Mutual compositions in even-dimensional affinely connected space without a torsion

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Abstract

Consider an affinely connected space without a torsion A^{2n} . [6] introduces pairs of mutual compositions $X^n \times X^n$ and $Y^n \times Y^n$, defined by the affinors a_a and b_a . This paper considers two new compositions given by the affinors $a = \sqrt{2}(a_a + b_a)$ and $c_a = \sqrt{2}(a_a - b_a)$. We show that these compositions define integrable structures in A^{2n} .

Keywords: affinely connected spaces, special compositions, affinors of compositions.

Subject Classification Index: 53B05

Abstrakt

Ky punim shkencor ka të bëj me prodhimet e ndërsjellta (të kundërta) me afinorët e lidhur në hapsirën pa tenzorë në A_{2n} [6], ku paraqet prodhimet reciproke $X_n \times \overline{X_n}$ dhe $Y_n \times \overline{Y_n}$ të definuar nga afinorët a_α^β dhe b_α^β bëhet fjalë për prodhimet : Karteziane-Kartiziane (K-K), Çobishevit-Çobishevit (Ç-Ç) dhe Gjeodezi-Gjeodezike (Gj-GJ).

$$\text{Afinorët } a_\alpha^\beta \text{ dhe } b_\alpha^\beta: a_\alpha^\beta = V_1^\beta V_\alpha^1 + V_2^\beta V_\alpha^2 - V_3^\beta V_\alpha^3 - V_4^\beta V_\alpha^4$$

$$b_\alpha^\beta = V_1^\beta V_\alpha^3 + V_3^\beta V_\alpha^1 - V_2^\beta V_\alpha^4 - V_2^\beta V_\alpha^4 .$$

Këtu fitohen dy prodhime të reja nga afinorët $d_\alpha^\beta = \frac{1}{\sqrt{2}}(a_\alpha^\beta + b_\alpha^\beta)$ dhe

$c_\alpha^\beta = \frac{1}{\sqrt{2}}(a_\alpha^\beta - b_\alpha^\beta)$. Ne tregojmë se këto prodhime c_α^β dhe d_α^β definojnë strukturat e integruara A_n

Апстракт

Овој научен труд се однесува на билатералните производи (спротивни) со афини поврзани во просторот без сензор во A_{2n} [6], каде што се појавуваат реципрочните продукти $X_n \times \overline{X_n}$ dhe $Y_n \times \overline{Y_n}$ дефинирани од афините a_α^β и b_α^β и станува збор за производите: Картизиане-Картизиане (К-К), Чобишеви-Чобишеви (С-С) и Геодедетски-Геодедетски (Г-Г).

$$\text{Афините } a_\alpha^\beta \text{ и } b_\alpha^\beta: a_\alpha^\beta = V_1^\beta V_\alpha^1 + V_2^\beta V_\alpha^2 - V_3^\beta V_\alpha^3 - V_4^\beta V_\alpha^4$$

$$b_\alpha^\beta = V_1^\beta V_\alpha^3 + V_3^\beta V_\alpha^1 - V_2^\beta V_\alpha^4 - V_2^\beta V_\alpha^4 .$$

Тука се добиваат две нови производи од афините $d_\alpha^\beta = \frac{1}{\sqrt{2}}(a_\alpha^\beta + b_\alpha^\beta)$ и $c_\alpha^\beta = \frac{1}{\sqrt{2}}(a_\alpha^\beta - b_\alpha^\beta)$.

Ние покажуваме дека овие производи c_α^β и d_α^β ги дефинираат интегрираните структури A_n .

1 Preliminary

Consider affinely connected space without a torsion A_N and denote with r its coefficients of connectedness. In A_N , let $X \times X_m (n + m = N)$ be a composition of two differential manifolds X and X_m . Along any point in the space of compositions $A_N (X \times X_m)$ there are two positions of the base manifolds, denoted by $P(X)$ and $P(X_m)$ [2]. Observe that defining a composition in A_N is equivalent to defining a field of the affinor a for which [1,2]:

$$a \cdot a = 6 \quad 1$$

Here a is called the affinor of the composition [2]. According to [2] and [4], the condition for integrability of the structure is:

$$a \cdot a^v - a^v \cdot a = 0 \quad 2$$

The projective affinors a and a^m , defined by $a = \frac{1}{6}(6 + a)$ and $a^m = \frac{1}{6}(6 - a)$, satisfy the conditions $a + a^m = 6$ and $a - a^m = a$ [2,3]. For any vector $v \in A_N (X \times X_m)$ we have $v = V + V^m$, where $V = a \cdot v \in P(X)$, $V^m = a^m \cdot v \in P(X_m)$.

