Articles

$2 \times 2 \times 2$ COLOR CUBES

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Abstract: *Using six colors, one per side, cubes can be colored in* 30 *unique ways. In this paper, a row and column pattern in Conway's matrix always leads to a selection of eight cubes to replicate one of the 30 cubes. Each cube in the set of* 30 *has a* $2 \times 2 \times 2$ *replica with inside faces of matching color. The eight cubes of each replica can be configured in two different ways.*

Keywords: coloring problems, 30 color cubes.

Introduction

Often mathematics is a delight when connections are discovered. One of these connections is that each of the distinct cubes painted with six colors can be replicated with eight other cubes from the set of thirty. Although the eight cubes in the solution do not change they can be arranged in two distinct ways [1].

How Many Different Color Cubes?

How many different ways can a cube be painted with six different colors, one per face? Different means that a cube cannot rotate to match another cube. Place the cube on a flat surface; the bottom face takes one color. The top face may be painted with five different colors. This leaves four colors for the four side faces. Fix a side face to prevent rotation equivalents which takes one color. Three side faces and three colors are left. The way the remaining colors are ordered is important resulting in six different possibilities (a permutation of three taken three at a time). Therefore, each of the top faces has six side possibilities resulting in 30 different color cubes.

Arrangement of Cubes

John Conway (University of Cambridge) set the 30 color cubes up in a 6×6 matrix (see appendix). I numbered the cubes in that matrix. The rows start with Cube 1, Cube 6, Cube 11, Cube 16, Cube 21, and Cube 26 on the left. In addition, the rows of the matrix can be labeled A − F and the columns a − f. No cubes are located along the main diagonal (Aa, Bb, Cc, Dd, Ee, and Ff). Conway had reasons for placing cubes in certain positions in the matrix. For example, cubes that are reflections of one another are strategically placed in the matrix. One of the benefits of using Conway's matrix is that the cubes for every $2 \times 2 \times 2$ color cube fall into a pattern.

$2 \times 2 \times 2$ First Solution

Each of the 30 color cubes can be replicated with eight other cubes of the 30 color cube set. An additional condition is that cubes meeting on the inside of the $2 \times 2 \times 2$ structure have the same color often called the domino condition.

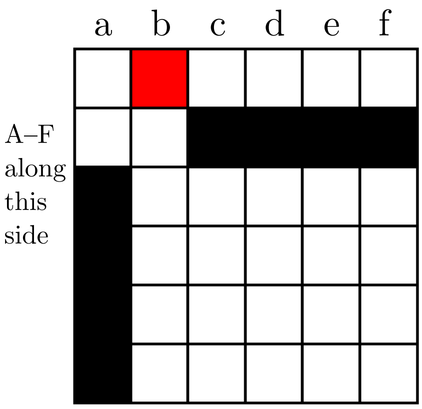

Figure 1: Conway's matrix.

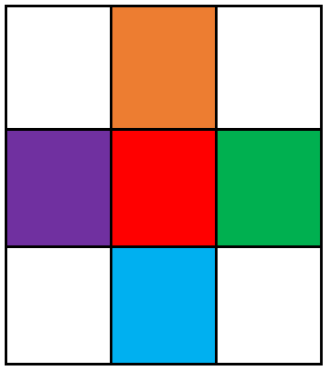

Figure 2: Cube 1, row A, column b (Ab) (top view).

By switching the row and column designations where the cube is positioned, the cubes used to construct a $2 \times 2 \times 2$ replica are located. For example, Cube 1 (Ab), shown in Figure 2, is in the first row, A and second column, b (red square in Figure 1). The cubes will be located in the second row, B and first column, a (black squares in Figure 1). Ba is not in the solution; the switched row/column designation cube is never in the solution. Aa and Bb are on the diagonal and contain no blocks. All other cubes in row B (Bc, Bd, Be, Bf) and column a (Ca, Da, Ea, Fa) form the solution. In general, a cube located in row X, column y (Xy) designates cubes from row Y, column x with Cube Yx not in the solution.

The configuration of the solution for the $2 \times 2 \times 2$ replica of Cube 1 (Ab) is shown below in Figure 3.

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Da	**Bc**	**Top layer**	**Be**	**Fa**	**Bottom layer**
Bf	**Ea**		**Ca**	**Bd**	

Front

Figure 3: Positions of cubes in $2 \times 2 \times 2$ solutions (top view).

The view is looking down on the cubes. Figure 4 shows the cubes and their orientation. Notice the top layer faces join with the color yellow and the bottom layer join with the color red. The rear top blocks, Da and Bc, have a bottom color of blue that corresponds with the top of Be and Fa. The front top blocks, Bf and Ea, have a bottom color of tan that corresponds with the top of Ca and Bd. The domino condition is satisfied and the $2 \times 2 \times 2$ block has the same face colors as Block 1 pictured in Figure 2.

Orientation of the eight color cubes in the first $2 \times 2 \times 2$ solution are shown in Figure 4.

