma)$, on the vector $\hat{\Theta}$ and on the covariance matrix $\mathbf{W}$. The $n \times 2k$ matrix $\mathbf{F}$ is the design matrix of a measurement. The $n \times 2N$ matrix $\mathbf{D}$ is the incidental matrix which characterizes the connection of the points $B_1, \dots, B_N$ with the points $P_1, \dots, P_k$. The covariance matrix Σ usually does not satisfy the requirements for the accuracy of the internal points of the structure; however, it satisfies the requirement for the accuracy of the state system.

3 ESTIMATOR OF THE VECTOR $\mathbf{b}$

Lemma 2.1 Let the rank of the matrix $\mathbf{F}$ be $r(\mathbf{F}) = 2k < n$, and let $\mathbf{W}$ and Σ be positive definite matrices. The best linear unbiased estimator (BLUE) of the vector $\mathbf{b}$ in the model

$$\begin{pmatrix} \hat{\Theta} \\ \mathbf{Y} \end{pmatrix} \sim_{2N+n} \left[\begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{D} & \mathbf{F} \end{pmatrix} \begin{pmatrix} \Theta \\ \mathbf{b} \end{pmatrix}, \begin{pmatrix} \mathbf{W} & \mathbf{0} \\ \mathbf{0} & \Sigma \end{pmatrix} \right]$$

is

$$\begin{aligned} \tilde{\mathbf{b}} &= [\mathbf{F}'(\Sigma + \mathbf{D}\mathbf{W}\mathbf{D}')^{-1}\mathbf{F}]^{-1}\mathbf{F}'(\Sigma + \mathbf{D}\mathbf{W}\mathbf{D}')^{-1}(\mathbf{Y} - \mathbf{D}\hat{\Theta}), \\ \text{Var}(\tilde{\mathbf{b}}) &= [\mathbf{F}'(\Sigma + \mathbf{D}\mathbf{W}\mathbf{D}')^{-1}\mathbf{F}]^{-1}. \end{aligned}$$

Proof. The BLUE $\tilde{\mathbf{b}}$ in this model is

$$\begin{aligned}\tilde{\mathbf{b}} &= (\mathbf{0}'_{2k, 2N}, \mathbf{I}_{2k}) \left[\begin{pmatrix} \mathbf{I}_{2N} & \mathbf{D}' \\ \mathbf{0} & \mathbf{F}' \end{pmatrix} \begin{pmatrix} \mathbf{W}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Sigma}^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{D} & \mathbf{F} \end{pmatrix} \right]^{-1} \\ &\quad \times \begin{pmatrix} \mathbf{I}_{2N} & \mathbf{D}' \\ \mathbf{0} & \mathbf{F}' \end{pmatrix} \begin{pmatrix} \mathbf{W}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Sigma}^{-1} \end{pmatrix} \begin{pmatrix} \hat{\Theta} \\ \mathbf{Y} \end{pmatrix} \\ &= (\mathbf{0}', \mathbf{I}) \begin{pmatrix} \mathbf{W}^{-1} + \mathbf{D}'\mathbf{\Sigma}^{-1}\mathbf{D} & \mathbf{D}'\mathbf{\Sigma}^{-1}\mathbf{F} \\ \mathbf{F}'\mathbf{\Sigma}^{-1}\mathbf{D} & \mathbf{F}'\mathbf{\Sigma}^{-1}\mathbf{F} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{W}^{-1}\hat{\Theta} + \mathbf{D}'\mathbf{\Sigma}^{-1}\mathbf{Y} \\ \mathbf{F}'\mathbf{\Sigma}^{-1}\mathbf{Y} \end{pmatrix} \\ &= ([21], [22]) \begin{pmatrix} \mathbf{W}^{-1}\hat{\Theta} + \mathbf{D}'\mathbf{\Sigma}^{-1}\mathbf{Y} \\ \mathbf{F}'\mathbf{\Sigma}^{-1}\mathbf{Y} \end{pmatrix},\end{aligned}$$

where

$$\begin{pmatrix} \mathbf{W}^{-1} + \mathbf{D}'\mathbf{\Sigma}^{-1}\mathbf{D} & \mathbf{D}'\mathbf{\Sigma}^{-1}\mathbf{F} \\ \mathbf{F}'\mathbf{\Sigma}^{-1}\mathbf{D} & \mathbf{F}'\mathbf{\Sigma}^{-1}\mathbf{F} \end{pmatrix}^{-1} = \begin{pmatrix} [11] & [12] \\ [21] & [22] \end{pmatrix}.$$

