RESEARCH PAPERS
FACULTY OF MATERIALS SCIENCE AND TECHNOLOGY IN TRNAVA
SLOVAK UNIVERSITY OF TECHNOLOGY IN BRATISLAVA

10.1515/rput-2016-0018 2016, Volume 24, Number 39

ON STABILITY CRITERION OF 4-TH ORDER QUASILINEAR DIFFERENTIAL EQUATIONS WITH QUASIDERIVATIVES

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Abstract

This paper deals with stability in the Liapunov sense of the 4-th order quasilinear differential equations with quasiderivatives. An asymptotic stability criterion is derived. An illustrative example is added.

Key words

Quasilinear differential equation, quasiderivative, 4-th order, stability in the Liapunov sense, asymptotic stability in the Liapunov sense

INTRODUCTION

In natural sciences as well as in engineering practice many dynamical systems are used. These systems are mostly based on classic derivatives. Because of the importance of the stability of their solutions, a large number of stability criteria have been established. However, some differential equations use so called quasiderivatives (see (E1) as well as (E2)). The first example is taken from thermodynamics. It deals with 2-nd order differential equations

(E1) $$\frac{d^2T}{dr^2}+\frac{1}{r}\frac{dT}{dr}=\frac{1}{r}\frac{d}{dr}\left(r\,\frac{dT}{dr}\right)=0$$

describing a stationary distribution of temperature in the wall of a circle tube (see [3], page 70). The second example is taken from mechanics. It is focused on 4-th order differential equations

(E2) $$EJ\,\frac{d^4v}{dx^4}+\frac{d}{dx}\left[\big(P+\gamma\,F\,(l-x)\big)\frac{dv}{dx}\right]=p(x)$$

describing an equilibrium state of a straight mass bar (see [3], page 327). The medium terms in (E1) resp. in (E2) are the quasiderivatives mentioned above (see their definition above Remark 1). The coefficients arising in the quasiderivatives *need not to be differentiable*. From this it follows that the equations with quasiderivatives cannot be, in general, expressed as differential equations with classic derivatives. It means that the classical stability criteria in this case cannot be used. For this reason, recently, several papers concerning the stability of differential equations with quasiderivatives have arisen – see, for example [4], [5], [6], where equations of the third order were investigated.

This article deals with the asymptotic stability criterion, in the Liapunov sense, of arbitrary solutions of a certain class of ordinary differential equations – so called quasilinear 4-th order differential equations with quasiderivatives

(L) $$L_4y + P_3(t)L_3y + P_2(t)L_2y + P_1(t)L_1y + P_0(t)L_0y = f(t,y),$$

where (a prime (or a dot) over a term means a derivative of this term owing to the independent variable t) $L_0y(t) = y(t),\ L_iy(t) = p_i(t)(L_{i-1}y(t))',\ i = 1,2,3, L_4y(t) = (L_3y(t))'$, $p_i(t),\ i = 1,2,3,\ P_j(t),\ j = 0,1,2,3$ are real-valued continuous functions defined on an half-closed real interval $I_a = [a,\infty),\ a \in E_1,\ f(t,y)$ are real-valued and continuous on an Cartesian product $I_a \times E_1$, where the symbol E_1 denotes the set of all real numbers. The terms $L_ky(t),\ k = 0,1,2,3,4$ are *k-th quasiderivatives* of a function $y(t)$.