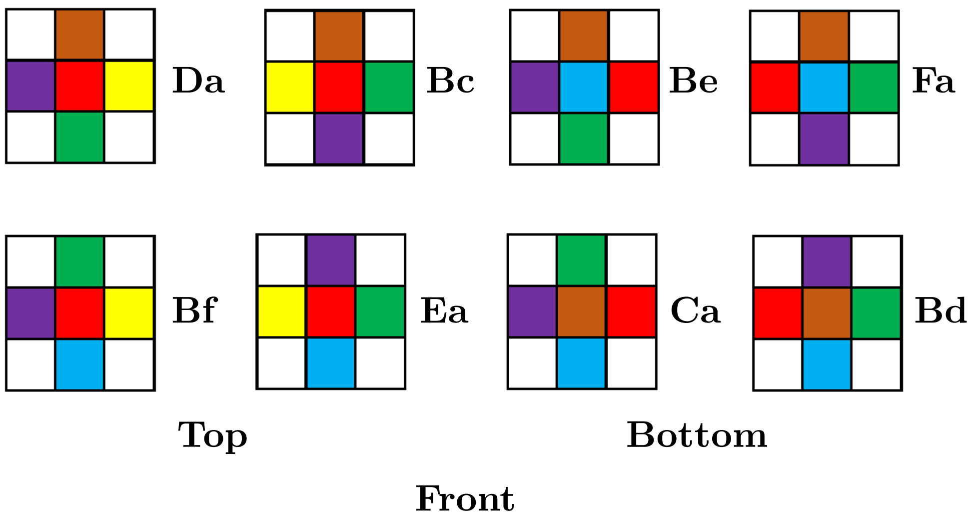

Figure 4: Individual color cubes oriented in the first solution (top view).

$2 \times 2 \times 2$ Second Solution

A second solution can be obtained by transforming the cubes of the first solution. The original top becomes the bottom and the bottom becomes the top. The second solution will produce a $2 \times 2 \times 2$ cube rotated 180 degrees from the original $2 \times 2 \times 2$ cube (Figure 5). Any $2 \times 2 \times 2$ replica will produce a second solution that is rotated 180 degrees. After the blocks are in the proper position the transformations in Table 1 are necessary to get the $2 \times 2 \times 2$ Cube 1 (Ab). The transformations in Table 1 are not a generalized process to obtain the second solution. Cubes are rotated backward, forward, left, or right followed by a clockwise or counterclockwise rotation. Looking at a model of the solution cube (Figure 5) makes the transformation process easier.

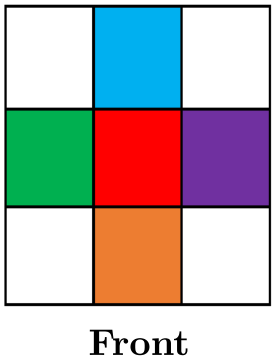

Figure 5: Cube 1 rotated 180 degrees (top view).

Original top → New bottom (looking from the top)	
Da roll right 90°; rotate clockwise 90°	Bc roll left 90°; rotate counterclockwise 90°
Bf roll right 90°; rotate counterclockwise 90°	Ea roll left 90°; rotate clockwise 90°

Original bottom → New top (looking from the top)	
Be roll left 90°; rotate clockwise 90°	Fa roll right 90°; rotate counterclockwise 90°
Ca roll left 90°; rotate counterclockwise 90°	Bd roll right 90°; rotate clockwise 90°

Table 1: A listing of the transformation leading to a second solution.

Figure 6 shows the orientation of the individual cubes in the second $2 \times 2 \times 2$ solution. Notice the cubes join faces with the same color. Be and Ca have a bottom color of purple, and Fa and Bd have a bottom color of green. The original top cubes are now the bottom cubes with yellow as a bottom color, the same bottom color as the first solution. The original bottom cubes are now the top cubes with red as the top color, the same top color as the original solution.