It is valid that

$$\begin{aligned}[21] &= -[\mathbf{F}'\mathbf{\Sigma}^{-1}\mathbf{F} - \mathbf{F}'\mathbf{\Sigma}^{-1}\mathbf{D}(\mathbf{W}^{-1} + \mathbf{D}'\mathbf{\Sigma}^{-1}\mathbf{D})^{-1}\mathbf{D}'\mathbf{\Sigma}^{-1}\mathbf{F}]^{-1}\mathbf{F}'\mathbf{\Sigma}^{-1}\mathbf{D} \\ &\quad \times (\mathbf{W}^{-1} + \mathbf{D}'\mathbf{\Sigma}^{-1}\mathbf{D})^{-1}, \\ [22] &= [\mathbf{F}'\mathbf{\Sigma}^{-1}\mathbf{F} - \mathbf{F}'\mathbf{\Sigma}^{-1}\mathbf{D}(\mathbf{W}^{-1} + \mathbf{D}'\mathbf{\Sigma}^{-1}\mathbf{D})^{-1}\mathbf{D}'\mathbf{\Sigma}^{-1}\mathbf{F}]^{-1} \\ &= [\mathbf{F}'(\mathbf{\Sigma} + \mathbf{DWD}')^{-1}\mathbf{F}]^{-1}.\end{aligned}$$

Thus

$$[21] = -[\mathbf{F}'(\mathbf{\Sigma} + \mathbf{DWD}')^{-1}\mathbf{F}]^{-1}\mathbf{F}'\mathbf{\Sigma}^{-1}\mathbf{D}(\mathbf{W}^{-1} + \mathbf{D}'\mathbf{\Sigma}^{-1}\mathbf{D})^{-1}$$

and

$$\begin{aligned}\tilde{\mathbf{b}} &= [\mathbf{F}'(\mathbf{\Sigma} + \mathbf{DWD}')^{-1}\mathbf{F}]^{-1} \{ \mathbf{F}'\mathbf{\Sigma}^{-1}\mathbf{Y} - \mathbf{F}'\mathbf{\Sigma}^{-1}\mathbf{D}(\mathbf{W}^{-1} + \mathbf{D}'\mathbf{\Sigma}^{-1}\mathbf{D})^{-1} \\ &\quad \times (\mathbf{W}^{-1}\hat{\Theta} + \mathbf{D}'\mathbf{\Sigma}^{-1}\mathbf{Y}) \} \\ &= [\mathbf{F}'(\mathbf{\Sigma} + \mathbf{DWD}')^{-1}\mathbf{F}]^{-1} \{ \mathbf{F}'\mathbf{\Sigma}^{-1}\mathbf{Y} - \mathbf{F}'(\mathbf{\Sigma} + \mathbf{DWD}')^{-1}\mathbf{D}\mathbf{W} \\ &\quad \times (\mathbf{W}^{-1}\hat{\Theta} + \mathbf{D}'\mathbf{\Sigma}^{-1}\mathbf{Y}) \} \\ &= [\mathbf{F}'(\mathbf{\Sigma} + \mathbf{DWD}')^{-1}\mathbf{F}]^{-1} \{ \mathbf{F}'\mathbf{\Sigma}^{-1}\mathbf{Y} - \mathbf{F}'(\mathbf{\Sigma} + \mathbf{DWD}')^{-1}\mathbf{D}\hat{\Theta} \\ &\quad - \mathbf{F}'(\mathbf{\Sigma} + \mathbf{DWD}')^{-1}(\mathbf{\Sigma} + \mathbf{DWD}' - \mathbf{\Sigma})\mathbf{\Sigma}^{-1}\mathbf{Y} \} \\ &= [\mathbf{F}'(\mathbf{\Sigma} + \mathbf{DWD}')^{-1}\mathbf{F}]^{-1}\mathbf{F}'(\mathbf{\Sigma} + \mathbf{DWD}')^{-1}(\mathbf{Y} - \mathbf{D}\hat{\Theta}).\end{aligned}$$

QED

The expression for $\text{Var}(\tilde{\mathbf{b}})$ is obvious. (For more detail, see [4].)

