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Making the Unfair Fair

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1 Introduction

In 1969, Walter Penney submitted the following problem to the Journal of Recreational Mathematics.

> `Although in a sequence of coin flips, any given consecutive set of, say, three flips is equally likely to be one of the eight possible, i.e., HHH, HHT, HTH, HTT, THH, THT, TTH, or TTT, it is rather peculiar that one sequence of three is not necessarily equally likely to appear` *`first`* `as another set of three. This fact can be illustrated by the following game: you and your opponent each ante a penny. Each selects a pattern of three, and the umpire tosses a coin until one of the two patterns appears, awarding the antes to the player who chose that pattern. Your opponent picks HHH; you pick HTH. The odds, you will find, are in your favor. By how much?`

We will leave it to the reader to determine how to arrive at the solution, but the odds are 3 : 2 in favor of HTH. The seemingly paradoxical nature of this and its popularization by Martin Gardner in his *Scientific American* column led to what is now known as Penney's Game.

Definition 1. *Penney's Game is a two-player game played via the flipping of a fair coin. Player I picks a sequence of Heads or Tails of length* 3 *(it can be any agreed-upon length, but in most all literature it is length* 3*) and makes his choice known. Player II then states her own sequence of length* 3*. An umpire then tosses the coin until one of the two sequences appears as a consecutive subsequence of the coin flips. The player whose sequence appears first is the winner.*

We now look at some of the odds associated with Penney's Game. The 3 : 2 odds from the original problem are actually not the best odds. If the opponent (from now on Player I) picks HHH, then Player II chooses THH, and the odds are 7 : 1 in Player II's favor.

Player I's Choice	Player II's Choice	Odds in Favor of Player II
HHH	THH	7 : 1
HHT	THH	3 : 1
HTH	HHT	2 : 1
HTT	HHT	2 : 1
THH	TTH	2 : 1
THT	TTH	2 : 1
TTH	HTT	3 : 1
TTT	HTT	7 : 1

These are the best odds for Player II and that there is a pattern here between Player I's decision and Player II's reaction. If Player I makes choices

Choice 1, Choice 2, Choice 3

then Player II's sequence is

opposite Choice 2, Choice 1, Choice 2

Thus Player I's HTH is countered with HHT by Player II.

Penney's game is one of the members of the collection of *non-transitive games*. Just like Rock-Paper-Scissors (or non-transitive dice), regardless of Player I's choice, Player II always has a choice that puts the odds in his/her favor.

Before we venture into calculating things, let us note that there are three typical methods for these calcuations:

- Direct calculation: We shall use this below. It is straightforward, but does not lend itself well to all types of generalizations.
- Leading numbers: Created by John H. Conway, leading numbers compares the Players' choices and turns these comparisons into binary numbers which themselves

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are put in a formula to give odds. This method has the advantage of being easily generalized for sequences longer than three tosses of a coin. More on this can be found in [1]. We will not make use of it here.

- Martingales: Used in probability theory, martingales were first applied to Penney's Game in [2]. Although not as easy as direct calculation, martingales works very well in this paper when the probability of Tails are fixed, but unknown.

These methods are used to answer two questions typically. The first is finding who wins when two players are involved in Penney's Game. The second question is when one person fixes a sequence. How many flips should one expect, on average, before this chosen sequence appears (Wait Times)?

2 Two of the Three Methods

We now look at how a few of these odds are computed. These first examples are the three types of direct calculations used in determining the eight entries in the chart above.

2.1 Direct Computation

Case 1 - Player I: TTT, Player II: HTT

In this situation, Player I can only win if the first three flips are TTT. If any Head appears, then before Player I can see TTT there is the sequence HTT and Player II is the winner. Thus

$$P(\text{ Player I wins }) = P(\text{ TTT }) = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

This gives us the odds in favor of Player II as $7:1$.

Case 2 - Player I: TTH, Player II: HTT

Player I wins here if the first three flips are TTH, or if the first four tosses are TTTH, of the first five are TTTTH, et cetera. This gives us the geometric series

$$P(\text{ Player I wins }) = \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^5 + \cdots = \frac{1/8}{1-1/2} = \frac{1}{4}$$

Thus odds in favor of Player II are $3:1$.