Remark 1. The differential equation (L) can be equivalently expressed in the form

(M) $$(p_3(t)(p_2(t)(p_1(t)y')')')' + P_3(t)p_3(t)(p_2(t)(p_1(t)y')')' +$$
$$+ P_2(t)p_2(t)(p_1(t)y')' + P_1(t)(p_1(t)y') + P_0(t)y = f(t,y).$$

Let us consider a 4-dimensional differential system of the first order

(S) $$\dot{y}_i = f_i(t, y_1, y_2, y_3, y_4), \quad i = 1,2,3,4.$$

Assumption 1. Let the system (S) be expressed in a vector form $\dot{\mathbf{y}} = \mathbf{f}(t,\mathbf{y})$. Then there exists a number *b* (real or $-\infty$) and an area $H \subset E_1^4,\ \mathbf{o} \in H,\ \mathbf{o} = (0,0,0,0)$, such that the function **f** is continuous on area $G = (b,\infty) \times H$ and for every point $(\tau,\mathbf{k}) \in G$ the following Cauchy problem

(1) $$\dot{\mathbf{y}} = \mathbf{f}(t,\mathbf{y}), \quad \mathbf{y}(\tau) = \mathbf{k},$$

admits the only solution. We also assume $\mathbf{f}(t, \mathbf{o}) = \mathbf{o}$ for all $t > b$, i.e. the Cauchy problem (1) admits for $\mathbf{k} = \mathbf{o}$ the trivial solution $\mathbf{o}(t) = \mathbf{o}$ for all $t > b$.

Definition 1. Let Assumption 1 hold. We say that the *trivial solution* **o** of the system (S) is *stable in the Liapunov sense*, if for every $\tau > b$ and every $\varepsilon > 0$ there exists $\delta = \delta(\tau,\varepsilon) > 0$ such that for every initial values $\mathbf{k} \in H,\ \|\mathbf{k}\| < \delta$ and for all $t \geq \tau$ it holds that the solution $\mathbf{u}(t,\tau,\mathbf{k})$ of the Cauchy problem (1) fulfils the following inequality

(2) $$\|\mathbf{u}(t,\tau,\mathbf{k})\| < \varepsilon.$$

Definition 2. Let Assumption 1 hold. We say that the *trivial solution* **o** of the system (S) *is asymptotically stable in the Liapunov sense*, if

(i) **o** is stable in the Liapunov sense (see Definition 1)

(ii) there exists a real number $\Delta > 0$ such that for all $\mathbf{k} \in H,\ \|\mathbf{k}\| < \Delta$ and for every $\tau > b$ it holds that $\lim_{t\to\infty} \|\mathbf{u}(t,\tau,\mathbf{k})\| = 0.$

Definition 3. The following equation (with a dependent variable $z(t)$)

(N) $$L_4z(t)+P_3(t)L_3z(t)+P_2(t)L_2z(t)+P_1(t)L_1z(t)+ \\ +P_0(t)L_0z(t)=f(t,u(t)+z(t))-f(t,u(t))$$
is called an $u(t)$-*equation competent to the equation* (L).

Definition 4. The following 4-dimensional differential system of the first order

(T) $$\dot{z}_i = z_{i+1}/p_i(t), \quad i=1,2,3, \\ \dot{z}_4 = f(t,u(t)+z_1)-f(t,u(t))-P_0(t)z_1-P_1(t)z_2-P_2(t)z_3-P_3(t)z_4$$
is called a *competent system to the equation* (N).

Remark 2. It can be easily shown that the function $z_1(t)$ is a solution of (N) if and only if a vector $(z_1(t),L_1z_1(t),L_2z_1(t),L_3z_1(t))$ is a solution of (T).

Definition 5. Let Assumption 1 hold for (T). Obviously, the function 0 is a solution of (N) on I_a. Then, according to Remark 2, a vector (0,0,0,0) is a solution of (T). We say 0 *is a stable solution of* (N) *in the Liapunov sense*, if (0,0,0,0) is a stable solution of (T) in the Liapunov sense.

Definition 6. Let Assumption 1 hold for (T). Then 0 is an *asymptotically stable solution of* (N) *in the Liapunov sense*, if (0,0,0,0) is an asymptotically stable solution of (T) in the Liapunov sense.