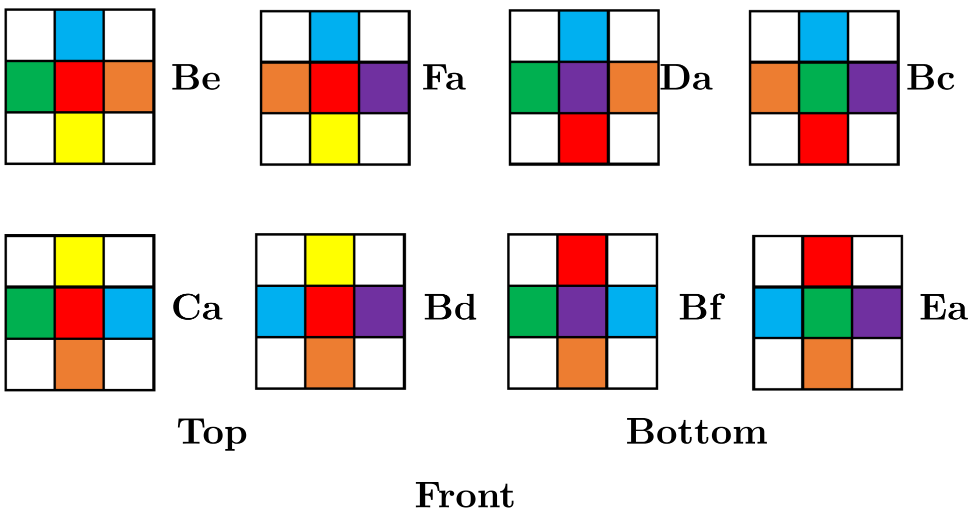

Figure 6: Individual color cubes oriented in the first solution (top view).

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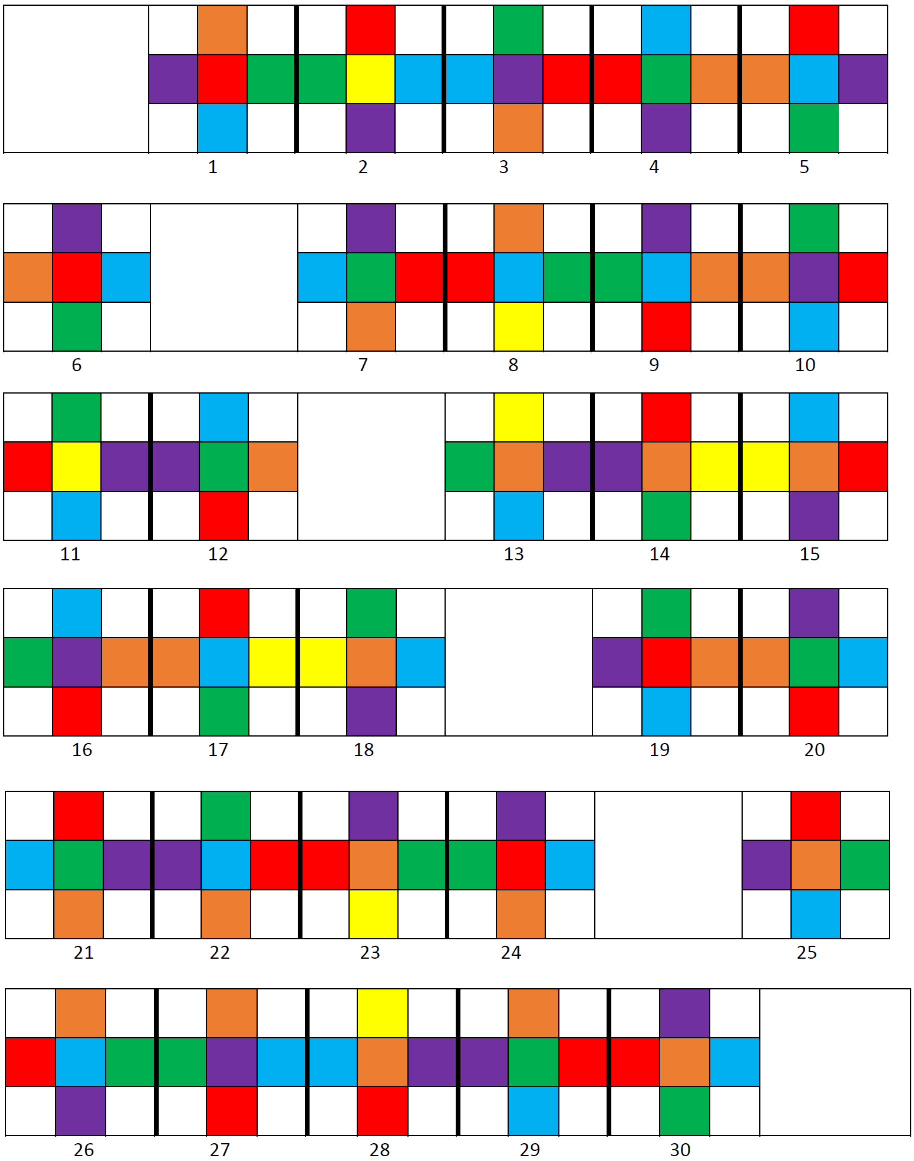
Conclusion

All thirty color cubes have two $2 \times 2 \times 2$ replica cubes with the domino condition for inside faces. The individual cubes in a $2 \times 2 \times 2$ replica cube always follow a row and column pattern in Conway's matrix. And, the first solution transforms into the second solution resulting in a rotation of 180 degrees while the top layer of cubes always switch with the bottom layer. The transformations are slightly different for each solution, but they are selected from a small set of possibilities.

References

[1] Gardner, M. "Mathematical games: Puzzling over a problem-solving matrix, cubes of many colors and three-dimensional dominoes", *Scientific American*, 239(1), 20–31, 1978.

Appendix



Conway's matrix ([1], page 22)

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