Case 3 - Player I: THT, Player II: TTH

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We will look at this situation from two perspectives.

Let x be the probability that THT appears *before* TTH. Since both sequences start with T, we can ignore any leading H's and the first appearance of T, as this give no player an advantage. So assuming we already have the first T, the sequence THT appears first if the next two tosses are HT. Otherwise, in order for THT to appear first we must have the next two flips after the T to be HH. For the sequence THH the game has essentially "reset" itself, starting from scratch. Thus

$$x = \frac{1}{4} + \frac{1}{4}x \implies x = P(\text{ Player I wins }) = \frac{1}{3}$$

or the odds in favor of Player II are $2:1$.

This can also be done from Player II's perspective. Again the leading H's are ignored. Once an T appears, Player II must win if the next flip is T. Otherwise, Player II needs a reset of HH. This translates to

$$y = \frac{1}{2} + \frac{1}{4}y$$

or $y = \frac{2}{3}$.

The second question to ask about coin flipping, which can also be solve via direct computation, is about Wait Times. If we fix a particular sequence, say HTH, how many flips should it take before our sequence appears? This is the question of wait times. Such a thing can be computer directly using $E(X)$, the expected number of flips before the sequence X appears. Let us work this out with $X = HTH$ and a fair coin.

The first flip can be H or T and each are equally likely. So $E(X)$ can be rewritten in terms of $E(X|H_1)$ (meaning the expected number of flips for X to appear given that the first flip is Heads) and $E(X|T_1)$. Each of these outcomes occur one-half of the time so we can write

$$E(X) = \frac{1}{2}E(X|H_1) + \frac{1}{2}E(X|T_1)$$

Now if Tails appears first this is no help getting the sequence to appear, so we start from scratch with the next flip; i.e., $E(X|T_1) = 1 + E(X)$. If Heads appears on the first flip we are 1/3 or the way to HTH so we continue, splitting things up according to the second flip.

$$E(X|H_1) = \frac{1}{2}E(X|H_1H_2) + \frac{1}{2}E(X|H_1T_2)$$

We do not need two H's at the front of our sequence, just one, so H_2 sets us back on step. Thus $E(X|H_1H_2) = 1 + E(X|H_1)$. The term $E(X|H_1T_2)$ needs to be broken down again.

$$E(X|H_1T_2) = \frac{1}{2}E(X|H_1T_2H_3) + \frac{1}{2}E(X|H_1T_2T_3)$$

The latter term does not help us obtain our sequence so $E(X|H_1T_2T_3) = 3 + E(X)$ and $E(X|H_1T_2H_3)$ is asking the wait time for HTH to appear if the first three flips are HTH, so the answer is 3.

Working our way backwards

$$E(X|H_1T_2) = \frac{1}{2} \cdot 3 + \frac{1}{2}(3 + E(X)) = 3 + \frac{1}{2}E(X)$$

So

$$E(X|H_1) = \frac{1}{2}(1 + E(X|H_1)) + \frac{1}{2}\left(3 + \frac{1}{2}E(X)\right)$$

therefore

$$E(X|H_1) = 4 + \frac{1}{2}E(X)$$

Finally,

$$E(X) = \frac{1}{2}\left(4 + \frac{1}{2}E(X)\right) + \frac{1}{2}(1 + E(X))$$

which gives us $E(X) = 10$.

Covering all the possibilities, HHH and TTT have a wait time of 14 flips (the longest), HTH and THT are both 10 tosses, and the rest the wait time is 8.

2.2 Martingales

A powerful technique for answering questions related to Penney's game involves the tool of martingales. Informally, a martingale refers to a process that evolves in increments that are random but have mean 0. The key idea of a martingale is that it represents a fair game, so under certain mild conditions, the expected value of a martingale does not change even in the long term.

Many of the probabilities we determined (Appendices 7.2 and 7.1) were calculated using martingales. We shall not go into deep detail, but will use examples to show how these functions of p are calculated. The key here is that the expected value of everything must be zero.