The main aim of the paper is to establish a criterion, which assures the stability of solutions of the equation (L). If we put $p_k(t)=1$ on I_a, $k=1,2,3$ in (L), we obtain a differential equation with classic derivatives. We note that the functions $p_k(t)$, $k=1,2,3$ are not, in general, assumed to be differentiable. From this it follows that we cannot use on (L) stability criteria derived for nonlinear differential equations with classic derivatives.

AUXILIARY ASSERTIONS

Now we introduce some auxiliary assertions, which will play an important role in our considerations. The first of them is the special case of the Hurwitz criterion when n = 4:

Theorem 1. Let us consider a polynomial

(3) $$\chi(z)=a_4z^4+a_3z^3+a_2z^2+a_1z+a_0,$$
where a_i, i = 0, 1, 2, 3, 4 are real numbers such that $a_0 > 0$, $a_4 \neq 0$. Then all zeros of the polynomial (3) admit negative real parts if and only if it holds that

(4) $$a_1 > 0, \qquad a_4 > 0,$$
(5) $$a_1a_2-a_0a_3 > 0,$$
(6) $$a_1a_2a_3-a_1^2a_4-a_0a_3^2 > 0.$$
Proof. The proof of this assertion can be found, for example, in [2], Section V.6.

The second assertion deals with the asymptotic stability criterion in the Liapunov sense for systems of differential equations of the first order. We note that we shall use in it a matrix norm of the form $\left\|\{a_{ij}\}_{i,j}\right\| = \sum_{i,j}|a_{ij}|$.

Theorem 2. Let us consider a system of differential equations of the first order presented in the following matrix form (a dot over a letter means the derivative owing to t; the symbol **o** is the null vector)

(7) $$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}(t)\mathbf{x} + \mathbf{g}(t,\mathbf{x}), \qquad \mathbf{g}(t,\mathbf{o}) = \mathbf{o},$$

where **A** is a real constant square matrix, **B**(t) is a real square matrix depending on t only, such that

(8) $$\lim_{t\to 0}\mathbf{B}(t) = \mathbf{0},$$

where **0** is a null matrix and **g** a real vector function continuous on an area $(b,\infty)\times H,$ where $b \in E_1,\ \mathbf{o} \in H \subset E_1^n$ satisfying a condition

(9) $$\|\mathbf{g}(t,\mathbf{x})\| / \|\mathbf{x}\| \to 0 \text{ uniformly for all } t \ge b \text{ as } \|\mathbf{x}\| \to 0.$$

If all eigenvalues of **A** have negative real parts, then the trivial solution of (7) is asymptotically stable in the Liapunov sense.

Proof. The proof of this assertion can be found in [1], Head XIII.

RESULTS

Lemma 1. Let $u(t)$ be a fixed solution of (L) on I_a. Let $y(t)$ be an arbitrary function defined on I_a. Let us put $z(t) = y(t) - u(t),\ t \in I_a$. Then an arbitrary $z(t)$ is a solution of (N) on I_a if and only if $y(t)$ is a solution of (L) on I_a.

Proof. An easy computing yields that $L_i,\ i = 0,1,2,3,4$ is a linear operator on a proper space of functions. Specially, from this for all proper functions $g(t)$, $h(t)$ it follows that

$$L_i(g(t)+h(t)) = L_i(g(t)) + L_i(h(t)).$$

The sufficient condition. Let $u(t)$, $y(t)$ be the solutions of (L). Then

$$L_4 y(t) + P_3(t)L_3 y(t) + P_2(t)L_2 y(t) + P_1(t)L_1 y(t) + P_0(t)L_0 y(t) = f(t, y(t)),$$
$$L_4 u(t) + P_3(t)L_3 u(t) + P_2(t)L_2 u(t) + P_1(t)L_1 u(t) + P_0(t)L_0 u(t) = f(t, u(t)).$$