This preliminary estimator $\tilde{\mathbf{b}}$ cannot be considered satisfactory in the state system, because of the structure of its covariance matrix $[\mathbf{F}'(\mathbf{\Sigma} + \mathbf{DWD}')^{-1}\mathbf{F}]^{-1}$. It produces non-admissibly large dispersions in an estimation of distances among the characteristic points of the new structure. The problem is also how to determine the coordinates of the non-characteristic points of the new structure in the state system. The solution is as follows:

We shall determine the shift and the rotation of the inner coordinate system in such a way that the new location of the points $P_1, \dots, P_k$ approximates the vector $\tilde{\mathbf{b}}$, i.e.,

$$\hat{\mathbf{b}}_i = \begin{pmatrix} \hat{t}_1 \\ \hat{t}_2 \end{pmatrix} + \begin{pmatrix} \widehat{\cos \varphi} & \widehat{\sin \varphi} \\ -\widehat{\sin \varphi} & \widehat{\cos \varphi} \end{pmatrix} \mathbf{a}_i, \quad i = 1, \dots, k,$$

whereas the shift $\begin{pmatrix} \hat{t}_1 \\ \hat{t}_2 \end{pmatrix}$ and the matrix of the rotation $\begin{pmatrix} \widehat{\cos \varphi} & \widehat{\sin \varphi} \\ -\widehat{\sin \varphi} & \widehat{\cos \varphi} \end{pmatrix}$ will be optimum in the sense of the least-squares theory. This transformation is linear and conforms, and it does not change the distances among the points P_i and P_j .

Let

$$\mathbf{T}_i = \begin{pmatrix} 1 & 0 & a_{i,1} & a_{i,2} \\ 0 & 1 & -a_{i,2} & a_{i,1} \end{pmatrix}, \quad i = 1, \dots, k,$$

$$c = \cos \varphi = c_0 + \delta c, \quad s = \sin \varphi = s_0 + \delta s,$$

$$t_1 = t_{1,0} + \delta t_1, \quad t_2 = t_{2,0} + \delta t_2,$$

where $t_{1,0}, t_{2,0}, c_0, s_0$ are the approximate values. After the linearization we obtain the following linear model with the type I constraints (for more detail, see [1]).

$$\begin{aligned}\boldsymbol{\eta} &= \begin{pmatrix} \tilde{\mathbf{b}}_1 \\ \vdots \\ \tilde{\mathbf{b}}_k \end{pmatrix} - \begin{pmatrix} \mathbf{T}_1 \\ \vdots \\ \mathbf{T}_k \end{pmatrix} \begin{pmatrix} t_{1,0} \\ t_{2,0} \\ c_0 \\ s_0 \end{pmatrix} \\ &\sim {}_{2k} \left[\begin{pmatrix} \mathbf{T}_1 \\ \vdots \\ \mathbf{T}_k \end{pmatrix} \begin{pmatrix} \delta t_1 \\ \delta t_2 \\ \delta c \\ \delta s \end{pmatrix}, [\mathbf{F}'(\mathbf{\Sigma} + \mathbf{DWD}')^{-1}\mathbf{F}]^{-1} \right],\end{aligned}$$

with the constraints

$$c_0 \delta c + s_0 \delta s = 0$$

(the term $(\delta c)^2 + (\delta s)^2$ is ignored).

Let

$$\mathbf{S} = [\mathbf{F}'(\mathbf{\Sigma} + \mathbf{DWD}')^{-1}\mathbf{F}]^{-1}, \quad \mathbf{X} = \begin{pmatrix} \mathbf{T}_1 \\ \vdots \\ \mathbf{T}_k \end{pmatrix}.$$