The properties of the following special following special compositions are studied in [2]:

The composition $(C - C)$ for which the positions $P(X)$ and $P(X_m)$ are translated parallelly along every line in A_N , is characterized by

$$a = 0 \quad 3$$

The composition $(Ch - Ch)$ for which the positions $P(X)$ and $P(X_m)$ are translated parallelly along the lines of X_m and X , is characterized by

$$a = 6 \quad 4$$

The composition $(G - G)$ for which the positions $P(X)$ and $P(X_m)$ are translated parallelly along the lines X and X_m , is characterized by

$$a \cdot a^v + a^v \cdot a = 0 \quad 5$$

Consider an even-dimensional space without a torsion A . Let v ($a = 1 \dots 2n$) are independent vectors. Define the covectors v with

$$v \cdot v = 6 \quad \text{and} \quad v \cdot v^v = 6^v \quad 6$$

Suppose that the indices take the following values:

$$\begin{aligned} a_{ij} &= \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \\ , &= \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \\ , &= \begin{pmatrix} n+1 & n+2 \\ n+1 & n+2 \end{pmatrix} \end{aligned} \quad 7$$

Following [5] and [6] consider the affinor

$$a = v_i^i v_j^j - v_i^j v_j^i \quad 8$$

From 6 it follows that $a \cdot a = 6$ i.e. the affinor (6) defines the composition $X \times X$ in A . Denote with $P(X)$ and $P(X)$ the positions of this composition.

Consider the vectors [6]

$$\begin{aligned} w_1 &= v_1 + v_{+1} & w_+ &= v_+ + v_1 & w_- &= v_- + v_{-1} \\ w_{+1} &= v_+ - v_1 & w_- &= v_- - v_{-1} & w &= v - v_+ \end{aligned} \quad 9$$

Define the covectors w from the following equalities

$$w \cdot w = 6 \quad \{ \because w \cdot w = 6^v \} \quad 10$$

Consider the affinor [6]

$$b = w_i^i w_j^j - w_i^j w_j^i \quad 11$$

From 10 and 11 we obtain $b \cdot b = 6$. It follows that the affinor b defines a composition in A , which we denote with $Y \times Y$. Let $P(Y)$ and $P(Y)$ be the positions of this composition.

According to [6] the compositions $X \times X$ and $Y \times Y$ are mutual.

2 Compositions defined by the affinors of two mutual compositions

Consider the mutual compositions $X \times X$ and $Y \times Y$. Consider the affinors

$$= \frac{1}{2}(a+b)C = \frac{1}{2}(a-b) \quad 12$$

From $a \cdot b = -b \cdot a$ [6] and 12 we have $\dots = 6$ and $C \cdot C = 6$. It therefore follows that the affnors 12 define compositions in A

Proposition 1 tto compositions deened bb the affnors $a \cdot b \cdot C$ are either $(C - C)$ or $(C h - C h)$ compositionss then the other tto compositions are either $(C - C)$ or $(C h - C h)$ compositions

Proof. From 12 it follows that $\dots = \frac{1}{2}(a+b)C = \frac{1}{2}(a-b)$
 $\dots = \frac{1}{2}(a+b)C = \frac{1}{2}(a-b)$ from where taking into account 3 and 4 the statement of the theorem follows.