We start with calculated the probability that Player I wins in the instance of HHT versus THH. We already know the answer to this, $P($ Player I wins $) = \frac{1}{4}$. Imagine a betting booth set up to take wagers on HHT and THH and two lines of people wishing to place bets. One line consists of bettors placing \$1 wagers on HHT. The second line has bettors place $-\$1$ on THH. This is so the expected value of the total from the nth bettor is 0 for all n. Each player, when it is his/her turn steps to the window and makes their first bet. A fair coin is flipped. If the bettor loses, this person is out of the game and leaves. If the bettor wins, since it is a fair coin the payoff is $1 - 1$ and that person now has twice their original bet (so \$2 or $-\$2$). They stay for the next flip in their sequence placing all of their money on the next outcome in the sequence *and* the person immediately behind steps up to place their first bet on the first outcome of the sequence. The placing of bets, betting the initial amounts and winnings, continues until one of the two sequence appears. Then the game is over. Anyone in line who has not placed a bet is out of the game and goes home.

So the focus of the game is on the last three bettors as the winning sequence of three occurs. Anyone before the first of the three flips has already lost and is out of the game and anyone after the last of the three flips does not get to place a bet as the game is over.

There are two circumstances to consider: HHT occurs first and THH occurs first.

Case 1 - HHT occurs first

For the players in line to bet on HHT, the third from last has won on the first H, the second H, and the T at the end, for a total of \$8. The person second from last bet the first H in the sequence during the second flip and won. Then she bet \$2 on the second H in the sequence, but during the third flip. Hence she lost and is out. The third bettor for HHT bet on H when the outcome was T and lost.

For the players in line to bet on THH, the third from last lost on the first bet (T bet on, H appears). The same outcome happens with the second bettor. The third bettor, however, places a $-\$1$ wager on T (the first outcome in his sequence) during the third coin flip (also a T). So he wins and, since the game is over, walks away with $-\$2$.

As far as the "House" (the betting booth) is concerned, in this instance they pay out $\$8 + -\$2 = \$6$.

Case 2 - THH occurs first

For the players in line to bet on HHT, the third from last places a bet on H and T occurs, thus losing. The second from last bettor, places \$1 on H (first in his sequence) and since the second coin flip is H, wins. He then bets \$2 on H (second in his sequence) and the H which is the last flip in THH comes up and he wins a total of \$4. In a similar vein, the last of the HTT bettors comes to the betting window for the third flip and wins \$2.

For the players in line to bet on THH, third from last wins all three bets for a total of $-\$8$. Second to last loses, since her first bet is T, but the second outcome is H. The same happens to the third bettor.

So in this instance the House's payout is $\$4 + \$2 + -\$8 = -\$2$.

Now if we let P represent the probability that Case 1 actually occurs (so $P($ Player I wins $))$ and keep in mind that for martingales it must be that the expected value is zero, we arrive at

$$6P + -2(1 - P) = 0$$

The solution is $P = \frac{1}{4}$, as we knew.

Now we repeat our Wait Times result using martingales instead. Say N people have bet when the last T in THT has finally appeared. We will write $E[N]$ for the expected amount of money the house has put out at the end of having these N people bet and the game being over. Our goal is to use $E[N]$ to determine the value of N.

Since this is a fair coin, for every \$1 bet, \$1 is paid out. Also, since the expected value has to be zero, the amount of money coming in has to equal the amount of money going ou. Out of these N bettors, 2 of them actually won money. The third from last person won \$7 on their initial \$1 bet. The last person made \$1 profit. So the house brought in \$1 each from $N - 2$ players and paid out \$8 total on two players. We can write this as

$$E[N - 2] = 8$$

This expected wait is a linear operator. The expected payout (wait time) for $N-2$ people is the expected payout for N people less 2; i.e. $E[N - 2] = E[N] - 2$. Thus

$$E[N - 2] = E[N] - 2 = 8$$

So $E[N] = 10$, and since $E[N]$ actually counts the number of \$1 bets which is the same as the number of flips, we have our solution.

Why do we want martingales when we already knew the answer? Because if we do not have a fair coin (say the probability of Tails is p) we easily use this method to determine the probability of Player I winning. There is one adjustment to make, if the coin is not fair, then for the expected payout of a bet to be zero (above it was $1 \cdot \frac{1}{2} + -\$1 \cdot \frac{1}{2}$) it turns out that betting \$1 on Tails the bettors comes away with a total of $\frac{1}{p}$ dollars and betting \$1 on Heads the bettor receives a total of $\frac{1}{1-p}$ dollars.