If we subtract the second equality from the first one, then we from the linearity of the operators L_i obtain

$$L_4(y(t)-u(t)) + P_3(t)L_3(y(t)-u(t)) + P_2(t)L_2(y(t)-u(t)) +$$
$$+ P_1(t)L_1(y(t)-u(t)) + P_0(t)L_0(y(t)-u(t)) = f(t, y(t)) - f(t, u(t)).$$

If we substitute $y(t)-u(t)$ by $z(t)$ and $y(t)$ by $z(t)+u(t)$ in the last equation, then we recieve $(z = z(t),\ u = u(t))$

$$L_4 z + P_3(t)L_3 z + P_2(t)L_2 z + P_1(t)L_1 z + P_0(t)L_0 z = f(t, u+z) - f(t,u),$$

i.e. $z(t)$ is a solution of (N). The necessary condition can be proved similarly and so it's proof is omitted.

Lemma 2. The solution $u(t)$ of (L) is stable in the Liapunov sense if and only if 0 is stable solution of (N) in the Liapunov sense.

Proof. From the proof of Lemma 1 it implies that $y(t) - u(t) = z(t) - 0,$ from which, owing to Definition 5, we obtain required assertion.

Lemma 3. The solution $u(t)$ of (L) is asymptotically stable in the Liapunov sense if and only if 0 is asymptotically stable solution of (N) in the Liapunov sense.

Proof. It follows from Lemma 2, Definition 6 as well as the equality $y(t)-u(t)=z(t)-0.$

Now we present the main result of the paper – the asymptotic stability criterion in the Liapunov sense of the quasilinear differential equation (L).

Theorem 3. Let us consider the differential equation (L) such that (a), (b), (c) hold, where

(a) $$\lim_{t\to\infty} p_i(t)=\pi_i>0,\ i=1,2,3, \qquad \lim_{t\to\infty} P_j(t)=\Pi_j,\ j=0,1,2,3,$$

(b) $$\Pi_0>0,\quad \Pi_1>0,\quad \pi_1\Pi_1\Pi_2-\pi_3\Pi_0\Pi_3>0,\quad \pi_1\pi_2\Pi_1\Pi_2\Pi_3-\pi_1\Pi_1^2-\pi_2\pi_3\Pi_0\Pi_3^2>0,$$

(c) $$\frac{|f(t,u(t)+z)-f(t,u(t))|}{|z|}\to 0 \text{ uniformly for all } t\ge b \text{ as } z\to 0,$$

where $u(t)$ is a fixed solution of (L) on I_a.

Then $u(t)$ is asymptotically stable in the Liapunov sense.

Proof. By Lemma 3 it suffices to prove the asymptotic stability in the Liapunov sense of the null solution of $u(t)$-equation (N), which is a competent to (L). The null solution of (N) is, according to Definition 6, asymptotically stable in the Liapunov sense, if and only if the solution (0,0,0,0) of the system (T) is asymptotically stable in the Liapunov sense. The system (T) can be rewritten into the form of a system (U), which is demanded by Theorem 2. The system (U) has a form

(U) $$\dot{\mathbf{z}}=\mathbf{A}\mathbf{z}+\mathbf{B}(t)\mathbf{z}+\mathbf{g}(t,\mathbf{z}), \qquad \mathbf{g}(t,\mathbf{o})=\mathbf{o},$$

where

$$\dot{\mathbf{z}}=\begin{bmatrix}\dot z_1\\ \dot z_2\\ \dot z_3\\ \dot z_4\end{bmatrix},\qquad \mathbf{A}=\begin{bmatrix}0 & 1/\pi_1 & 0 & 0\\ 0 & 0 & 1/\pi_2 & 0\\ 0 & 0 & 0 & 1/\pi_3\\ -\Pi_0 & -\Pi_1 & -\Pi_2 & -\Pi_3\end{bmatrix},\qquad \mathbf{z}=\begin{bmatrix}z_1\\ z_2\\ z_3\\ z_4\end{bmatrix},$$