Theorem 1 The compositions given bb the affnors and C s deene integrable structures in the space A

Proof. Set the net $(v_1 v \dots v)$, defined by the independent vectors v , as a coordinate one. Then we have:

$$(13) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} v \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} v \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \quad 13$$

From 13 the matrices of the affnors a and b in the chosen coordinate net are

$$(a) = \begin{pmatrix} 6_i & & \\ & \ddots & \\ & & -6_i \end{pmatrix} \quad (b) = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \quad 14$$

From 2 the condition for integrability of the stucture defined by is

$$v \dots v = \quad 15$$

Taking into account 12 the above is equivalent to

$$(a + b)(a^v + b^v) - (a + b)(a^v + b^v) - \quad 16$$

$$+(a + b)(a^v + b^v) + (a + b)(a^v + b^v) =$$

In the coordinate net the left hand side has of 16 takes the form

$$G^v = (a + b)(r_p^v(a^p + b^p) - r_p^v(a^p + b^p) +$$

$$+(a + b)(-r_p^v(a^p + b^p) + r_p^v(a^p + b^p)) \quad 17$$

Finally from 17, taking into account that $r = r$ and $a^v = 6^v$, $b^v =$

6^v , $a^v = -b^v$, it follows that $G^v =$ i.e. the composition given by the affnor defines integrable structure in A . That the structure given by C is integrable follows similarly.

Theorem 2 three of the compositions given by the affnors a and b are

$(G - G)$ compositions then the fourth is $(G - G)$ composition as well

Proof. From 12 it follows that

$$G^v + G^v = \frac{1}{2}(a + a^v + a + a^v) + \quad 18$$

$$+ \frac{1}{2}(b + b^v + b + b^v) + \frac{1}{2}G^v$$

$$C + C + C + C^v = \frac{1}{2}(a + a^v + a + a^v) + \quad 19$$

$$+ \frac{1}{2}(b + b^v + b + b^v) - \frac{1}{2}G^v$$

where $G^v = a + b^v + a + b^v + b + a^v + b + a^v$ From 18 we have

$$G^v + G^v + C + C^v = a + a^v + a + a^v + b + b^v + b + b^v \quad 20$$

from where, taking into account 5, the statement follows.

Proposition 2 For vectors in $P(X)$, $P(X)$, $P(Y)$ and $P(Y)$ the following hold respectively

1. For any $v \in P(X)$ we have that $v \in P(Y)$ and $C v \in P(Y)$;
2. For any $v \in P(X)$ we have that $v \in P(Y)$ and $C v \in P(Y)$;
3. For any $v \in P(Y)$ we have that $v \in P(X)$ and $C v \in P(X)$;
4. For any $v \in P(Y)$ we have that $v \in P(X)$ and $C v \in P(X)$

Proof. From 8, 9, 10 and 11 it follows that

$$b = \frac{1}{2} (v_1^2 + v_{+1}^2 + v_1^2 + v_{+1}^2 + v_1^2 + v_{+1}^2 + v_1^2 + v_{+1}^2 + v_1^2 + v_{+1}^2) \quad (21)$$

For $v \in P(X)$ we have $a = \frac{1}{2} (v_1^2 + v_{+1}^2)$. From (6), (8), (9), (10) and (11) we find that

$$v = \frac{1}{2} (v_1^2 + v_{+1}^2) \quad C v = \frac{1}{2} (v_1^2 - v_{+1}^2) \quad +i$$

so that $v \in P(Y)$ and $C v \in P(Y)$, which establishes part 1. The remaining parts follow similarly.

Let a, a^m , and b, b^m are the projective affinors of the compositions $X \times X$ and $Y \times Y$ respectively. From (8) and (21) $a = \frac{1}{2} (v_1^2 + v_{+1}^2)$, $a^m = \frac{1}{2} (v_1^2 - v_{+1}^2)$ we have

$$b = \frac{1}{2} (6 + v_{+1}^2 + v_1^2 + v_{+1}^2 + v_1^2) \\ b^m = \frac{1}{2} (6 - v_{+1}^2 - v_1^2 - v_{+1}^2 - v_1^2)$$

$m \quad m$

Corollary 1 For the affinors a, b and C the following equalities hold

$$\begin{aligned} b^m a_v &= b a_v^m \\ a^m b_v &= a b_v^m \\ b C a_v &= b C a_v^m \\ a C b_v &= a C b_v^m \end{aligned}$$

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Proof. Consider an arbitrary vector v^v . Then $a_v v^v \in P(X)$. From Proposition we have $a_v v^v \in P(Y)$ so that $b^m a_v v^v = 0$. Since v^v is an arbitrary vector, it follows that the first line of 22 holds. The rest follow similarly.

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