3 Making Things Fair

Now we wish to look at unfair coins where $p(T)$, the probability that the coin lands on Tails, is $p \in [0, 1[$. Our first question is the straightforward one of, "Is there a way to make this in some way 'fair'? " That is, if we look at each of the three cases in Section 2.1 and want to make

$$P(\text{ Player I wins }) = P(\text{ Player II wins }) = \frac{1}{2}$$

what value of p do we need *and* how does this affect our other two cases?

3.1 Case 1 is Fair

To make things fair between the two players, TTT and HTT, if $P(\text{ Tails }) = \frac{1}{\sqrt[3]{2}}$. Then for Case 2, Player I wins with probability

$$\sum_{n=2}^{\infty}\left(\frac{1}{\sqrt[3]{2}}\right)^n\left(1-\frac{1}{\sqrt[3]{2}}\right) = \left(\frac{1}{\sqrt[3]{2}}\right)^2\left(1-\frac{1}{\sqrt[3]{2}}\right)\sum_{n=0}^{\infty}\left(\frac{1}{\sqrt[3]{2}}\right)^n = \left(\frac{1}{\sqrt[3]{2}}\right)^2 \approx 0.63$$

This gives the advantage to Player I.

For Case 3 we get the equation

$$x = \frac{1}{\sqrt[3]{2}} + \left(1-\frac{1}{\sqrt[3]{2}}\right)^2 x$$

which has solution

$$x = \frac{-2}{\sqrt[3]{4}-4} \approx 0.8289$$

3.2 Case 2 is Fair

If we make Case 2 fair, then we end up with the equation

$$\frac{1}{2} = \sum_{n=2}^{\infty} p^n(1-p) = (1-p)p^2\sum_{n=0}^{\infty} p = (1-p)p^2\frac{1}{1-p} = p^2$$

where $p = P(\text{ Tails })$. Thus $p = \frac{1}{\sqrt{2}}$.

For Case 1, we have

$$P(\text{ Player I wins }) = \left(\frac{1}{\sqrt{2}}\right)^3 = \frac{1}{2\sqrt{2}} \approx 0.3535$$

and

$$P(\text{ Player II wins }) = 1 - \frac{1}{2\sqrt{2}} \approx 0.6464$$

In Case 3, the equation becomes

$$x = \frac{1}{2\sqrt{2}} + \left(1-\frac{1}{2\sqrt{2}}\right)^2 x$$

Then

$$x = \frac{\sqrt{2}}{2\sqrt{2}-1}$$

So $P(\text{ Player I wins }) \approx 0.2265$ and $P(\text{ Player II wins }) \approx 0.7734$.

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3.3 Case 3 is Fair

Case 3 is actually an implausible situation. The equation

$$\frac{1}{2} = p + (1-p)^2\frac{1}{2}$$

has only $p = 0$ for a solution.

4 Wait Times with an Unfair Coin

Previously, when talking about wait times we had a collection of equations where the coefficients where $\frac{1}{2}$, the probability of getting a Head and the probability of getting a Tails. When these probabilities are replaces with $1-p$ and p, respectively the problems get more difficult to solve. It is this situation where the strength of the martingale really comes to play.

First we need to talk about the payouts. If p is the probability of Tails, then of course the probability of Heads is $1-p$. With a \$1 bet on Tails in order for the expected value to be zero the amount the dealer pays a winning bet is A and

$$A \cdot p + (-1) \cdot (1-p) = 1$$

and $A = \frac{1-p}{p} = \frac{1}{p} - 1$. For martingale purposes, if bettor places a \$1 bet on Tails and Tails appears, that bettor is paid off with $\$1/p$; the payout plus the return of the original \$1 bet. Similarly, a bettor placing a bet of \$1 on Heads walks away with a total of $\frac{1}{1-p}$ dollars. Winning bets in sequence have the payouts multiplied together. So when a player places bets on THT and all three win, the total the player wins is

$$\frac{1}{p} \cdot \frac{1}{1-p} \cdot \frac{1}{p}$$

We now can put all this together to solve out probability questions in an unfair Penney's Game.