Lemma 2.2 The BLUEs of the quantities $t_1, t_2, \cos \varphi, \sin \varphi$ are $\hat{t}_1 = t_{1,0} + \delta t_1, \hat{t}_2 = t_{2,0} + \delta t_2, \widehat{\cos \varphi} = c_0 + \delta c, \widehat{\sin \varphi} = s_0 + \delta s$, where

$$\begin{pmatrix} \widehat{\delta t_1} \\ \widehat{\delta t_2} \\ \widehat{\delta c} \\ \widehat{\delta s} \end{pmatrix} = \mathbf{C}^{-1} \mathbf{X}' \mathbf{S}^{-1} \boldsymbol{\eta} - \mathbf{C}^{-1} \mathbf{B}' (\mathbf{B} \mathbf{C}^{-1} \mathbf{B}')^{-1} \mathbf{B} \mathbf{C}^{-1} \mathbf{X}' \mathbf{S}^{-1} \boldsymbol{\eta},$$

$$\mathbf{C} = \mathbf{X}' \mathbf{S}^{-1} \mathbf{X}, \quad \mathbf{B} = (0, 0, c_0, s_0),$$

$$\text{Var} \begin{pmatrix} \widehat{\delta t_1} \\ \widehat{\delta t_2} \\ \widehat{\delta c} \\ \widehat{\delta s} \end{pmatrix} = \mathbf{C}^{-1} - \mathbf{C}^{-1} \mathbf{B}' (\mathbf{B} \mathbf{C}^{-1} \mathbf{B}')^{-1} \mathbf{B} \mathbf{C}^{-1} = \mathbf{V}.$$

For the proof, see [4].

QED

The resulting location of the point P_i in the state system is given by the vector

$$\hat{\mathbf{b}}_i = \begin{pmatrix} \hat{t}_1 & 0 & \widehat{\cos \varphi} & \widehat{\sin \varphi} \\ 0 & \hat{t}_2 & -\widehat{\sin \varphi} & \widehat{\cos \varphi} \end{pmatrix} \mathbf{a}_i = \mathbf{T}_i \begin{pmatrix} \hat{t}_1 \\ \hat{t}_2 \\ \widehat{\cos \varphi} \\ \widehat{\sin \varphi} \end{pmatrix}, \quad i = 1, \dots, k.$$

The expression for $\hat{\mathbf{b}}_i$ can be rewritten as follows

$$\begin{aligned}\hat{\mathbf{b}}_i &= \mathbf{T}_i \{ \mathbf{I} - \mathbf{C}^{-1} \mathbf{B}' [\mathbf{B} \mathbf{C}^{-1} \mathbf{B}']^{-1} \mathbf{B} \} \mathbf{C}^{-1} \mathbf{X}' \mathbf{S}^{-1} (\tilde{\mathbf{b}} - \mathbf{b}_0) \\ &= \mathbf{T}_i \{ \mathbf{I} - \mathbf{C}^{-1} \mathbf{B}' [\mathbf{B} \mathbf{C}^{-1} \mathbf{B}']^{-1} \mathbf{B} \} \mathbf{C}^{-1} \mathbf{X}' \mathbf{F}' (\mathbf{\Sigma} + \mathbf{DWD}')^{-1} (\mathbf{Y} - \mathbf{D}\hat{\Theta}),\end{aligned}$$

where

$$\mathbf{b}_0 = \begin{pmatrix} \mathbf{T}_1 \\ \vdots \\ \mathbf{T}_k \end{pmatrix} \begin{pmatrix} t_{1,0} \\ t_{2,0} \\ c_0 \\ s_0 \end{pmatrix}.$$

The estimator $\hat{\mathbf{b}}$ is the final estimator of the coordinates of the characteristic points of the structure in the state coordinate system. The transformation can be used for the non-characteristic points of the structure as well.

Let

$$\mathbf{Z}_i = \mathbf{T}_i \{ \mathbf{I} - \mathbf{C}^{-1} \mathbf{B}' [\mathbf{B} \mathbf{C}^{-1} \mathbf{B}']^{-1} \mathbf{B} \} \mathbf{C}^{-1} \mathbf{X}' \mathbf{F}' (\mathbf{\Sigma} + \mathbf{DWD}')^{-1}.$$

Lemma 2.3 The covariance matrix of the vector $(\widehat{\Theta}', \widehat{\mathbf{b}}_i')$ is

$$\text{cov}(\widehat{\Theta}, \widehat{\mathbf{b}}_i) = -\mathbf{W}\mathbf{D}'\mathbf{Z}_i'$$

and the covariance matrix of the vector $(\widehat{\mathbf{b}}_i', \widehat{\mathbf{b}}_j')$

$$\text{cov}(\widehat{\mathbf{b}}_i, \widehat{\mathbf{b}}_j) = \mathbf{T}_i\mathbf{V}\mathbf{T}_j'.$$