$$\mathbf{B}(t)=\begin{bmatrix}0 & 1/p_1(t)-1/\pi_1 & 0 & 0\\ 0 & 0 & 1/p_2(t)-1/\pi_2 & 0\\ 0 & 0 & 0 & 1/p_3(t)-1/\pi_3\\ \Pi_0-P_0(t) & \Pi_1-P_1(t) & \Pi_2-P_2(t) & \Pi_3-P_3(t)\end{bmatrix},$$

$$\mathbf{g}(t,\mathbf{z})=\begin{bmatrix}0\\ 0\\ 0\\ f(t,u(t)+z_1(t))-f(t,u(t))\end{bmatrix},\qquad \mathbf{o}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}.$$

If $\|\mathbf{z}\|\ne 0,\ z_1\ne 0,$ then

$$0\le\frac{\|\mathbf{g}(t,\mathbf{z})\|}{\|\mathbf{z}\|}=\frac{|f(t,u(t)+z_1)-f(t,u(t))|}{|z_1|+|z_2|+|z_3|+|z_4|}=$$

$$= \frac{|f(t,u(t)+z_1) - f(t,u(t))|}{|z_1|} \cdot \frac{|z_1|}{|z_1|+|z_2|+|z_3|+|z_4|} \leq$$

$$\leq \frac{|f(t,u(t)+z_1) - f(t,u(t))|}{|z_1|}.$$

If $\|\mathbf{z}\| \neq 0,\ z_1 = 0,$ then $\dfrac{\|\mathbf{g}(t,\mathbf{z})\|}{\|\mathbf{z}\|} = 0$.

From the two last possibilities as well as (c) it implies that (9) hold. We can easily observe the validity of the conditions (7), (8) in Theorem 2. A validity of the conditions (a), (b) assures that (4), (5), (6) hold in Theorem 1. Then Theorem 1 yields that all eigenvalues of **A** have negative real parts. From this it follows that Theorem 2 hold as $n = 4$. Consequently, Theorem 2 yields required stability of the solution (0,0,0,0) of the system (T).

Example 1. Let us consider the equation (L) in the form (M), where

$$p_1(t) = 3 + 1/t, \quad p_2(t) = 1 + 1/t, \quad p_3(t) = 1 - 1/t,$$

$$P_0(t) = 1 + 1/t, \quad P_1(t) = 1 - 1/t, \quad P_2(t) = 3 + 1/t, \quad P_3(t) = 1 + 1/t,$$

$$f(t,y) = 1/t - 2/t^2 + 20/t^3 + 16/t^4 + 54/t^5 + 96/t^6 - 90/t^7 - 90/t^8 + (y - 1/t)^2.$$

It can be taken $a = b = 2$. An easy computing yield that a function $u(t) = 1/t$ is a solution of (L) on I_2 as well as $\pi_1 = 3$, $\pi_2 = 1$, $\pi_3 = 1$, $\Pi_0 = 1$, $\Pi_1 = 1$, $\Pi_2 = 3$, $\Pi_3 = 1$. From this immediately follows the validity of (a) and (b). Moreover, if $z \neq 0,$ then

$$\frac{|f(t,u(t)+z) - f(t,u(t))|}{|z|} = \frac{z^2}{|z|} = |z|.$$

From this it implies that

$$\frac{|f(t,u(t)+z) - f(t,u(t))|}{|z|} = |z| \to 0 \text{ uniformly for all } t \geq 2 \text{ as } z \to 0,$$

because $|z|$ does not explicitly depend on the variable *t*. Thus, the condition (c) hold, too. Then, owing to Theorem 3, the function $u(t) = 1/t$ is an asymptotically stable solution of (L) in the Liapunov sense.

Acknowledgement

This publication is the result of implementation of the project: "UNIVERSITY SCIENTIFIC PARK: CAMPUS MTF STU - CAMBO" (ITMS: 26220220179) supported by the Research & Development Operational Program funded by the EFRR.

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