Let us look at the previously example, THT. When THT finally appears, the third from last player has won three bets in a row for a total payout of $\frac{1}{p(1-p)p}$, for Tails, then Heads, then Tails again. The second player lost on their first bet (betting on T when Heads is tossed), and the third player wins $\frac{1}{p}$ for correctly betting T when the last flip is made. So the total of the house's money paid out (recall, the original \$1 bets are not the house's money) is

$$\left[\frac{1}{p^2(1-p)} - 1\right] + \left[\frac{1}{p} - 1\right]$$

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If N is the number of players (also the number of flips) the house collected on $N-2$ people and the payout and amount collected should be equal. Thus

$$E[N-2] = E[N] - 2 = \frac{1}{p^2(1-p)} + \frac{1}{p} - 2$$

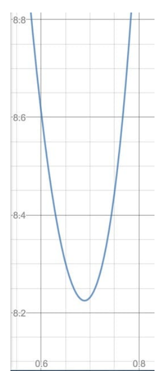
Thus the expected number of tosses is

$$E[N] = \frac{1}{p^2(1-p)} + \frac{1}{p}$$

Note that (a) the 2 from the two winners and 2 from the original dollars each player bets cancel each other out, and (b) this does give us the previous answer of 10 when $p = \frac{1}{2}$ for a fair coin.

Below is a section of the graph of $E[N]$ for THT where we see there is a minimum wait time. The minimum for THT occurs at the real root of $p^3 - 2p^2 - 2p + 2$. This is at

$$p = \sqrt[3]{-\frac{1}{27} + \sqrt{-\frac{32}{27}}} + \sqrt[3]{-\frac{1}{27} - \sqrt{-\frac{32}{27}}} + \frac{2}{3}$$



Since during our calculations the constants cancel out, we can just determine expected wait time as the sum of the products each of the winning bettor's winning sides. For a larger example, for the sequence TTT each of the last three bettors wins with the products being $\frac{1}{ppp}$, $\frac{1}{pp}$, and $\frac{1}{p}$. Thus the function of p that models the expected with time for TTT is

$$\frac{1}{p^3} + \frac{1}{p^2} + \frac{1}{p} = \frac{p^2+p+1}{p^3}$$

When Tails is certain ($p = 1$) the wait for three tails to appear is three flips, and as $p \to 0$ the wait time approaches infinitely long.

An appendix with all the wait times based on p, the probability the coin lands on Tails can be round in Appendix 7.1.

5 Penney's Game with an Unfair Coin

Now we wish to generalize things, looking at Penney's game where the probabilities are written as functions of p where p is the probability that the coin lands on heads. We will again use martingales. As with Section 4 the payout for Tails is $\frac{1}{p}$ and for Heads $1/(1-p)$ and for a sequence of bets that are won the probabilities are multiplied together.

Let us illustrate how we can find the probability for any Player I versus Player II with an example: Here we will look at Player I - HTH and Player II HHT. When $p = \frac{1}{2}$, the probability that Player I wins is $\frac{1}{3}$. When HTH appears first in the case of Player I's bettors, the third from last and last bettors win

$$\frac{1}{p(1-p)^2} \text{ and } \frac{1}{1-p}$$

respectively.

When HTH appears first in the case of Player II's bettors (who are paid in negative dollars) only the last bettor wins and the amount paid out is

$$-\frac{1}{1-p}$$

Thus the total the house pays out for HTH is

$$\frac{1}{p(1-p)^2} + \frac{1}{1-p} + \frac{-1}{1-p} = \frac{1}{p(1-p)^2}$$

In the case where HHT appears first the second from past bettor for HTH wins the amount

$$\frac{1}{p(1-p)}$$

For the HHT bettors, third from last is the only bettor who wins and the amount is

$$-\frac{1}{p(1-p)^2}$$

So this situation has a total house payout of

$$\frac{1}{p(1-p)} + -\frac{1}{p(1-p)^2}$$

Now let P be the probability that Player I wins. So the first house payout occurs with probability P and the second house payout occurs with probability $1-P$. Since the expected value is zero we have

$$P \cdot \frac{1}{p(1-p)^2} + (1-P) \cdot \left(\frac{1}{p(1-p)} + -\frac{1}{p(1-p)^2} \right) = 0$$

We can solve this for P and find that

$$P = f(p) = \text{ the probability that Player I wins } = \frac{p^2 - 2p + 1}{p(p-2)}$$

The chart in Appendix 7.2 summarizes the probabilities for all the possible Player I versus Player II matchups.