Thus

$$\text{Var} \begin{pmatrix} \widehat{\Theta} \\ \widehat{\mathbf{b}}_1 \\ \vdots \\ \widehat{\mathbf{b}}_k \end{pmatrix} = \begin{pmatrix} \mathbf{W}, & -\mathbf{W}\mathbf{D}'\mathbf{Z}_1', & \dots, & -\mathbf{W}\mathbf{D}'\mathbf{Z}_k' \\ -\mathbf{Z}_1\mathbf{D}\mathbf{W}, & \mathbf{T}_1\mathbf{V}\mathbf{T}_1', & \dots, & \mathbf{T}_1\mathbf{V}\mathbf{T}_k' \\ \dots & \dots & \dots & \dots \\ -\mathbf{Z}_k\mathbf{D}\mathbf{W}, & \mathbf{T}_k\mathbf{V}\mathbf{T}_1', & \dots, & \mathbf{T}_k\mathbf{V}\mathbf{T}_k' \end{pmatrix}.$$

Proof. Since $\widehat{\mathbf{b}}_i = \mathbf{Z}_i(\mathbf{Y} - \mathbf{D}\widehat{\Theta})$, it is valid that

$$\text{cov}(\widehat{\Theta}, \widehat{\mathbf{b}}_i) = -\mathbf{W}\mathbf{D}'\mathbf{Z}_i', \quad i = 1, \dots, k,$$

and

$$\text{cov}(\widehat{\mathbf{b}}_i, \widehat{\mathbf{b}}_j) = \mathbf{Z}_i\text{Var}(\mathbf{Y} - \mathbf{D}\widehat{\Theta})\mathbf{Z}_j'$$

$$= \mathbf{T}_i\mathbf{V}\mathbf{X}'\mathbf{F}'(\Sigma + \mathbf{D}\mathbf{W}\mathbf{D}')^{-1}(\Sigma + \mathbf{D}\mathbf{W}\mathbf{D}') \times (\Sigma + \mathbf{D}\mathbf{W}\mathbf{D}')^{-1}\mathbf{F}\mathbf{X}\mathbf{V}\mathbf{T}_j' \\ = \mathbf{T}\mathbf{V}\mathbf{X}'\mathbf{S}^{-1}\mathbf{X}\mathbf{V}\mathbf{T}_j' = \mathbf{T}_i\mathbf{V}\mathbf{C}\mathbf{V}\mathbf{T}_j' = \mathbf{T}_i\mathbf{V}\mathbf{T}_j',$$

since

$$\mathbf{V}\mathbf{C}\mathbf{V} = \left\{ \mathbf{C}^{-1} - \mathbf{C}^{-1}\mathbf{B}'[\mathbf{B}\mathbf{C}^{-1}\mathbf{B}']^{-1}\mathbf{B}'\mathbf{C}^{-1} \right\} \mathbf{C} \\ \times \left\{ \mathbf{C}^{-1} - \mathbf{C}^{-1}\mathbf{B}'[\mathbf{B}\mathbf{C}^{-1}\mathbf{B}']^{-1}\mathbf{B}'\mathbf{C}^{-1} \right\} = \mathbf{V}.$$

QED

Theorem 2.4 The distances among the characteristic points $P_1, \dots, P_k$ with the coordinate vectors $\mathbf{a}_1, \dots, \mathbf{a}_k$ and the points $C_1, \dots, C_l$, with the internal coordinate vectors $\mathbf{c}_1, \dots, \mathbf{c}_l$ are not changed by the transformation with the parameters

$$\widehat{t}_1, \widehat{t}_2, \widehat{\cos\varphi}, \widehat{\sin\varphi},$$

i.e.,

$$\text{Var}(\|\widehat{\mathbf{b}}_i - \widehat{\mathbf{d}}_j\|^2) = 0,$$

where $\widehat{\mathbf{d}}_j$ is the transformed location of point C_j .