6 Conclusions

There are many more things to note about Penney's *Coin* Game and some question:

- The way Penney's Game is played, Player I makes a choice and then Player II chooses. We know the second player's choice if the coin is fair. If the coin is unfair, with p being the probability of Tails, then Appendix 7.3 contains information on Player Two's best choice. For example, if Player I chooses HHH, Player II should always choose THH, but if Player I chooses HTH, then Player II should choose HHT only if $p \leqslant .524$. After that, Player II should switch to TTH.

- Not all of the probability functions, $f(p)$ the probability that Player I wins, have a range of $[0, 1]$. Four of the functions do not.

Player I's Choice	Player II's Choice	$f(p)$	Range
HHT	HTH	$\dfrac{1}{1+p}$	$\left[0, \frac{1}{2}\right]$
HTH	THH	$\dfrac{p^2 - p + 1}{p+1}$	$\left[\frac{1}{2}, 1\right]$
TTH	THT	$\dfrac{1}{2-p}$	$\left[\frac{1}{2}, 1\right]$
THT	HTT	$\dfrac{-p^2 + p - 1}{p-2}$	$\left[\frac{1}{2}, 1\right]$

- How does this game apply to other objects, such as dice or spinners where each sector is not of uniform area?

- Suppose one is waiting for the first occurrence of more than one sequence. What then is the expected wait time?

References

[1] Steve Humble and Yutaka Nishiyama. "Hunble-Nishiyama Randomness Game: A New Variation on Penney's Coin Game". In: *IMA Mathematics Today* 46 (2010), pp. 194–195.

[2] Aaron M. Montgomery and Robert W. Vallin. "Penney's game from multiple perspectives". In: *Proceedings of the Recreational Mathematics Colloquium V—Gathering for Gardner Europe*.

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7 Appendices

7.1 Wait Times

Let p be the probability that the coin lands on Tails.

Player's Choice	Wait Time	Minimum Wait Time $(p, W(p))$
HHH	$\frac{p^2-3p+3}{-p^3+3p^2-3p+1}$	none
THH	$\frac{1}{p(1-p)^2}$	$\left(\frac{1}{3}, 6.75\right)$
HHT	$\frac{1}{p(1-p)^2}$	$\left(\frac{1}{3}, 6.75\right)$
HTH	$\frac{1}{p(1-p)^2}+\frac{1}{1-p}$	$(\approx 0.312, 8.225)$
TTH	$\frac{1}{p^2(1-p)}$	$\left(\frac{2}{3}, 6.75\right)$
THT	$\frac{1}{p^2(1-p)}+\frac{1}{p}$	$(\approx 0.688, 8.225)$
HTT	$\frac{1}{p^2(1-p)}$	$\left(\frac{2}{3}, 6.75\right)$
TTT	$\frac{1+p+p^2}{p^3}$	none

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7.2 Penney's Game

Let p be the probability that the coin lands on Tails.

Player I's Choice	Player II's Choice	Probability that Player I wins
HHH	THH	$(1-p)^3$
HHH	TTT	$\dfrac{p^5-2p^4+p^3-p^2+2p-1}{p^4-2p^3-p^2+2p-1}$
HHH	HHT	$1-p$
HHH	HTH	$\dfrac{p-1}{p^2-p-1}$
HHH	HTT	$\dfrac{p-1}{p^2-p-1}$
HHH	THT	$\dfrac{p^3-4p^2+5p-2}{-p^2+2p-2}$
HHH	TTH	$\dfrac{p^4-2p^2+2p-1}{p^3-3p^2+2p-1}$
HHT	HTH	$\dfrac{1}{p+1}$
HHT	THH	$(p-1)^2$
HHT	TTH	$\dfrac{p^3-p^2-p+1}{p^3-p^2+1}$
HHT	THT	p^3-2p^2+1
HHT	HTT	$\dfrac{1-p}{p^2-p+1}$
HHT	TTT	$\dfrac{p^4-p^3-p+1}{p^3-p+1}$
HTH	THH	$\dfrac{p^2-p+1}{p+1}$
HTH	TTH	p^3-p^2-p+1
HTH	THT	$\dfrac{p^3-p^2-p+1}{p^2-p+1}$
HTH	HTT	$-p+1$
HTH	TTT	$\dfrac{p^4-p^3-p+1}{p^2-p+1}$