Proof. The location of point P_i with the coordinate vector $\mathbf{a}_i$ and point C_j with coordinate vector $\mathbf{c}_j$ after transformation is given by the vectors

$$\widehat{\mathbf{b}}_i = (\mathbf{I}, \mathbf{A}_i) \begin{pmatrix} \widehat{t}_1 \\ \widehat{t}_2 \\ \widehat{\cos\varphi} \\ \widehat{\sin\varphi} \end{pmatrix}, \quad \widehat{\mathbf{d}}_j = (\mathbf{I}, \mathbf{C}_j) \begin{pmatrix} \widehat{t}_1 \\ \widehat{t}_2 \\ \widehat{\cos\varphi} \\ \widehat{\sin\varphi} \end{pmatrix},$$

where

$$\mathbf{A}_i = \begin{pmatrix} a_{i,1} & a_{i,2} \\ -a_{i,2} & a_{i,1} \end{pmatrix}, \quad \mathbf{C}_j = \begin{pmatrix} c_{j,1} & c_{j,2} \\ -c_{j,2} & c_{j,1} \end{pmatrix}.$$

It means that

$$\|\widehat{\mathbf{b}}_i - \widehat{\mathbf{d}}_j\|^2 = \left\| [(\mathbf{I}, \mathbf{A}_i) - (\mathbf{I}, \mathbf{C}_j)] \begin{pmatrix} \widehat{t}_1 \\ \widehat{t}_2 \\ \widehat{\cos\varphi} \\ \widehat{\sin\varphi} \end{pmatrix} \right\|^2 \\ = (\widehat{\cos\varphi}, \widehat{\sin\varphi})(\mathbf{A}_i - \mathbf{C}_j)'(\mathbf{A}_i - \mathbf{C}_j) \begin{pmatrix} \widehat{\cos\varphi} \\ \widehat{\sin\varphi} \end{pmatrix} \\ = (a_{i,1} - c_{j,1})^2 + (a_{i,2} - c_{j,2})^2 = \|\mathbf{a}_i - \mathbf{c}_j\|^2. \quad \text{QED}$$

Remark 2.5 Since the distances among the points $P_1, \dots, P_k$, $C_1, \dots, C_l$ are not changed after the transformation, the angles do not change as well, i.e., they are error-free. However, it is necessary to remark that the covariance matrix of the estimators of the coordinates of the characteristic and non-characteristic points of the new structure

is not a zero matrix, i.e., the dispersions of the coordinate estimators are not zero as well. The dispersions are of the same size as the dispersions of the coordinates of points $B_1, \dots, B_N$.

Remark 2.6 There is also another approach to the determination of the state coordinates of the characteristic points of the structure. However, it is optimum in another sense. The question is about the H - optimum estimator (for more detail, see [6]).

4 EXAMPLE

The following example shows that the idea given in the preceding sections can be used in the case of levelling networks as well.

Let a levelling network consist of the points B_1, B_2 with estimated heights equal to $\widehat{\Theta}_1$ and $\widehat{\Theta}_2$, respectively,

$$\begin{pmatrix} \widehat{\Theta}_1 \\ \widehat{\Theta}_2 \end{pmatrix} \sim_2 \left[\begin{pmatrix} \Theta_1 \\ \Theta_2 \end{pmatrix}, \begin{pmatrix} w_{1,1} & w_{1,2} \\ w_{2,1} & w_{2,2} \end{pmatrix} \right].$$

The points P_1, P_2 are of unknown heights b_1 and b_2 , respectively. Let a measurement of the unknown heights be given by the following model:

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \sim_2 \left[\begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} \Theta_1 \\ \Theta_2 \\ b_1 \\ b_2 \end{pmatrix}, \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} \right],$$

i.e.

$$\begin{pmatrix} \widehat{\Theta} \\ \widehat{\mathbf{Y}} \end{pmatrix} \sim_4 \left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}, \begin{pmatrix} w_{1,1} & w_{1,2} & 0 & 0 \\ w_{2,1} & w_{2,2} & 0 & 0 \\ 0 & 0 & \tau_1 & 0 \\ 0 & 0 & 0 & \tau_2 \end{pmatrix} \right].$$

Thus (Lemma 2.1)

$$\begin{pmatrix} \tilde{b}_1 \\ \tilde{b}_2 \end{pmatrix} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \left[\begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} w_{1,1} & w_{1,2} \\ w_{2,1} & w_{2,2} \end{pmatrix} \right]^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}^{-1} \\ \times \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \left[\begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} w_{1,1} & w_{1,2} \\ w_{2,1} & w_{2,2} \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right]^{-1} \\ \times \left[\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} - \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \widehat{\Theta}_1 \\ \widehat{\Theta}_2 \end{pmatrix} \right] = \begin{pmatrix} \widehat{\Theta}_1 + Y_1 \\ \widehat{\Theta}_2 - Y_2 \end{pmatrix}$$

and

$$\text{Var} \begin{pmatrix} \tilde{b}_1 \\ \tilde{b}_2 \end{pmatrix} = \begin{pmatrix} \sigma_{1,1} & \sigma_{1,2} \\ \sigma_{2,1} & \sigma_{2,2} \end{pmatrix} = \begin{pmatrix} w_{1,1} + \tau_1 & w_{1,2} \\ w_{2,1} & w_{2,2} + \tau_2 \end{pmatrix} = \Sigma.$$

If $b_2 - b_1 = a$ is known, then an analogy of the Helmert transformation in 1D is

$$\begin{pmatrix} \tilde{b}_1 \\ \tilde{b}_2 \end{pmatrix} \sim_2 \left[\begin{pmatrix} 0 \\ a \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \beta, \Sigma \right],$$

i.e. $b_1 = \beta$ and $b_2 = \beta + a$.

The BLUE of the transformation parameter β in this model is

$$\hat{\beta} = \left[(1, 1)\Sigma^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right]^{-1} (1, 1)\Sigma^{-1} \begin{pmatrix} \tilde{b}_1 \\ \tilde{b}_2 - a \end{pmatrix} \\ = \frac{1}{\sigma_{1,1} + \sigma_{2,2} - 2\sigma_{1,2}} [(\sigma_{2,2} - \sigma_{1,2})\tilde{b}_1 + (\sigma_{1,1} - \sigma_{1,2})(\tilde{b}_2 - a)].$$

Thus

$$\hat{b}_1 = \hat{\beta},$$

$$\hat{b}_2 = \hat{\beta} + a = \frac{1}{\sigma_{1,1} + \sigma_{2,2} - 2\sigma_{1,2}} [(\tilde{b}_1 + a)(\sigma_{2,2} - \sigma_{1,2}) + \tilde{b}_2(\sigma_{1,1} - \sigma_{1,2})].$$

Since

$$\text{cov}(\hat{\Theta}_1, \hat{b}_1) = \frac{1}{w_{1,1} + w_{2,2} + \tau_1 + \tau_2 - 2w_{1,2}} [(w_{2,2} + \tau_2 - w_{1,2})w_{1,1} + (w_{1,1} + \tau_1 - w_{1,2})w_{1,2}] = \text{cov}(\hat{\Theta}_1, \hat{b}_2) = c_{1,2}$$

$$\text{cov}(\hat{\Theta}_2, \hat{b}_1) = \frac{1}{w_{1,1} + w_{2,2} + \tau_1 + \tau_2 - 2w_{1,2}} [(w_{2,2} + \tau_2 - w_{1,2})w_{1,2} + (w_{1,1} + \tau_1 - w_{1,2})w_{2,2}] = \text{cov}(\hat{\Theta}_2, \hat{b}_2) = c_{2,2},$$

the covariance matrix $\text{Var} \begin{pmatrix} \hat{\Theta}_1 \\ \hat{\Theta}_2 \\ \hat{b}_1 \\ \hat{b}_2 \end{pmatrix}$ can be written in the form

$$\text{Var} \begin{pmatrix} \hat{\Theta}_1 \\ \hat{\Theta}_2 \\ \hat{b}_1 \\ \hat{b}_2 \end{pmatrix} = \begin{pmatrix} w_{1,1} & w_{1,2} & c_{1,1} & c_{1,1} \\ w_{2,1} & w_{2,2} & c_{2,1} & c_{2,1} \\ c_{1,1} & c_{2,1} & \boxed{1} & \boxed{1} \\ c_{1,1} & c_{2,1} & \boxed{1} & \boxed{1} \end{pmatrix},$$

where

$$\boxed{1} = \frac{(w_{1,1} + \tau_1)(w_{2,2} + \tau_2) - w_{1,2}^2}{w_{1,1} + w_{2,2} + \tau_1 + \tau_2 - 2w_{1,2}}.$$

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