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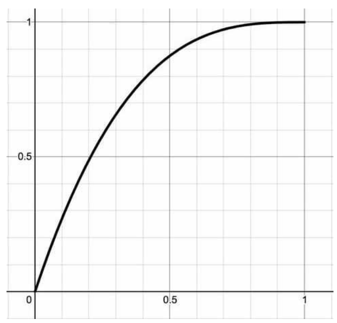
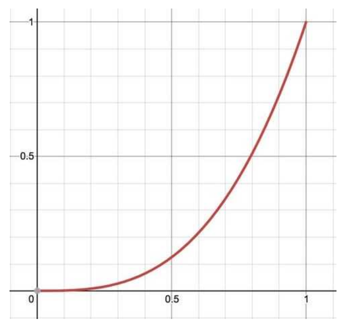
Player I's Choice	Player II's Choice	Probability that Player I wins
THH	TTH	$\dfrac{p^2-2p+1}{p^2-p+1}$
THH	THT	$-p+1$
THH	HTT	$-p+1$
THH	TTT	$\dfrac{p^3-p^2-p+1}{p^3-p+1}$
TTH	THT	$\dfrac{1}{2-p}$
TTH	HTT	p^2
TTH	TTT	$1-p$
THT	HTT	$\dfrac{-p^2+p-1}{p-2}$
THT	TTT	$\dfrac{p^2-1}{p^2-p-1}$
HTT	TTT	$1-p^3$

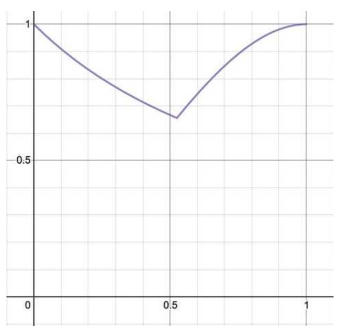
7.3 Penney's Game and Optimal Player II Choices



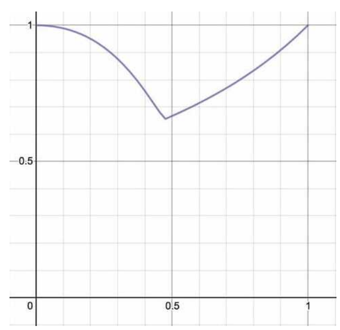
HHH
THH always



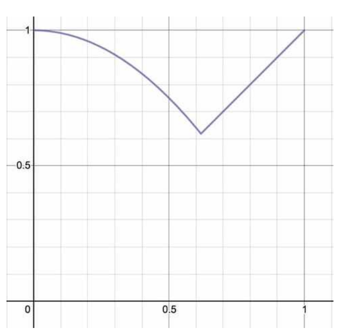
TTT
HTT always



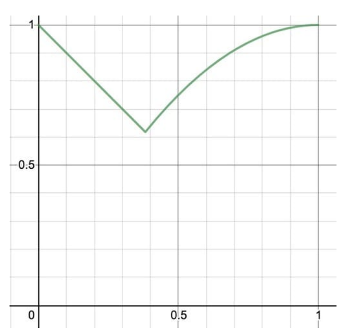
HTH
HHT, then TTH



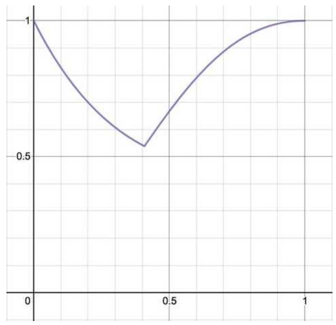
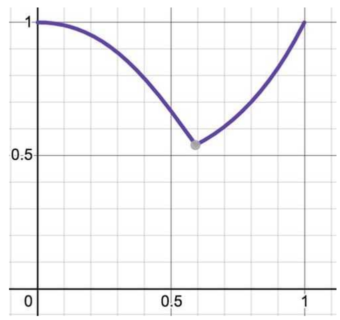
THT
HHT, then TTH



TTH
HTT, then TTT



HHT
HHH, then THH

	
THH HTH, then TTH	HTT HHT, then THT

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